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Some Fejér Type Integral Inequalities for Geometrically-Arithmetically-Convex Functions with Applications

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Abstract. In this paper, the notion of geometrically symmetric functions is introduced. A new identity involving geometrically symmetric functions is established, and by using the obtained identity, the Hölder integral inequality and the notion of geometrically-arithmetically convexity, some new Fejér type integral inequalities are presented. Applications of our results to special means of positive real numbers are given as well.

1. Introduction

The classical or the usual convexity is defined as follows:
A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

A number of papers have been written on inequalities using the classical convexity and one of the most fascinating inequalities in mathematical analysis is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Both the inequalities hold in reversed direction if f is concave. The inequalities stated in (1) are known as Hermite-Hadamard inequalities.

For more results on (1) which provide new proofs, noteworthy extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [2, 3, 6, 9, 10, 20, 21] and the references therein.

The usual notion of convex functions have been generalized in diverse manners. One of them is the so called GA-convex functions and is stated in the definition below.

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Definition 1.1. *[13, 14] A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^{\lambda} y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda} y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda) f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

The definition of GA-convexity is further generalized as GA-s-convexity in the second sense as follows.

Definition 1.2. *[7] A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be s-GA-convex function on I if*

$$f\left(x^{\lambda} y^{1-\lambda}\right) \leq \lambda^{s} f(x) + (1-\lambda)^{s} f(y)$$

holds for all $x, y \in I$, $\lambda \in [0,1]$ and for some $s \in (0,1]$.

For the properties of GA-convex functions and GA-s-convex functions, we refer the reader to [5, 7, 13, 14, 23, 24] and the reference there in.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for GA-convex and for GA-s-convex functions.

Zhang et al. in [24] established the following Hermite-Hadamard type integral inequalities for GA-convex function.

Theorem 1.3. *[24] Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^{\circ}$ with $a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^{q}$ is GA-convex on $[a,b]$ for $q \geq 1$, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{\left[(b-a) A(a,b)\right]^{1-\frac{1}{q}}}{2^{\frac{1}{q}}} \left\{ \left[L\left(a^2, b^2\right) - a^2\right] \left|f'(a)\right|^{q} + \left[b^2 - L\left(a^2, b^2\right)\right] \left|f'(b)\right|^{q} \right\}^{\frac{1}{q}}. \quad (2)$$

Theorem 1.4. *[24] Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a function differentiable function on I° and $a, b \in I^{\circ}$ with $a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^{q}$ is GA-convex on $[a,b]$ for $q > 1$, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq (\ln b - \ln a) \left[L\left(a^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}}\right) \right]^{1-\frac{1}{q}} \left[A\left(\left|f'(a)\right|^{q} \left|f'(b)\right|^{q} \right) \right]^{\frac{1}{q}}. \quad (3)$$

Theorem 1.5. *[24] Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^{\circ}$ with $a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^{q}$ is GA-convex on $[a,b]$ for $q \geq 1$, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{(2q)^{\frac{1}{q}}} \left[L\left(a^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}}\right) \right]^{1-\frac{1}{q}}$$
$$\times \left\{ \left[L\left(a^{2q}, b^{2q}\right) - a^{2q}\right] \left|f'(a)\right|^{q} + \left[b^{2q} - L\left(a^{2q}, b^{2q}\right)\right] \left|f'(b)\right|^{q} \right\}^{\frac{1}{q}}. \quad (4)$$

Theorem 1.6. *[24] Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^{\circ}$ with $a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^{q}$ is GA-convex on $[a,b]$ for $q > 1$ and $2q > p > 0$. Then*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{p^{\frac{1}{q}}} \left[L\left(a^{\frac{2q-p}{q-1}}, b^{\frac{2q-p}{q-1}}\right) \right]^{1-\frac{1}{q}}$$
$$\times \left\{ \left[L\left(a^{p}, b^{p}\right) - a^{p}\right] \left|f'(a)\right|^{q} + \left[b^{p} - L\left(a^{p}, b^{p}\right)\right] \left|f'(b)\right|^{q} \right\}^{\frac{1}{q}}. \quad (5)$$

Applications of the above results to special means are given in [24] as well.

İşcan [7], proved the following result for GA-s-convex functions in the second sense.

Theorem 1.7. *Suppose that $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is s-GA-convex in the second sense and $a, b \in I$ with $a < b$. If $f \in L([a,b])$, then one has the inequalities:*

$$2^{s-1} f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{f(a)+f(b)}{s+1}. \tag{6}$$

If f in Theorem 1.7 is GA-convex function, then we get the following inequalities.

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{f(a)+f(b)}{2}. \tag{7}$$

For more results on GA-convex functions and s-GA-convex functions see e.g. [5], [7], [11], [13], [22], [23] and [24].

