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# Calibrating Linear Continuous-Time Dynamical Systems via Perturbation Analysis

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**Abstract.** This work considered the continuous-time linear dynamical systems described by the matrix differential equations, and aimed at studying the perturbation analysis via solving perturbed linear dynamical systems. In specific, we solved Riccati differential equations and continuous-time algebraic Riccati equations with finite and infinite times respectively. Moreover, we stated some assumptions on the existence and uniqueness of the solutions of the perturbed Riccati equations. Similar techniques were applied to the discrete-time linear dynamical systems. Two numerical examples illustrated the efficiency and accuracy.

## 1. Introduction

By discretizing the partial differential equations (PDEs), we obtain the continuous-time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \quad 0 \le t \le t_1, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

with coefficient matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times m}$, state $x(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^r$. Traditionally, the optimal control $u$ is used to influence the state $x$ for output $y$, and the Riccati differential equations (RDEs) can be solved via the minimization of the cost function using an optimal control method. The resolution of the continuous-time algebraic Riccati equation (CARE) with infinite time can the be obtained.

In this paper, we propose an alternative strategy to get accurate solutions of RDEs and CAREs associated with the stochastic model by applying the Hamiltonian transformation and the structure-preserving doubling algorithm (SDA). An efficient method via the stochastic model of RDEs and CAREs with time-independent coefficients is used instead of the expected values and the variances by a huge number of simulations. Such method is more efficient than other methods such as polynomial chaos methods [6]. We also provide sufficient conditions to guarantee that the RDEs and CAREs have the unique and stable solutions. Similar phenomena also work well in the discrete-time linear systems of Riccati difference equations (RdEs) and discrete-time algebraic Riccati equations (DAREs).

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We organize this paper as follows. We introduce the continuous-time linear system and discuss methods of solving RDEs and CAREs in Section 2, with some corresponding properties on the existence and uniqueness of the solutions of the perturbed RDEs and CAREs. Section 3 discusses the discrete-time linear system, the RdEs and DAREs in specific. We present in Section 4 some illustrative examples that are found from some practical applications.

## 2. Continuous-time Linear Dynamical System

We first assume $D = 0$ without loss of generality and add perturbation parameters $p = (p_1, p_2, \cdots, p_q) \in \Pi \subseteq \mathbb{R}^q$ to each coefficient matrices $A$, $B$ and $C$ stepwise, then consider three kinds of cases to discuss the sensitivity with respect to $p$. We only list one case when $A$ is perturbed.

$$\begin{aligned} \dot{x}(t;p) &= A(p)x(t;p) + Bu(t;p), \; x(0;p) = \tilde{x}_0, \\ y(t;p) &= Cx(t;p), \end{aligned} \tag{2}$$

where $A(p) = A_0 + \sum_{j=1}^{q} a_j(p_j)A_j$, with constant matrices $A_0$, $A_1$, $\cdots$, $A_q \in \mathbb{R}^{n\times n}$ and continuous linear or nonlinear functions $a_j$: $\mathbb{R} \to \mathbb{R}$. The perturbation parameter $p_j$ comes from a traditional distribution such as uniform type, Gaussian type or beta type. We set $A_0 = A$ and divide $A$ into some submatrices according to nonzero entries in positions such as $A = \sum_{j=1}^{q} A_j$.

### 2.1. Continuous-time Riccati Differential Equation

Assume that $\det(sI_n - A(p)) \neq 0$, for some $s$, the linear system (2) consists of a unique parameterized state vector $x(t;p)$. Furthermore, (2) produces only one output vector $y(t;p)$. There are some methods to solve (2) and we consider a common method from the control theory that chooses the parameterized optimal control $u(t;p)$ to minimize a cost function

$$J(u, \tilde{Q}_1, t_1) \equiv \frac{1}{2}\int_0^{t_1} [x^\top(t;p)Hx(t;p) + u^\top(t;p)Ru(t;p)]dt + \frac{1}{2}x^\top(t_1;p)\tilde{Q}_1 x(t_1;p), \; \tilde{Q}_1 = Q_1 + \Delta Q_1,$$

where $\tilde{Q}_1$ is a symmetric positive semidefinite (s.p.s.d.) matrix and $t_1$ is a terminal time. The optimal control $u$ is found by solving the parameterized Riccati differential equation (pRDE):

