Filomat 32:6 (2018), 2219–2245
https://doi.org/10.2298/FIL1806219A

Published by Faculty of Sciences and Mathematics,
University of Nivs, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Synchronization of Coupled Stochastic Systems Driven by Symmetric α-Stable Process and Brownian Motion

Shahad Al-Azzawi[a], Jicheng Liu[a], Xianming Liu[a]

[a]*School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, P.R.China*

Abstract. The synchronization of stochastic differential equations (SDEs) driven by symmetric α-stable process and Brownian Motion is investigated in pathwise sense. This coupled dynamical system is a new mathematical model, where one of the systems is driven by Gaussian noise, another one is driven by non-Gaussian noise. In this paper, we prove that the synchronization still persists for this coupled dynamical system. Examples and simulations are given.

1. Introduction

A stochastic dynamical system is a dynamical system subjected to the effects of noise where the effect of noise in dynamical systems is a very important area of research. Such effects of fluctuations have been of interest for over a century since the seminar work of Einstein. Synchronization of coupled dynamical systems is a wildly-known phenomenon that has been observed in many sciences like biology, physics and other areas, In general, these systems are subjected to different types of noise, and deal with coupled dynamical systems that have common dynamical features in an asymptotic sense. A readable descriptive account of its diversity of occurrence can be found in the Strogatz book [33], which contains an extensive list of references. The synchronization of coupled dissipative systems in the case of autonomous systems has been investigated mathematically in [10] both for asymptotically stable equilibria and general attractors, such as chaotic attractors. Analogous results also hold for nonautonomous systems, but require a new concept of a nonautonomous attractor. Recently, The authors in [8, 13] provided that appropriate concepts of random attractors and stochastic stationary solutions are used instead of their deterministic counterparts.

Gaussian and non-Gaussian processes have been widely used to model fluctuations in engineering and science. Brownian motion is one example of Gaussian processes, where the particle driven by Brownian motion has continuous sample paths in time almost surely and the probability density function decays exponentially in space [26], the mean square displacement increases linearly in time. Lévy processes arise as models for fluctuations in many systems, for example, a passive tracer particle may subjected to a series of "pauses", when the particle is trapped by a vortex for a random time period, and "jumps" or " flights".

2010 *Mathematics Subject Classification*. Primary 60H10, 34F05
Keywords. Synchronization, random attractor, one-sided Lipschitz dissipative condition, symmetric α-stable process.
Received: 05 January 2017; Revised: 02 May 2017; Accepted: 19 November 2017
Communicated by Miljana Jovanović
Email addresses: shahad_math2014@yahoo.com (Shahad Al-Azzawi), jcliu@hust.edu.cn (Jicheng Liu), xmliu@hust.edu.cn (Xianming Liu)

In [7] (or see [2]), Caraballo and Kloeden showed that synchronization persists under additive noise, provided asymptotically stable stochastic stationary solutions are considered rather than asymptotically stable steady state solutions. Specifically, they considered two Itô stochastic differential equations in $\mathbb{R}^d$

$$\begin{aligned} dX_t &= f(X_t)dt + \alpha dW_t^{(1)}, \\ dY_t &= g(Y_t)dt + \beta dW_t^{(2)}, \end{aligned} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}_+^d$ are constant vectors with no components equal to zero, $W_t^{(1)}, W_t^{(2)}$ are independent two-sided scalar Wiener processes, and the functions f, g satisfy the one-sided dissipative Lipschitz conditions

$$\begin{aligned} \langle x_1 - x_2, f(x_1) - f(x_2)\rangle &\leqslant -L|x_1 - x_2|^2, \\ \langle y_1 - y_2, g(y_1) - g(y_2)\rangle &\leqslant -L|y_1 - y_2|^2. \end{aligned} \tag{2}$$