In Section 2, we will introduce a new notion of geometrically symmetric functions and by using this notion we prove a weighted generalization of (7). In Section 2, we will also establish a new weighted identity to provide more general and better estimates for the difference between the right most and the middle terms of the weighted version of (7).

2. Main Results

Throughout this section we take $U(t) = a^{(1-t)/2} b^{(1+t)/2}$ and $L(t) = a^{(1+t)/2} b^{(1-t)/2}$. The Beta function and the integral from of the hypergeometric function are defined as follows to be used in the sequel of the paper

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt, \alpha > 0, \beta > 0$$

and

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt$$

for $|z| < 1, \gamma > \beta > 0$.

The notion of geometrically symmetric functions is given in following definition.

Definition 2.1. *A function $g : [a,b] \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be geometrically symmetric with respect to $\sqrt{ab}$ if*

$$g\left(\frac{ab}{x}\right) = g(x)$$

holds for all $x \in [a,b]$.

Theorem 2.2. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx \leq \int_a^b \frac{f(x)g(x)}{x}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx. \tag{8}$$

Proof. By the GA-convexity of f, we have

$$\begin{aligned} f\left(\sqrt{ab}\right)\int_0^1 g\left(a^{1-t}b^t\right)dt &\leq \int_0^1 \left[\frac{1}{2}f\left(a^t b^{1-t}\right) + \frac{1}{2}f\left(a^{1-t}b^t\right)\right] g\left(a^{1-t}b^t\right)dt \\ &= \frac{1}{2}\int_0^1 f\left(a^t b^{1-t}\right) g\left(a^{1-t}b^t\right)dt + \frac{1}{2}\int_0^1 f\left(a^{1-t}b^t\right) g\left(a^{1-t}b^t\right)dt. \end{aligned} \tag{9}$$

By geometrical symmetry of g with respect to $\sqrt{ab}$, we also have

$$\int_0^1 f\left(a^t b^{1-t}\right) g\left(a^{1-t} b^t\right) dt = \int_0^1 f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) dt. \tag{10}$$

Hence by using (10) in (9) and by the change of variables $x = a^t b^{1-t}$ and $y = a^{1-t} b^t$, we obtain

$$\begin{aligned}
\frac{f\left(\sqrt{ab}\right)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx &\le \frac{1}{2} \int_0^1 f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) dt + \frac{1}{2} \int_0^1 f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) dt \\
&= \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(x) g(x)}{x} dx + \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(y) g(y)}{y} dy \\
&= \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(x) g(x)}{x} dx + \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(x) g(x)}{x} dx = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x) g(x)}{x} dx. \quad (11)
\end{aligned}$$

By the GA-convexity on $[a, b]$ and geometrical symmetry of g with respect to $\sqrt{ab}$, we have

$$f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) \le [(1-t) f(a) + t f(b)]\, g\left(a^{1-t} b^t\right) \tag{12}$$

and

$$f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) \le [(1-t) f(b) + t f(a)]\, g\left(a^{1-t} b^t\right). \tag{13}$$

Adding (12) and (13) and integrating with respect to t over $[0, 1]$, we obtain

$$\int_0^1 f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) dt + \int_0^1 f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) dt \le [f(a) + f(b)] \int_0^1 g\left(a^{1-t} b^t\right) dt. \tag{14}$$

By the change of variables $x = a^t b^{1-t}$ and $y = a^{1-t} b^t$ in (14), we get

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x) g(x)}{x} dx \le \frac{[f(a) + f(b)]}{2(\ln b - \ln a)} \int_a^b \frac{g(x)}{x} dx. \tag{15}$$

Combining the inequalities (11) and (15), we get the required result. □

Now we prove a weighted integral identity which play a key role in establishing our main results.

Lemma 2.3. *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. If $f' \in L([a, b])$, then the following equality holds*

$$\begin{aligned}
&\frac{f(b) + f(a)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \\
&\qquad = \frac{\ln b - \ln a}{4} \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx \right) \left[U(t) f'(U(t)) - L(t) f'(L(t)) \right] dt. \quad (16)
\end{aligned}$$

Proof. Let

$$I_1 = \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx \right) U(t) f'(U(t))\, dt$$

and

$$I_2 = \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx \right) L(t) f'(L(t))\, dt.$$