$$\dot{X}(t;p) = -H - A^\top(p)X(t;p) - X(t;p)A(p) + X(t;p)GX(t;p), \; X(t_1;p) \equiv \tilde{Q}_1, \; 0 \leq t \leq t_1, \tag{3}$$

with $H = C^\top C$, $G = BR^{-1}B^\top$ and the parameterized optimal control has resulted

$$u(t;p) = -R^{-1}B^\top X(t;p)x(t;p), \;\; 0 \leq t \leq t_1. \tag{4}$$

The RDEs arise in many quantitative and mathematical problems. Before solving the pRDEs (3), we describe some properties on the existence and uniqueness of solutions.

**Theorem 2.1.** *[3, Theorem 8] Let $X(t;p)$ be the solution of (3) with parameter dependency that passes through $\tilde{Q}_1$ at $t = t_1$. Then the parameterized solution $X(t;p)$ exists and is unique on $0 \leq t \leq t_1$ regardless of $\tilde{Q}_1$ at some parameter $p$ for any given $t$.*

In this paper, we apply the Bernoulli substitution technique to transform pRDEs (3) into an equivalent parameterized linear differential Hamiltonian system.

Consider the parameterized Hamiltonian differential equation (pHDE):

$$\begin{bmatrix} \dot{N}_1(t;p) \\ \dot{N}_2(t;p) \end{bmatrix} = \begin{bmatrix} A(p) & -G \\ -H & -A^\top(p) \end{bmatrix} \begin{bmatrix} N_1(t;p) \\ N_2(t;p) \end{bmatrix}, \tag{5}$$

where $N_1(t;p)$, $N_2(t;p) \in \mathbb{R}^{n\times n}$. We describe the relation between the solutions of pRDEs and pHDEs as follows.

**Theorem 2.2.** *[2] Let $N_1(t;p)$, $N_2(t;p)$: $[0,t_1] \to \mathbb{R}^{n\times n}$ be the solutions of the pHDEs. Then*

1. *For all $t \in [0,t_1]$, $N_1(t;p)$ is non-singular;*
2. *The pRDEs (3) have the parameterized solution $X(t;p) = N_2(t;p)N_1^{-1}(t;p)$, $t \in [0,t_1]$.*

If we can solve the pHDEs (5), then we can also solve the pRDEs (3) using the Theorem 2.2. Let

$$\Phi(p) = \begin{bmatrix} A(p) & -G \\ -H & -A^\top(p) \end{bmatrix}$$

be the Hamiltonian matrix and we apply the diagonalization to the parameterized Hamiltonian matrix $\Phi(p)$ so that

$$\Phi(p) = R(p)\Psi(p)R^{-1}(p),$$

where each column vector of $R(p) \in \mathbb{R}^{2n\times 2n}$ is each eigenvector of $\Phi(p)$ and $\Psi(p) \in \mathbb{R}^{2n\times 2n}$ is the diagonal matrix and its diagonal elements are eigenvalues of $\Phi(p)$. Therefore, the solution of pHDE (5) can be generally represented as

$$\begin{bmatrix} N_1(t;p) \\ N_2(t;p) \end{bmatrix} = e^{\Phi(p)t}C = e^{\Phi(p)(t-t_1)} \begin{bmatrix} I_n \\ \tilde{Q}_1 \end{bmatrix},$$

where $C \in \mathbb{R}^{2n\times n}$ is an integration constant that depends on the boundary condition. If we simplify the formula by partitions

$$e^{\Phi(p)(t-t_1)} = \begin{bmatrix} e_1(t;p) & e_2(t;p) \\ e_3(t;p) & e_4(t;p) \end{bmatrix},$$

where $e_i(t;p) \in \mathbb{R}^{n\times n}$, for $i = 1,2,3,4$, the solutions of pRDEs (3)

$$X(t;p) = N_2(t;p)N_1^{-1}(t;p) = (e_3(t;p) + e_4(t;p)\tilde{Q}_1)(e_1(t;p) + e_2(t;p)\tilde{Q}_1)^{-1}.$$