The synchronized system corresponding to SDEs (1)

$$\begin{aligned} dX_t &= f(X_t)dt + \nu(Y_t - X_t)dt + \alpha dW_t^{(1)}, \\ dY_t &= g(Y_t)dt + \nu(X_t - Y_t)dt + \beta dW_t^{(2)}, \end{aligned}$$

has a unique stationary solution $(\bar{X}^\nu \circ \theta_t, \bar{Y}^\nu \circ \theta_t)$, which is pathwise globally asymptotically stable with

$$(\bar{X}^\nu(\theta_t\omega), \bar{Y}^\nu(\theta_t\omega)) \to (\bar{Z}^\infty(\theta_t\omega), \bar{Z}^\infty(\theta_t\omega)), \text{ as } \nu \to \infty,$$

pathwise on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$, where $\bar{Z}_t^\infty$ is the unique pathwise globally asymptotically stable stationary solution of the "averaged" SDEs

$$dZ_t = \frac{1}{2}[f(Z_t) + g(Z_t)]dt + \frac{1}{2}\alpha dW_t^{(1)} + \frac{1}{2}\beta dW_t^{(2)}.$$

In [23], Liu et al. showed that synchronization persists for coupled dynamical systems driven by α-stable multiplicative noises, provided asymptotically stable stochastic stationary solutions are considered rather than asymptotically stable steady state solution. They considered two Marcus canonical equations in $\mathbb{R}^d$

$$\begin{aligned} dX_t &= f(X_t)dt + aX_t \diamond dL_t^{(1)}, \\ dY_t &= g(Y_t)dt + bY_t \diamond dL_t^{(2)}, \end{aligned} \tag{3}$$

where a, b are constants in $\mathbb{R}$, $L_t^{(1)}$ and $L_t^{(2)}$ are independent two-sided scalar α-stable processes as in Lemma 2 in [23], and the vector fields f and g are sufficiently regular to ensure the existence and uniqueness of local solution, and additionally satisfy one-sided dissipative Lipschitz conditions (2). The synchronized system corresponding to SDEs (3) is

$$\begin{aligned} dX_t &= f(X_t)dt + \nu(e^{2\eta_t}Y - X)dt + aX_t \diamond dL_t^{(1)}, \\ dY_t &= g(Y_t)dt + \nu(e^{-2\eta_t}X - Y)dt + bY_t \diamond dL_t^{(2)}, \end{aligned}$$

where $2\eta_t = O_t^{(1)} - O_t^{(2)}$ and $O_t^{(1)}, O_t^{(2)}$ are two Ornstein-Uhlenbeck processes with respect to $aL_t^{(1)}$ and $bL_t^{(2)}$ respectively. The coupled random system has a unique stationary solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$. It is pathwise globally asymptotically stable with $(\bar{X}_t^\nu, \bar{Y}_t^\nu) \to (\bar{Z}_t^\infty e^{-O_t^{(1)}}, \bar{Z}_t^\infty e^{-O_t^{(2)}})$ as $\nu \to \infty$, pathwise on finite time-intervals $[T_1, T_2]$. Note that $\bar{Z}_t^\infty$ is the unique pathwise global asymptotically stable stationary solution of the "averaged" RODE in $\mathbb{R}^d$

$$\frac{dz}{dt} = \frac{1}{2}[e^{-\eta_t} f(e^{\eta_t} z_t) + e^{-\eta_t} g(e^{-\eta_t} z_t) + aX_t \diamond dL_t^{(1)} + bY_t \diamond dL_t^{(2)}].$$

In this paper, we will consider two coupled stochastic equations in $\mathbb{R}^d$

$$\begin{aligned} dX_t &= f(X_t)dt + (a_1X_t + b_1) \diamond dL_t^\alpha, \\ dY_t &= g(Y_t)dt + (a_2Y_t + b_2) \circ dW_t. \end{aligned} \tag{4}$$

Here L_t^α is a two sided scalar α-stable process and W_t is a two-sided scalar Wiener process independent of L_t^α. b_1 and b_2 are constant vectors in $\mathbb{R}^d$, a_1 and a_2 are constants in $\mathbb{R}$, $\diamond$ and $\circ$ denote the Marcus integral and Stratonovich integral respectively. The functions f and g are sufficiently regular to ensure the existence and uniqueness of local solution, and additionally satisfy one-side dissipative Lipschitz condition (2). The aim of this paper is to test the synchronization phenomenon for coupled dynamical systems (4). We will transform the SDEs to random ordinary differential equations (RODEs) and prove that the system is asymptotical stable. When $\alpha_1, \alpha_2 \neq 0$, finally, we obtain

$$\begin{aligned} dX_t = &\quad [f(X_t) + \nu(e^{2\eta_t}Y_t - X_t) + \nu(\tfrac{b_2}{a_2}e^{2\eta_t} - \tfrac{b_1}{a_1})]dt + (a_1X_t + b_1) \diamond dL_t^\alpha, \\ dY_t = &\quad [g(Y_t) + \nu(e^{-2\eta_t}X_t - Y_t) + \nu(\tfrac{b_1}{a_1}e^{-2\eta_t} - \tfrac{b_2}{a_2})]dt + (a_2Y_t + b_2) \circ dW_t. \end{aligned}$$