Since $g : [a, b] \to [0, \infty)$ is geometrically symmetric to $\sqrt{ab}$, hence $g(U(t)) = g(L(t))$ for all $t \in [0, 1]$. By this, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx \right) U(t) f'(U(t))\, dt \\
&= \frac{2}{\ln b - \ln a} \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx \right) d\left[f(U(t)) \right] \\
&= \frac{2}{\ln b - \ln a} \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx \right) f(U(t)) \Big|_0^1 - \int_0^1 \left[g(U(t)) + g(L(t)) \right] f(U(t))\, dt \\
&= \frac{2f(b)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx - 2 \int_0^1 g(U(t)) f(U(t))\, dt \\
&= \frac{2f(b)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx - \frac{4}{\ln b - \ln a} \int_{\sqrt{ab}}^b \frac{g(x) f(x)}{x} dx. \qquad (17)
\end{aligned}$$

Analogously, we have

$$-I_2 = \frac{2f(a)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx - \frac{4}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{g(x) f(x)}{x} dx. \qquad (18)$$

Adding (17) and (18) and multiplying the result by $\frac{\ln b - \ln a}{4}$, we get the required identity. This completes the proof of the Lemma. □

Lemma 2.4. *For $u, v > 0$, we have*

$$\int_0^1 u^{(1-t)/2} v^{(1+t)/2} dt = \sqrt{v} L\left(\sqrt{u}, \sqrt{v} \right),$$

$$\int_0^1 u^{(1+t)/2} v^{(1-t)/2} dt = \sqrt{u} L\left(\sqrt{u}, \sqrt{v} \right),$$

$$\Psi(u, v) \triangleq \frac{1}{2} \int_0^1 t u^{(1-t)/2} v^{(1+t)/2} dt = \begin{cases} \frac{v - \sqrt{v} L\left(\sqrt{u}, \sqrt{v} \right)}{\ln v - \ln u}, & u \neq v \\ \frac{1}{4} u, & u = v, \end{cases}$$

and

$$\Phi(u, v) \triangleq \frac{1}{2} \int_0^1 t^2 u^{(1-t)/2} v^{(1+t)/2} dt = \begin{cases} \frac{4\sqrt{v} L\left(\sqrt{u}, \sqrt{v} \right) - 4v + v(\ln v - \ln u)}{(\ln v - \ln u)^2}, & u \neq v \\ \frac{1}{6} u, & u = v. \end{cases}$$

Proof. The proof follows from a straightforward computation. □

We now establish new Fejér type inequalities for GA-convex functions, which provide weighted generalization of some of the results established in recent literature concerning GA-convex functions.

Theorem 2.5. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$ such that $f' \in L([a,b])$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{(\ln b - \ln a)^2}{2^{1+1/q}}\|g\|_\infty \Big\{[\Psi(a,b)]^{1-1/q}\left([\Psi(a,b)-\Phi(a,b)]\left|f'(a)\right|^q + [\Psi(a,b)+\Phi(a,b)]\left|f'(b)\right|^q\right)^{1/q}$$
$$+ [\Psi(b,a)]^{1-1/q}\left([\Psi(b,a)+\Phi(b,a)]\left|f'(a)\right|^q + [\Psi(b,a)-\Phi(b,a)]\left|f'(b)\right|^q\right)^{1/q}\Big\}, \quad (19)$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$.

Proof. From Lemma 2.3 and Hölder's inequality , we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{\ln b - \ln a}{4}\int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x}dx\right)\left[U(t)\left|f'(U(t))\right| + L(t)\left|f'(L(t))\right|\right]dt$$
$$\leq \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \int_0^1 \left[tU(t)\left|f'(U(t))\right| + tL(t)f'(L(t))\right]dt$$
$$\leq \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \left\{\left(\int_0^1 tU(t)dt\right)^{1-1/q}\left(\int_0^1 tU(t)\left|f'(U(t))\right|^q dt\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 tL(t)dt\right)^{1-1/q}\left(\int_0^1 tL(t)\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \quad (20)$$

By the GA-convexity of $|f'|^q$ on $[a,b]$ for $q \geq 1$ and by using Lemma 2.4, we have

$$\int_0^1 tU(t)\left|f'(U(t))\right|^q \leq \left|f'(a)\right|^q \int_0^1 t\left(\frac{1-t}{2}\right)a^{(1-t)/2}b^{(1+t)/2}dt + \left|f'(b)\right|^q \int_0^1 t\left(\frac{1+t}{2}\right)a^{(1-t)/2}b^{(1+t)/2}dt$$
$$= [\Psi(a,b)-\Phi(a,b)]\left|f'(a)\right|^q + [\Psi(a,b)+\Phi(a,b)]\left|f'(b)\right|^q \quad (21)$$

and

$$\int_0^1 tL(t)\left|f'(L(t))\right|^q \leq \left|f'(a)\right|^q \int_0^1 t\left(\frac{1+t}{2}\right)a^{(1+t)/2}b^{(1-t)/2}dt + \left|f'(b)\right|^q \int_0^1 t\left(\frac{1-t}{2}\right)a^{(1+t)/2}b^{(1-t)/2}dt$$
$$= [\Psi(b,a)+\Phi(b,a)]\left|f'(a)\right|^q + [\Psi(b,a)-\Phi(b,a)]\left|f'(b)\right|^q. \quad (22)$$