As a result, the parameterized linear systems (2) can be expressed using (4)

$$\begin{aligned} \dot{x}(t;p) &= [A(p) - BR^{-1}B^\top X(t;p)]x(t;p), \\ &\equiv M(t;p)x(t;p), \end{aligned} \tag{6}$$

where $M(t;p) \equiv A(p) - BR^{-1}B^\top X(t;p) \in \mathbb{R}^{n\times n}$. Let $S(t;p)$ be the anti-derivative of $M(t;p)$, we divide the parameterized state vector $x(t;p)$ on the both side in the differential equation (6) and take an indefinite integral:

$$x(t;p) = e^{S(t;p)} \cdot K(p),$$

where $K(p) \in \mathbb{R}^{n\times 1}$ is the constant vector with the parameter $p$. We apply the eigen-decomposition to $S(t;p)$ and get

$$S(t;p) = V(t;p)\Lambda(t;p)V^{-1}(t;p),$$

where $V(t;p)$ and $\Lambda(t;p) \in \mathbb{R}^{n\times n}$. Hence, the representation of the parameterized state vector $x(t;p)$ associated with the parameterized initial state vector $\tilde{x}_0$ can be rewritten into

$$x(t;p) = V(t;p)e^{\Lambda(t;p)}V^{-1}(t;p)K(p),$$

where $K(p) \equiv [e^{S(0;p)}]^{-1}\tilde{x}_0$. The following parameterized output vector can be obtained via (2)

$$y(t;p) = CV(t;p)e^{\Lambda(t;p)}V^{-1}(t;p)K(p).$$

### 2.2. Continuous-time Algebraic Riccati Equation

We derive the parameterized CARE (pCARE) from the pRDE with $t_1 \to \infty$:

$$\mathcal{C}(X(p)) \equiv H + A^\top(p)X(p) + X(p)A(p) - X(p)GX(p) = 0. \tag{7}$$

We make the following assumptions for a unique stabilizing solution:

**Assumption**

- The parameterized pair $(A(p), B)$ in pCARE (7) is c-stabilizable.
- The parameterized pair $(A(p), C)$ in pCARE (7) is c-detectable.

We seek to solve the parameterized continuous-time linear optimal control problem

$$\min_u J = \frac{1}{2}\int_0^\infty [x^\top(t;p)Hx(t;p) + u^\top(t;p)Ru(t;p)]dt,$$

and the following parameterized optimal control is derived

$$u(t;p) = -R^{-1}B^\top(p)X(p)x(t;p), 0 \le t \le t_1, \tag{8}$$

where $X(p)$ is the s.p.s.d. solution to the pCARE (7).

We adapt in this paper the SDA [1] to get the parameterized s.p.s.d. solution $X(p)$ of the pCARE (7), then substitute the parameterized optimal control $u(t;p)$ (8) in (2) and get

$$\begin{aligned} \dot{x}(t;p) &= (A(p) - BR^{-1}B^\top X(p))x(t;p), \\ &= W(p)x(t;p), \end{aligned} \tag{9}$$

where $W(p) \equiv A(p) - BR^{-1}B^\top X(p) \in \mathbb{R}^{n\times n}$. We apply the eigen-decomposition to $W(p)$ and get

$$W(p) = T(p)\varphi(p)T^{-1}(p),$$

where $T(p)$ and $\varphi(p) \in \mathbb{R}^{n\times n}$. As a result, we obtain the solution of (9)

$$x(t;p) = T(p)e^{\varphi(p)t}T^{-1}(p)L(p),$$

with $L(p) \equiv \tilde{x}_0$. The following parameterized output vector is represented

$$y(t;p) = CT(p)e^{\varphi(p)t}T^{-1}(p)L(p).$$

$B(P)$ and $C(p)$ are derived in the similar ways.

## 3. Discrete-Time Linear Dynamical System

The discrete-time linear dynamical system is stated below:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k,\ k = 0, 1, 2, \ldots, \text{given } x_0, \\ y_k &= Cx_k + Du_k, \end{aligned} \tag{10}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times m}$. Assume that $D = 0$ in (10), then we add some perturbation parameters to coefficient matrices $A$, $B$ and $C$ stepwise. We only discuss the perturbed matrix $A(p)$ in (10) and obtain the perturbed discrete-time linear system

$$\begin{aligned} x_{k+1}(p) &= A(p)x_k(p) + Bu_k(p),\ k = 0, 1,\ \ldots,\ \text{given } x_0(p), \\ y_k(p) &= Cx_k(p). \end{aligned} \tag{11}$$