The above system has a unique stationary stochastic solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$, which is pathwise globally asymptotically stable with

$$(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega)) \to (\bar{Z}_t(\omega)e^{\eta_t} - \frac{b_1}{\alpha_1}, \bar{Z}_t(\omega)e^{-\eta_t} - \frac{b_2}{a_2}), \text{ as } \nu \to \infty,$$

where $\bar{Z}_t(\omega)$ is the stationary solution of

$$dZ_t = \quad \tfrac{1}{2}[e^{-\eta_t}f(e^{\eta_t}Z_t - \tfrac{b_1}{a_1}) + e^{\eta_t}g(e^{-\eta_t}Z_t - \tfrac{b_2}{a_2})]dt + \tfrac{1}{2}a_1Z_t \diamond dL_t^\alpha + \tfrac{1}{2}a_2Z_t \circ dW_t,$$

with

$$\eta_t = \frac{1}{2}(a_1O_t^{(1)} - a_2O_t^{(2)}).$$

When $\alpha_2 = 0$, equation (4) becomes

$$\begin{aligned} dX_t &= f(X_t)dt + (a_1X_t + b_1) \diamond dL_t^\alpha, \\ dY_t &= g(Y_t)dt + b_2dW_t. \end{aligned} \tag{5}$$

The other aim of this paper is to test the synchronization phenomenon for coupled dynamical systems (5). By using another transformation which is different from the transformation that is used in(4),we will transform the SDEs to random ordinary differential equations (RODEs)and prove that the system is asymptotical stable. Finally, we obtain

$$\begin{aligned} dX_t &= [f(X_t) + \nu(e^{a_1O_t^{(1)}}Y_t - X_t) - \nu(\tfrac{b_1}{a_1} - b_2O_t^{(2)}e^{a_1O_t^{(1)}})]dt + (a_1X_t + b_1) \diamond dL_t^\alpha, \\ dY_t &= [g(Y_t) + \nu(e^{-a_1O^{(1)}}X_t - Y_t) + \nu(b_2O_t^{(2)} - \tfrac{b_1}{a_1}e^{-a_1O_t^{(1)}})]dt + b_2dW_t. \end{aligned}$$

Then this system has a unique stationary stochastic solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$, which is pathwise globally asymptotically stable with

$$(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega)) \to (\bar{z}_t(\omega)e^{a_1O_t^{(1)}} - \frac{b_1}{a_1}, \bar{z}_t(\omega) + b_2O_t^{(2)}), as \quad \nu \to \infty,$$

with

$$\bar{z}_t = e^{-\frac{1}{2}a_1O_t^{(1)}}(\bar{Z}_t + \frac{b_1}{a_1}) - \frac{1}{2}b_2O_t^{(2)},$$

where $\bar{Z}_t(\omega)$ is the stationary solution of

$$\begin{aligned} dZ_t = &\quad \tfrac{1}{2}[f(Z_t - \tfrac{1}{2}b_2O_t^{(2)}e^{a_1O_t^{(1)}}) + e^{a_1O_t^{(1)}}g(e^{-\frac{1}{2}a_1O_t^{(1)}}(Z_t + \tfrac{b_1}{a_1}) + \tfrac{1}{2}b_2O_t^{(2)})]dt \\ &+\tfrac{1}{2}[(a_1Z_{t-} + b_1) \diamond dL_t^{(\alpha)} + b_2e^{a_1O_t^{(1)}}dW_t]. \end{aligned}$$

The main contributions of this paper are two aspects: firstly, we consider a new model, which may be considered as a combination of those in [2], [7], [8], [22] and [23], our result means that the synchronization still persists under two different kinds of environmental noises. On the other hand, our noises are general linear noises, whereas all the noises in references mentioned above are pure additive, or pure multiplicative noise. Different methods are adapted to two kinds of noises. Hence their methods could not directly apply to general linear noises. Moreover, we found that our first result can directly apply to the pure multiplicative noise, but can also not directly apply to the pure additive case, because the same transformation does not work. In view of this reason, It is necessary to consider the synchronization of systems, where one of systems is driven by pure multiplicative noise, another one is driven by pure additive noise. This leads to our second result. In addition, we also give an examples with simulation for two results, respectively.