Using (21) and (22) in (20), we get the required result. This completes the proof of the theorem. □

Corollary 2.6. *Suppose the assumptions of Theorem 2.5 are satisfied. If $q = 1$, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \Big\{[\Psi(a,b)+\Psi(b,a)-\Phi(a,b)+\Phi(b,a)]\left|f'(a)\right|$$
$$+ [\Psi(a,b)+\Psi(b,a)+\Phi(a,b)-\Phi(b,a)]\left|f'(b)\right|\Big\}, \quad (23)$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$.

Corollary 2.7. *If $g(x) = \frac{1}{\ln b - \ln a}$, for all $x \in [a,b]$ in Theorem 2.5, then*

$$\left|\frac{f(b)+f(a)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{(\ln b - \ln a)}{2^{1+1/q}}\left\{[\Psi(a,b)]^{1-1/q}\left([\Psi(a,b)-\Phi(a,b)]\left|f'(a)\right|^q + [\Psi(a,b)+\Phi(a,b)]\left|f'(b)\right|^q\right)^{1/q}\right.$$
$$\left. + [\Psi(b,a)]^{1-1/q}\left([\Psi(b,a)+\Phi(b,a)]\left|f'(a)\right|^q + [\Psi(b,a)-\Phi(b,a)]\left|f'(b)\right|^q\right)^{1/q}\right\}. \quad (24)$$

Corollary 2.8. *If $q = 1$ in Corollary 2.7, then we get the following inequality*

$$\left|\frac{f(b)+f(a)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{(\ln b - \ln a)}{4}\left\{[\Psi(a,b)+\Psi(b,a)+\Phi(b,a)-\Phi(a,b)]\left|f'(a)\right|\right.$$
$$\left. + [\Psi(b,a)+\Psi(a,b)+\Phi(a,b)-\Phi(b,a)]\left|f'(b)\right|\right\}. \quad (25)$$

Theorem 2.9. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$ such that $f' \in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q > 1$, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \leq \frac{(\ln b - \ln a)^{2-1/q}\left\|g\right\|_\infty}{4\cdot q^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]\left|f'(a)\right|^q + \left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'(b)\right|^q\right)^{1/q}\right.$$
$$\left. + a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]\left|f'(a)\right|^q + \left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'(b)\right|^q\right)^{1/q}\right\}, \quad (26)$$

where $\left\|g\right\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$.

Proof. From Lemma 2.3 and Hölder's inequality , we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{\ln b - \ln a}{4}\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x}dx\right)\left[U(t)\left|f'(U(t))\right| + L(t)\left|f'(L(t))\right|\right]dt$$
$$\leq \frac{(\ln b - \ln a)^2}{4}\left\|g\right\|_\infty\int_0^1\left[tU(t)\left|f'(U(t))\right| + tL(t)f'(L(t))\right]dt \leq \frac{(\ln b - \ln a)^2}{4}\left\|g\right\|_\infty\left(\int_0^1 t^{q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 [U(t)]^q\left|f'(U(t))\right|^q dt\right)^{1/q} + \left(\int_0^1 [L(t)]^q\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \quad (27)$$

Since

$$\int_0^1 [U(t)]^q\left|f'(U(t))\right|^q = \int_0^1 a^{q(1-t)/2}b^{q(1+t)/2}\left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|$$
$$\leq \left|f'(a)\right|^q\int_0^1\left(\frac{1-t}{2}\right)a^{q(1-t)/2}b^{q(1+t)/2}dt + \left|f'(b)\right|^q\int_0^1\left(\frac{1+t}{2}\right)a^{q(1-t)/2}b^{q(1+t)/2}dt$$
$$= \frac{b^{q/2}\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]}{q(\ln b - \ln a)}\left|f'(a)\right|^q + \frac{b^{q/2}\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]}{q(\ln b - \ln a)}\left|f'(b)\right|^q \quad (28)$$

and

$$\int_0^1 [L(t)]^q \left|f'(L(t))\right|^q = \int_0^1 a^{q(1+t)/2} b^{q(1-t)/2} \left|f'\left(a^{(1+t)/2} b^{(1-t)/2}\right)\right|$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{1+t}{2}\right) a^{q(1+t)/2} b^{q(1-t)/2} dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{1-t}{2}\right) a^{q(1+t)/2} b^{q(1-t)/2} dt$$
$$= \frac{a^{q/2}\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right]}{q(\ln b - \ln a)} \left|f'(a)\right|^q + \frac{a^{q/2}\left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right]}{q(\ln b - \ln a)} \left|f'(b)\right|^q. \quad (29)$$