*3.1. Discrete-Time Riccati Difference Equation*

Analogously, we solve the parameterized Riccati difference equation (pRdE) in order to get the parameterized optimal control via minimizing the parameterized cost function:

$$X_{k-1}(p) = A^\top(p)X_k(p)A(p) - A^\top(p)X_k(p)B(B^\top X_k(p)B + R)^{-1}B^\top X_k(p)A(p) + H, \tag{12}$$

for $k = N, N-1, \cdots, 1$, $H = C^\top C$ and $X_N(p) = Q_2$. Applying the Sherman-Morrison-Woodbury formula (SMWF), we simplify the pRdE (12) into

$$X_{k-1}(p) = B_2^\top(p)X_k(p)(I_n + C_2(p)R^{-1}C_2^\top(p)X_k(p))^{-1}B_2(p) + H,\ X_N(p) = Q_2,\ k = N, \dots, 1. \tag{13}$$

Before we solve pRdE (13), We state some assumptions about its unique solution.

**Assumption**

- The parameterized pair $(A(p), B)$ in pRdE (13) is d-stabilizable.
- The parameterized pair $(A(p), C)$ in pRdE (13) is d-detectable.

We use the back substitution to get the parameterized solution $X_{k-1}(p)$ with boundary condition $X_N(p) = Q_2$, for $k = N, \dots, 1$ in (13). Consequently, the parameterized state in (11) can be obtained via the parameterized optimal control

$$x_{k+1}(p) = [A(p) - B(R + B^\top X_k(p)B)^{-1}B^\top X_k(p)A(p)]x_k(p),\ \text{given } x_0(p).$$

The following parameterized output vector can be also obtained:

$$y_k(p) = Cx_k(p).$$

*3.2. Discrete-Time Algebraic Riccati Equation*

With $N \to \infty$, we solve the parameterized discrete-time algebraic Riccati equation (pDARE)

$$\mathcal{D}(X(p)) = A^\top(p)X(p)A(p) - X(p) - A^\top(p)X(p)B(R + B^\top X(p)B)^{-1}B^\top X(p)A(p) + H = 0. \tag{14}$$

We adapt the efficient method called SDA to obtain the unique parameterized s.p.s.d. solution $X(p)$ to the pDARE (14) and apply the similar technique to get the parameterized state vector

$$x_{k+1}(p) \quad = \quad [A(p) - B(R + B^\top X(p)B)^{-1}B^\top X(p)A(p)]x_k(p),\ \text{given } x_0(p),$$

for $k = 0, 1, \dots$. Moreover, the parameterized output vector $y_k(p)$ is also obtained. $B(p)$ and $C(p)$ are derived similarly.

## 4. Numerical Experiments

The simulations were carried out on an Acer desktop and the codes are written in MATLAB [4] Version R2016a. The desktop processor is 3.40 GHz Intel Core 2 Duo, the memory is 32 GBs, and the machine accuracy is $eps = 2.22 \times 10^{-16}$.

We have selected two representative examples:

(1) The continuous-time linear dynamical system in Example 4.1 is constructed as in [7].

(2) The discrete-time linear dynamical system in Example 4.2 is constructed as in [5].

For the numerical results, we present the state vectors $x(t)$, $xk$ and output vectors $yk$, arising from RDEs, RdEs, CAREs, DAREs with finite and infinite times in the continuous- and discrete-time linear systems, presented in Figures 1, 2 and 3.

*Example 4.1 [7, Example 1]*

The continuous-time linear dynamical system is modified from [7] with $n = m = r = 2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, R = I_2,\ C = 0_{2\times 2}.$$

We set $t_1 = \frac{\Pi}{2}$ and time $t \in [0, 2]$ for RDEs and $t \in [0, 5]$ for CAREs, then we divide time into 20 parts and draw two figures that relate the state vectors $x(t)$ and time $t$, depicting in the Figure 1 solved by RDEs (left) and CAREs (right), respectively. We only consider the trends of state vectors with respect to time since output vectors are zeros.