The structure of the paper is as follows: in Section 2, we will recall some basic facts about Lévy process and Brownian process. In Section 3, we will review some concepts in random dynamical systems. In Section 4, we will explain the method of the transformation from SDEs to RODEs. In Section 5, we will prove that the uncoupled system has a unique stationary solution, which is globally asymptotically stable. In Section 6, we will show that the asymptotic behaviours of the coupled synchronized system is uniformly boundedness. In Section 7, synchronization persists for coupled system, that is,the stationary solutions of coupled synchronized system converge to the unique pathwise globally asymptotically stable stationary solution of the "averaged" system. We will test our theory by an example and simulations. In Section 8, we will prove the synchronization persists under two different noises where each stochastic differential equations is subjected to a noise which differs from the other. We will test our theory by an example and simulations.

2. Brownian motion and symmetric α-stable process

Lévy process L_t, taking values in $\mathbb{R}^d$, is characterized by a drift parameter $\mathbf{b} \in \mathbb{R}^d$, an $n \times n$ non-negative covariance matrix $\mathbf{A}$ and a Borel measure γ, defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and concentrated on $\mathbb{R}^d \backslash \{0\}$, that satisfies

$$\int_{\mathbb{R}^d \backslash \{0\}} (y^2 \wedge 1)\gamma(dy) < \infty,$$

or equivalently

$$\int_{\mathbb{R}^d \backslash \{0\}} \frac{y^2}{1+y^2}\gamma(dy) < \infty.$$

This measure γ is the so called Lévy jump measure of the Lévy process. A Lévy process L_t has the following Lévy-Itô decomposition [3, 10]

$$L_t = \mathbf{b}t + B_t + \int_{\|x\|<1} x\tilde{N}(t, dx) + \int_{\|x\|\geqslant 1} xN(t, dx) \tag{6}$$

where $N(dt, dx)$ is Poisson random measure, $\tilde{N}(dt, dx) = N(dt, dx) - \gamma(dx)dt$ is the compensated Poisson random measure, and B_t is an independent Brownian motion d-dimensional Brownian motion with covariance matrix $\mathbf{A}$. A Lévy process with the generating triplet $(\mathbf{b}, \mathbf{A}, \gamma)$.

Lévy process has independent and stationary increments, and is thought to be appropriate models for non-Gaussian processes fluctuations [5, 28]. Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L_t - L_{t_0}| \geqslant \delta) \to 0$ as $t \to t_0$ for any positive δ. With a suitable modification [26], these paths may be taken as *càdlàg*, that is, paths are continuous on the right and have limits on the left. This continuity is weaker than the usual continuity in time. In fact, a *càdlàg* function has finite or at most countable discontinuities on any time interval.

Brownian motion B_t is a special case of Lévy process, being a Gaussian process, is characterized by its mean vector (taken to be the zero vector) and its covariance matrix (taken to be the identity matrix). Additionally, (i) almost every sample path is continuous in time in the usual sense, and (ii) the increments are Gaussian distributed.

As another special case of Lévy process, the symmetric α-stable process plays an important role among stable processes like Brownian motion among Gaussian processes. For the definition of symmetric α-stable process with $0 < \alpha < 2$, see [29, 30]. Its jump measure in $\mathbb{R}^d$ is $\gamma(dy)=\frac{dy}{\|y\|^{d+\alpha}}$. When $\alpha = 2$, we have the standard Brownian motion.

Marcus canonical stochastic differential equations were introduced by Marcus [24] with semimartingales as the driving processes. For a Lévy process L_t it can be written as

$$dx(t) = b(x(t))dt + \sigma(x(t-)) \diamond dL_t, \tag{7}$$

where $\diamond$ denotes the Marcus integral, which for a scalar Lévy process is given by

$$\begin{aligned} dx(t) &= b(x(t))dt + \sigma(x(t-)) \circ dL_c(t) + \sigma(x(t-))dL_d(t) \\ &\quad + \sum_{0\leqslant s\leqslant t} [\varphi(x(s-)), \triangle L(s) - x(s-) - \sigma(x(s-))\triangle L(s)], \end{aligned}$$