The inequality (26) is proved by applying (28) and (29) in (27). □

Corollary 2.10. *If the assumptions of Theorem 2.9 are satisfied and if* $g(x) = \frac{1}{\ln b - \ln a}$ *for all* $x \in [a,b]$*, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} dx\right| \leq \frac{(\ln b - \ln a)^{1-1/q}}{4 \cdot q^{1/q}} \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times \left\{ b^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2}\right] \left|f'(a)\right|^q + \left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right] \left|f'(b)\right|^q\right)^{1/q}\right.$$
$$\left. + a^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right] \left|f'(a)\right|^q + \left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right] \left|f'(b)\right|^q\right)^{1/q}\right\}. \quad (30)$$

Theorem 2.11. *Let* $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I^\circ$ *with* $a < b$ *and let* $g : [a,b] \to [0,\infty)$ *be continuous positive mapping and geometrically symmetric to* $\sqrt{ab}$ *such that* $f' \in L([a,b])$*. If* $\left|f'\right|^q$ *is GA-convex on* $[a,b]$ *for* $q > 1$*, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx\right| \leq \frac{(\ln b - \ln a)^{2-1/q} \left\|g\right\|_\infty}{2(4q)^{1/q}} \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times \left\{[L(a^q, b^q) - a^q] \left|f'(a)\right|^q + [b^q - L(a^q, b^q)] \left|f'(b)\right|^q\right\}^{1/q}, \quad (31)$$

where $\left\|g\right\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$.

Proof. From Lemma 2.3 and Hölder's inequality , we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx\right|$$
$$\leq \frac{\ln b - \ln a}{4} \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x} dx\right) \left[U(t)\left|f'(U(t))\right| + L(t)\left|f'(L(t))\right|\right] dt$$
$$\leq \frac{(\ln b - \ln a)^2}{4} \left\|g\right\|_\infty \int_0^1 \left[tU(t)\left|f'(U(t))\right| + tL(t) f'(L(t))\right] dt \leq \frac{(\ln b - \ln a)^2}{4} \left\|g\right\|_\infty \left(\int_0^1 t^{q/(q-1)} dt\right)^{1-1/q}$$
$$\times \left\{\left(\int_0^1 [U(t)]^q \left|f'(U(t))\right|^q dt\right)^{1/q} + \left(\int_0^1 [L(t)]^q \left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \quad (32)$$

By the power-mean inequality ($a^r + b^r \leq 2^{1-r}(a+b)^r$ for $a > 0, b > 0$ and $r < 1$), we have

$$\left(\int_0^1 [U(t)]^q \left|f'(U(t))\right|^q dt\right)^{1/q} + \left(\int_0^1 [L(t)]^q \left|f'(L(t))\right|^q dt\right)^{1/q}$$
$$\leq 2^{1-1/q} \left(\int_0^1 [U(t)]^q \left|f'(U(t))\right|^q dt + \int_0^1 [L(t)]^q \left|f'(L(t))\right|^q dt\right)^{1/q}. \quad (33)$$

Since $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q>1$

$$\begin{aligned}
&\int_0^1 [U(t)]^q \left|f'(U(t))\right|^q dt + \int_0^1 [L(t)]^q \left|f'(L(t))\right|^q dt \\
&\le \left|f'(a)\right|^q \left[\int_0^1 \left(\frac{1-t}{2}\right) a^{q(1-t)/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1+t}{2}\right) a^{q(1+t)/2} b^{q(1-t)} dt\right] \\
&+ \left|f'(b)\right|^q \left[\int_0^1 \left(\frac{1+t}{2}\right) a^{q(1-t)/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1-t}{2}\right) a^{q(1+t)/2} b^{q(1-t)} dt\right] \\
&= \left[\frac{2L(a^q,b^q)-2a^q}{q(\ln b-\ln a)}\right]\left|f'(a)\right|^q + \left[\frac{2b^q-2L(a^q,b^q)}{q(\ln b-\ln a)}\right]\left|f'(b)\right|^q. \quad (34)
\end{aligned}$$

Using (33) in (34), we get

$$\begin{aligned}
&\left(\int_0^1 [U(t)]^q \left|f'(U(t))\right|^q dt\right)^{1/q} + \left(\int_0^1 [L(t)]^q \left|f'(L(t))\right|^q dt\right)^{1/q} \\
&\le 2^{1-2/q}\left(\left[\frac{L(a^q,b^q)-a^q}{q(\ln b-\ln a)}\right]\left|f'(a)\right|^q + \left[\frac{b^q-L(a^q,b^q)}{q(\ln b-\ln a)}\right]\left|f'(b)\right|^q\right)^{1/q} \quad (35)
\end{aligned}$$