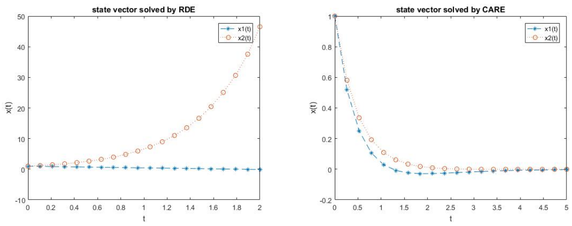


Figure 1: Continuous-Time System: State vectors solved by RDE (Left) and CARE (Right)

In Figure 1 (left), two elements start around 1, then the first element $x_1(t)$ decreases slowly to 0 but the second element $x_2(t)$ goes up drastically to about 47 when $t = 2$. However, two elements in Figure 1 (right) decay together from $t = 0$ to 1.8, then the second element $x_2(t)$ keeps decreasing but the first element $x_1(t)$ slightly increases before they reach to 0.

*Example 4.2 [5]*

This example is about an inverted pendulum on a cart. We apply the finite difference method with the sampling period $T = 6 \times 10^{-3}s$ to get coefficient matrices of the discrete-time system:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -0.000012155 & 0.0053894 & -4.6894 \times 10^{-9} \\ 0 & 1.0030 & 0.0095869 & 0.0060010 \\ 0 & -0.0039102 & 0.80361 & -5.7031 \times 10^{-6} \\ 0 & 0.98480 & 3.0841 & 1.0013 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ B &= [0.00060449\ -0.0094909\ 0.19442\ -3.0532]^\top,\ R = [0.01],\ x_0 = [1\ 1\ 1\ 1]^\top,\ D = [0\ 0]^\top, \\ Q &= \begin{bmatrix} 1+a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ X_{20} = \begin{bmatrix} 246.35 & 187.01 & 97.138 & 6.6726 \\ 187.01 & 976.62 & 268.88 & 19.886 \\ 97.138 & 268.88 & 124.67 & 8.6264 \\ 6.6726 & 19.886 & 8.6264 & 1.6057 \end{bmatrix}. \end{aligned}$$

We set time $k = 20$ and $a = 2$, then draw four figures. Figures 2 and 3 describe the time-dependent relationships of state vectors $x_k$ and the vectors $y_k$ respectively. In each case. the left figure shows the solution of RdEs and the right figure shows that of DAREs.

In the Figure 2, the trends are similar. The first and fourth components $xk_1$ and $xk_4$ increase with an increase of time, but the second component $xk_2$ decreaess. Furthermore, the third component $xk_3$ solved by RdEs continues to increase to the end. However, the component $xk_3$ solved by DAREs increases until $k = 17$, then it slightly decays at the end.

In the Figure 3, the first component $yk_1$ always increases from 1 to around 1.24 when $k = 22$ but the second component $yk_2$ decreases to 0.19 on the left figure (RdEs) and 0.06 on the right figure (DAREs), respectively.

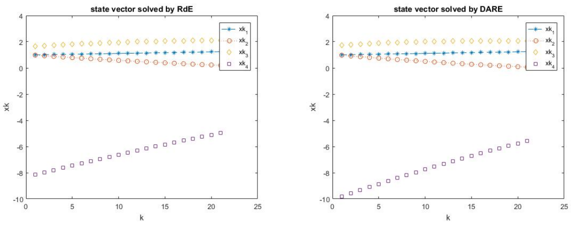


Figure 2: Discrete-Time System: State vectors solved by RdE (Left) and DARE (Right)

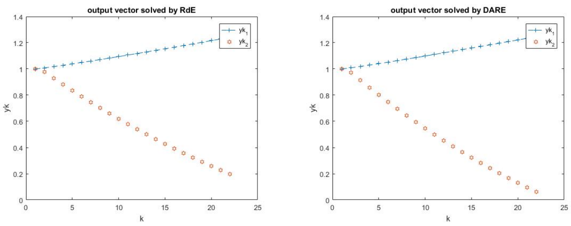


Figure 3: Discrete-Time System: Output vectors solved by RdE (Left) and DARE (Right)

## 5. Conclusions

We have discussed a stochastic model of the parameter-dependence system for continuous- and discrete-time linear dynamical systems, using solutions of RDEs (finite time), CAREs (infinite time) and RdEs (finite time), DAREs (infinite time). Our numerical experiments show that some perturbation matrices with small stochastic parameters will make different results compared to the original continuous-time linear system due to the physical parameters structure within the matrices. Such situation also holds for discrete-time linear system.

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