where L_c and L_d are the usual continuous and discontinuous parts of L, $\circ$ denotes the Stratonovich stochastic integral, and $\varphi(u, v)$ is the solution or flow of the ordinary differential equation

$$\frac{d\varphi(u, v)}{dv} = \sigma(\varphi), \quad (u, 0) = u.$$

With the help of Lévy-Itô decomposition of Lévy processes, the Marcus canonical equation also admits an Itô interpretation, see [3, 19]. The solution of Marcus canonical SDE (7) defines a stochastic flow, or actually a cocycle, of homeomorphisms or diffeomorphisms [12, 19], when the coefficients b and σ satisfy appropriate conditions. For more details about the Marcus integral and canonical equation, see [11, 20, 25].

3. Random Dynamical Systems

Let (Ω, F, P) be a probability space. Following Arnold [4], a random dynamical system (RDS) (θ, ϕ) on $\Omega \times \mathbb{R}^d$ consists of a metric dynamical system θ on Ω and a cocycle mapping $\Phi:\mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, namely, φ satisfies the conditions

$$\varphi(0, \omega) = id_{\mathbb{R}^d}, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$$

for all $\omega \in \Omega$ and all $s, t \in \mathbb{R}$. This cocycle is required to be at least measurable from the σ-field $\mathcal{B}(\mathbb{R})\times F\times\mathcal{B}(\mathbb{R}^d)$ to the σ-field $\mathcal{B}(\mathbb{R}^d)$.

For random dynamical systems driven by noise process, we take $\Omega = (\mathbb{R}, \mathbb{R}^d)$ with the Skorohod metric as the canonical sample space and denote by $F := \mathcal{B}(D(\mathbb{R}, \mathbb{R}^d))$ the associated Borel σ-field.

A family $\hat{A} = \{A(\omega), \omega \in \Omega\}$ of nonempty measurable compact subset $A(\omega)$ of $\mathbb{R}^d$ is called ϕ-invariant if $\phi(t, \omega, A(\omega))=$ A$(\theta_t\omega)$ for all $t \geqslant 0$ and is called a random attractor if in addition it is pathwise pullback attracting in the sense that

$$H^*_d(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) \to 0, \quad \text{as } t \to +\infty$$

for all suitable (i.e. in a given attracting universe, for instance, in [5, 25]) families $\hat{D} = \{D(\omega), \omega \in \Omega\}$ of nonempty measurable bounded subsets $D(\omega)$ of $\mathbb{R}^d$. Here H^*_d is the Hausdorff semi-distance on $\mathbb{R}^d$.

Theorem 3.1. *Let (θ, ϕ) be an RDS on $\Omega \times \mathbb{R}^d$. If there exists a family $\hat{B} =\{B(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $B(\omega)$ of $\mathbb{R}^d$ and a $T_{\hat{D},\omega} \geqslant 0$ such that*

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad \forall t \geqslant T_{\hat{D},\omega}$$

for all families $\hat{D} =\{D(\omega), \omega \in \Omega\}$ in the given attracting universe, then the RDS (θ, ϕ) has a random attractor $\hat{A} =\{A(\omega), \omega \in \Omega\}$ with the component subsets defined for each $\omega \in \Omega$ by

$$A(\omega) = \bigcap_{s>o} \overline{\bigcup_{t\geqslant s} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}.$$

If the random attractor consists of singleton sets, i.e. $A(\omega) = X^*(\omega)$ for some random variable X^*, then $X_t^*(\omega) := X^*(\theta_t\omega)$ is a stationary stochastic process, if the driving system θ_t is a stationary process the prove of this theorem can be found in [6, 18]. We will need the following lemmas (see [7, 9, 22]).