Applying (35) in (32), we obtain the required inequality (31). □

Corollary 2.12. *If the assumptions of Theorem 2.11 are satisfied and if $g(x)=\frac{1}{\ln b-\ln a}$ for all $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2} - \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\le \frac{(\ln b-\ln a)^{1-1/q}}{2(4q)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{\left[L(a^q,b^q)-a^q\right]\left|f'(a)\right|^q + \left[b^q-L(a^q,b^q)\right]\left|f'(b)\right|^q\right\}^{1/q}. \quad (36)
\end{aligned}$$

Theorem 2.13. *Let $f: I\subseteq \mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,\ b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$ such that $f'\in L([a,b])$. If $\left|f'\right|$ is GA-convex on $[a,b]$, then the following inequality holds for $q>1$*

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \le \frac{(\ln b-\ln a)^2\left[L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left\|g\right\|_\infty}{8} \\
&\times\Bigg\{\left(b^{1/2}\left[B(q+1,q+1)\right]^{1/q} + a^{1/2}\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(a)\right| \\
&+\left(a^{1/2}\left[B(q+1,q+1)\right]^{1/q} + b^{1/2}\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(b)\right|\Bigg\}, \quad (37)
\end{aligned}$$

where $\left\|g\right\|_\infty=\sup_{x\in[a,b]} g(x)<\infty$ and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function.

Proof. From Lemma 2.3 and the GA-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x}dx-\int_a^b \frac{f(x)g(x)}{x}dx\right| \\
&\le \frac{\ln b-\ln a}{4}\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x}dx\right)\left[U(t)\left|f'(U(t))\right|+L(t)\left|f'(L(t))\right|\right]dt \\
&\le \frac{(\ln b-\ln a)^2}{4}\|g\|_\infty\int_0^1\left[tU(t)\left|f'(U(t))\right|+tL(t)f'(L(t))\right]dt \\
&\le \frac{(\ln b-\ln a)^2}{4}\|g\|_\infty\left\{\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt\right. \\
&\qquad\left.+\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\right\}. \qquad (38)
\end{aligned}$$

Using Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt\le\left(\int_0^1 a^{q(1-t)/[2(q-1)]}b^{q(1+t)/[2(q-1)]}dt\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1 t^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 t^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}=\frac{1}{2}\left[b^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q} \\
&\qquad\times\left\{\left[B(q+1,q+1)\right]^{1/q}\left|f'(a)\right|+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(b)\right|\right\}. \qquad (39)
\end{aligned}$$

Similarly, we one have

$$\begin{aligned}
&\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\le\left(\int_0^1 a^{q(1+t)/[2(q-1)]}b^{q(1-t)/[2(q-1)]}dt\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1 t^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 t^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}=\frac{1}{2}\left[a^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q} \\
&\qquad\times\left\{\left[B(q+1,q+1)\right]^{1/q}\left|f'(b)\right|+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(a)\right|\right\}. \qquad (40)
\end{aligned}$$

Using (39) and (40) in (38), we obtain the required inequality (37). □

Corollary 2.14. *Under the assumptions of Theorem 2.13, if $g(x)=\frac{1}{\ln b-\ln a}$ for all $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\le\frac{(\ln b-\ln a)\left[L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}}{2} \\
&\times\left\{\left(b^{1/2}\left[B(q+1,q+1)\right]^{1/q}+a^{1/2}\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(a)\right|\right. \\
&\qquad\left.+\left(a^{1/2}\left[B(q+1,q+1)\right]^{1/q}+b^{1/2}\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(b)\right|\right\}, \qquad (41)
\end{aligned}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function.

3. Applications to Special Means

In this section we apply some of the above established inequalities of Hermite-Hadamard type involving the product of a geometrically-arithmetically convex
function and a geometrically symmetric function to construct inequalities for special means.
For positive numbers $a > 0$ and $b > 0$ with $a \neq b$

$$A(a,b) = \frac{a+b}{2}, L(a,b) = \frac{b-a}{\ln b - \ln a}, G(a,b) = \sqrt{ab}, H(a,b) = \frac{2ab}{a+b}$$

and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0 \\ L(a,b), & p = -1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0 \end{cases}$$

are the arithmetic mean, the logarithmic mean, geometric mean, harmonic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$ respectively. For further information on means, we refer the readers to [17–19] and the references therein.

Now let $f(x) = x^r$ for $x > 0$, $r \in \mathbb{R}$ with $r \neq 0$. Then

$$\left|f'\left(x^\lambda y^{1-\lambda}\right)\right|^q = |r|^q \left[x^{q(r-1)}\right]^\lambda \left[y^{q(r-1)}\right]^{1-\lambda} \leq |r|^q \left[\lambda x^{q(r-1)} + (1-\lambda) y^{q(r-1)}\right]$$

for $\lambda \in [0,1]$, $x, y > 0$ and $q \geq 1$. That is $\left|f'(x)\right|^q = |r|^q x^{q(r-1)}$ is geometrically-arithmetically convex on $[a,b]$ for $q \geq 1$ and $r \neq 1$, where $a, b > 0$.