Lemma 3.2. *Let $\{x_n\}$ be a sequence in a complete metric space (X, d) such that every subsequence $\{x_{n_i}\}$ has a subsequence $\{x_{n_i}\}$ converging to a common limit x^*. Then the sequence $\{x_n\}$ converges to x^*.*

Lemma 3.3. *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant subset $\bar{\Omega} \in F$ of $\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{1}{t} \parallel \omega(t) \parallel = 0 \quad for \quad \omega \in \bar{\Omega},$$

and there exist random variable $\bar{O}^{(1)}$ and $\bar{O}^{(2)}$ such that

$$\bar{O}^{(1)}(\theta_t\omega) = \bar{O}_t^{(1)}(\omega) \quad and \quad \bar{O}^{(2)}(\theta_t\omega) = \bar{O}_t^{(2)}(\omega) \quad for \quad \omega \in \bar{\Omega}.$$

Moreover, we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \bar{O}^{(1)}(\theta_\tau\omega)d\tau = \lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \bar{O}^{(2)}(\theta_\tau\omega)d\tau = 0 \quad for \quad \omega \in \bar{\Omega}.$$

In what follows, we consider θ defined on $\bar{\Omega}$ instead of Ω. This mapping has the same properties as the original one if we choose for F the trace σ-algebra with respect to $\bar{\Omega}$.

4. Transformation of Systems to Random Differential Equation

Now consider a Marcus stochastic differential equation with linear noise

$$\begin{aligned} dX_t &= f(X_t)dt + (a_1X_t + b_1) \diamond dL_t^\alpha, \\ dY_t &= g(Y_t)dt + (a_2Y_t + b_2) \circ dW_t. \end{aligned} \tag{8}$$

Here L_t^α is independent two sided scalar α-stable processes and W_t is independent two-sided scalar Wiener processes, with b_1, b_2 is constant in $\mathbb{R}^d$ and a_1, a_2 is constant in $\mathbb{R}$. The functions f, g are sufficiently regular to ensure the existence and uniqueness of local solution, and additionally satisfy one-side dissipative Lipschitz conditions (2).

Using the transformation

$$x(t, \omega) = e^{-a_1O_t^{(1)}(\omega)}\left(X_t(\omega) + \frac{b_1}{a_1}\right)$$

and

$$y(t, \omega) = e^{-a_2O_t^{(2)}(\omega)}\left(Y_t(\omega) + \frac{b_2}{a_2}\right),$$

where

$$O_t^{(1)} = e^{-t}\int_{-\infty}^t e^u dL_u^\alpha, \quad O_t^{(2)} = e^{-t}\int_{-\infty}^t e^u dW_u, \quad t \in \mathbb{R}$$

are two stationary Ornstein-Uhlenbeck processes. By Itô's formula, we get the pathwise random ordinary differential equation (RODE)

$$\begin{aligned} \tfrac{dx}{dt} &= F(x, O_t^{(1)}) := e^{-a_1O_t^{(1)}} f(e^{a_1O_t^{(1)}}x - \tfrac{b_1}{a_1}) + a_1O_t^{(1)}x, \\ \tfrac{dy}{dt} &= G(y, O_t^{(2)}) := e^{-a_2O_t^{(2)}} g(e^{a_2O_t^{(2)}}y - \tfrac{b_2}{a_2}) + a_2O_t^{(2)}y. \end{aligned} \tag{9}$$

We will show in the next section that each of the stochastic systems in (8) a pathwise asymptotically stable and has random attractor which consists of a single stationary stochastic process. Therefor, the use of the stationary Ornstein-Uhlenbeck process in the transformation will be essential. Then we will study their behavior after synchronization by linear cross coupling, i.e, we will consider the coupled RODE

$$\frac{dx}{dt} = F(x, O_t^{(1)}(\omega)) + \nu(y - x),$$
$$\frac{dy}{dt} = G(y, O_t^{(2)}(\omega)) + \nu(x - y),$$

we will also show above system has a pathwise asymptotically stable and has random attractor consist of a single stationary stochastic process $(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega))$. In particular, $(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega)) \to (\bar{z}(\omega), \bar{z}(\omega))$ as $\nu \to \infty$ where $\bar{z}(\omega)$ is the pathwise asymptotically stable solution of the averaged RODE

$$\frac{dz}{dt} = \frac{1}{2}[F(z, O_t^{(1)}) + G(z, O_t^{(2)})],$$

that is

$$\frac{dz}{dt} = \frac{1}{2}[e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} z - \frac{b_1}{a_1}) + e^{-a_2 O_t^{(2)}} g(e^{a_2 O_t^{(2)}} z - \frac{b_2}{a_2}) + (a_1 O_t^{(1)} + a_2 O_t^{(2)})z].$$