Let $g : [a,b] \to \mathbb{R}_0$ be defined as

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2, x \in [a,b].$$

It is obvious that

$$g\left(\frac{ab}{x}\right) = g(x)$$

for all $x \in [a,b]$. Hence $g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2$, $x \in [a,b]$ is geometrically symmetric with respect to $x = \sqrt{ab}$.

Now applications of our results are given in the following theorems to come.

Theorem 3.1. *Let $0 < a < b$, $r \in \mathbb{R}\setminus\{-2,0,1,2\}$ and $q \geq 1$. Then*

$$\begin{aligned} &\left|\frac{2A(a^r,b^r)L\left(a^2,b^2\right) - L\left(a^{r+2},b^{r+2}\right)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{r-2},b^{r-2}\right) + 2L(a^r,b^r) - 2A(a^r,b^r)\right| \\ &\leq \frac{|r|(b-a)^2}{2^{1+1/q}G(a,b)L(a,b)}\left\{[\Psi(a,b)]^{1-1/q}\left[2\Psi(a,b)A\left(a^{q(r-1)},b^{q(r-1)}\right) + \Phi(a,b)\left(b^{q(r-1)} - a^{q(r-1)}\right)\right]^{1/q}\right. \\ &\qquad \left. + [\Psi(b,a)]^{1-1/q}\left[2\Psi(b,a)A\left(a^{q(r-1)},b^{q(r-1)}\right) + \Phi(b,a)\left(a^{q(r-1)} - b^{q(r-1)}\right)\right]^{1/q}\right\}, \quad (42) \end{aligned}$$

where $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are defined as in Lemma 2.4.

Proof. Applying Theorem 2.5 to the functions

$$f(x) = x^r \text{ for } x > 0,\ r \in \mathbb{R} \setminus \{-2, 0, 1, 2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x} \right)^2, x \in [a, b]$$

we get the desired result. □

Corollary 3.2. *Suppose the assumptions of Theorem 3.1 are satisfied and if $r = -1$, the the following inequality holds*

$$\begin{aligned}
&\left| \frac{2L\left(a^2, b^2\right)[H(a,b)]^{-1} + L(a,b)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{-3}, b^{-3}\right) - 2[H(a,b)]^{-1} \right| \\
&\leq \frac{(b-a)^2}{2^{1+1/q} G(a,b) L(a,b)} \left\{ [\Psi(a,b)]^{1-1/q} \left[2\Psi(a,b) A\left(a^{-2q}, b^{-2q}\right) + \Phi(a,b)\left(b^{-2q} - a^{-2q}\right) \right]^{1/q} \right. \\
&\qquad \left. + [\Psi(b,a)]^{1-1/q} \left[2\Psi(b,a) A\left(a^{-2q}, b^{-2q}\right) + \Phi(b,a)\left(a^{-2q} - b^{-2q}\right) \right]^{1/q} \right\}, \quad (43)
\end{aligned}$$

where $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are defined as in Lemma 2.4.

Corollary 3.3. *Under the assumptions of Theorem 3.1, the following inequality holds true for $q = 1$*

$$\begin{aligned}
&\left| \frac{2A(a^r, b^r) L\left(a^2, b^2\right) - L\left(a^{r+2}, b^{r+2}\right)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{r-2}, b^{r-2}\right) + 2L(a^r, b^r) - 2A(a^r, b^r) \right| \\
&\leq \frac{|r|(b-a)^2}{4G(a,b)L(a,b)} \left\{ L\left(\sqrt{a}, \sqrt{b}\right) A\left(a^{r-1}, b^{r-1}\right) \left[2A\left(\sqrt{a}, \sqrt{b}\right) - L\left(\sqrt{a}, \sqrt{b}\right) \right] \right. \\
&\qquad \left. + 2(r-1)\left(L\left(\sqrt{a}, \sqrt{b}\right)\left[L\left(\sqrt{a}, \sqrt{b}\right) - 2A\left(\sqrt{a}, \sqrt{b}\right) \right] + A(a,b) \right) L\left(a^{r-1}, b^{r-1}\right) \right\}. \quad (44)
\end{aligned}$$

Corollary 3.4. *If we take $r = -1$ in Corollary 3.3, then the following inequality holds valid*

$$\begin{aligned}
&\left| \frac{2L\left(a^2, b^2\right)[H(a,b)]^{-1} + L(a,b)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{-3}, b^{-3}\right) - 2[H(a,b)]^{-1} \right| \\
&\leq \frac{(b-a)^2}{4G(a,b)L(a,b)} \left\{ L\left(\sqrt{a}, \sqrt{b}\right) A\left(a^{-2}, b^{-2}\right) \left[2A\left(\sqrt{a}, \sqrt{b}\right) - L\left(\sqrt{a}, \sqrt{b}\right) \right] \right. \\
&\qquad \left. - 4\left(L\left(\sqrt{a}, \sqrt{b}\right)\left[2A\left(\sqrt{a}, \sqrt{b}\right) - L\left(\sqrt{a}, \sqrt{b}\right) \right] - A(a,b) \right) L\left(a^{-2}, b^{-2}\right) \right\}. \quad (45)
\end{aligned}$$

Theorem 3.5. *Let $0 < a < b$, $r \in \mathbb{R} \setminus \{-2, 0, 1, 2\}$ and $q > 1$. Then*

$$\begin{aligned}
&\left| \frac{2A(a^r, b^r) L\left(a^2, b^2\right) - L\left(a^{r+2}, b^{r+2}\right)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{r-2}, b^{r-2}\right) + 2L(a^r, b^r) - 2A(a^r, b^r) \right| \\
&\leq \frac{(b-a)^{2-1/q}|r|}{4 \cdot q^{1/q} G(a,b) [L(a,b)]^{1-1/q}} \left(\frac{2q-1}{q-1} \right)^{1-\frac{1}{q}} \\
&\times \left\{ b^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2} \right] a^{q(r-1)} + \left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right) \right] b^{q(r-1)} \right)^{1/q} \right. \\
&\qquad + a^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2} \right] a^{q(r-1)} + \left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right) \right] b^{q(r-1)} \right)^{1/q}. \quad (46)
\end{aligned}$$

Proof. Applying Theorem 2.9 to the functions

$$f(x) = x^r \text{ for } x > 0, r \in \mathbb{R}\setminus\{-2,0,1,2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2, x \in [a,b]$$

we get the desired result. □

Corollary 3.6. *Suppose the assumptions of Theorem 3.5 are fulfilled and if $r = -1$, the following inequality holds true*

$$\begin{aligned}
&\left|\frac{2[H(a,b)]^{-1} L\left(a^2,b^2\right) + L(a,b)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{-3},b^{-3}\right) - 2[H(a,b)]^{-1}\right| \\
&\leq \frac{(b-a)^{2-1/q}}{4\cdot q^{1/q} G(a,b)[L(a,b)]^{1-1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \\
&\times\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right) - a^{q/2}\right]a^{-2q} + \left[2b^{q/2} - a^{q/2} - L\left(a^{q/2},b^{q/2}\right)\right]b^{-2q}\right)^{1/q}\right. \\
&\qquad \left. + a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right]a^{-2q} + \left[b^{q/2} - L\left(a^{q/2},b^{q/2}\right)\right]b^{-2q}\right)^{1/q}\right\}. \quad (47)
\end{aligned}$$

Theorem 3.7. *Let $0 < a < b$, $r \in \mathbb{R}\setminus\{-2,0,1,2\}$ and $q > 1$. Then*

$$\begin{aligned}
&\left|\frac{2A(a^r,b^r) L\left(a^2,b^2\right) - L\left(a^{r+2},b^{r+2}\right)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{r-2},b^{r-2}\right) + 2L(a^r,b^r) - 2A(a^r,b^r)\right| \\
&\leq \frac{|r|(b-a)^2}{2^{2/q+1} G(a,b) L(a,b)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[rL(a^{qr},b^{qr}) - (r-1) L\left(a^{q(r-1)},b^{q(r-1)}\right) L(a^q,b^q)\right]^{1/q}. \quad (48)
\end{aligned}$$

Proof. Applying Theorem 2.11 to the functions

$$f(x) = x^r \text{ for } x > 0, r \in \mathbb{R}\setminus\{-2,0,1,2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2, x \in [a,b]$$

we get the desired result. □

Corollary 3.8. *Suppose the assumptions of Theorem 3.7 are satisfied and if $r = -1$, the the following inequality holds valid*

$$\begin{aligned}
&\left|\frac{2[H(a,b)]^{-1} L\left(a^2,b^2\right) + L(a,b)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{-3},b^{-3}\right) - 2[H(a,b)]^{-1}\right| \\
&\leq \frac{(b-a)^2}{2^{2/q+1} G(a,b) L(a,b)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[2L\left(a^{-2q},b^{-2q}\right) L\left(a^{-1},b^{-1}\right) - L\left(a^{-q},b^{-q}\right)\right]^{1/q}. \quad (49)
\end{aligned}$$

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