The equivalent stochastic differential equation is given by

$$dZ_t = \frac{1}{2}[e^{-\eta_t} f(e^{\eta_t} Z_t - \frac{b_1}{a_1}) + e^{\eta_t} g(e^{-\eta_t} Z_t - \frac{b_2}{a_2})]dt + \frac{1}{2}a_1 Z_t \diamond dL_t^\alpha + \frac{1}{2}a_2 Z_t \circ dW_t, \tag{10}$$

where

$$Z_t = e^{\frac{1}{2}(a_1 O_t^{(1)} + a_2 O_t^{(2)})} z_t$$

and

$$\eta_t = \frac{1}{2}(a_1 O_t^{(1)} - a_2 O_t^{(2)}).$$

In terms of the original system of Marcus stochastic differential equations (8), the coupled random equations take the form

$$\begin{aligned} dX_t &= [f(X_t) + \nu(e^{2\eta_t} Y_t - X_t) + \nu(\tfrac{b_2}{a_2} e^{2\eta_t} - \tfrac{b_1}{a_1})]dt + (a_1 X_t + b_1) \diamond dL_t^\alpha, \\ dY_t &= [g(Y_t) + \nu(e^{-2\eta_t} X_t - Y_t) + \nu(\tfrac{b_1}{a_1} e^{-2\eta_t} - \tfrac{b_2}{a_2})]dt + (a_2 Y_t + b_2) \circ dW_t. \end{aligned} \tag{11}$$

Then this system has a unique stationary stochastic solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$, which is pathwise globally asymptotically stable with

$$(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega)) \to (\bar{Z}_t(\omega) e^{\eta_t} - \frac{b_1}{\alpha_1}, \bar{Z}_t(\omega) e^{-\eta_t} - \frac{b_2}{a_2}), \text{ as } \nu \to \infty,$$

where $\bar{Z}_t(\omega)$ is the stationary solution of (10).

5. The Uncoupled System with α-Stable Noise and Wiener Noise

In this section, we will prove the uncoupled equations SDE (8) has unique stochastic stationary solutions, which are

$$\begin{aligned} dX_t &= f(X_t)dt + (a_1 X_t + b_1) \diamond dL_t^\alpha, \\ dY_t &= g(Y_t)dt + (a_2 Y_t + b_2) \circ dW_t, \end{aligned} \tag{12}$$

where f, g are continuously differential, satisfy the one-sided dissipative Lipschitz conditions (2). Its solution paths are generally not differentiable. Thus we rewrite the above equations as

$$\begin{aligned} dX_t &= [f(X_t) + O_t^{(1)}(a_1 X_t + b_1)]dt + (a_1 X_t + b_1) \diamond dO_t^{(1)}, \\ dY_t &= [g(Y_t) + O_t^{(2)}(a_2 X_t + b_2)]dt + (a_2 X_t + b_2) \circ dO_t^{(2)}, \end{aligned} \tag{13}$$

where $O_t^{(1)}$ and $O_t^{(2)}$, $t \in \mathbb{R}$, is the stationary solution of

$$dO_t^{(1)} = -O_t^{(1)}dt + dL_t^{\alpha}$$

and

$$dO_t^{(2)} = -O_t^{(2)}dt + dW_t.$$

That is

$$O_t^{(1)} = e^{-t}\int_{-\infty}^{t} e^u dL_u^{\alpha}, \quad O_t^{(2)} = e^{-t}\int_{-\infty}^{t} e^u dW_u, \quad t \in \mathbb{R}.$$

Then we transform (13) to the pathwise random ordinary differential equation

$$\begin{aligned} \tfrac{dx}{dt} &= F(x, O_t^{(1)}) := e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} x - \tfrac{b_1}{a_1}) + a_1 O_t^{(1)} x, \\ \tfrac{dy}{dt} &= G(y, O_t^{(2)}) := e^{-a_2 O_t^{(2)}} g(e^{a_2 O_t^{(2)}} y - \tfrac{b_2}{a_2}) + a_2 O_t^{(2)} y. \end{aligned} \tag{14}$$

The vector-field function

$$\tilde{f}(x, z) = e^{-a_1 z} f(e^{a_1 z} x - \frac{b_1}{a_1})$$

and

$$\tilde{g}(x, z) = e^{-a_2 z} g(e^{a_2 z} y - \frac{b_2}{a_2})$$

in the system (14) satisfies a one-sided Lipschitz condition in its first variable uniformly in the second with the same constant as the original drift coefficient f, g, since we have

$$\langle x_1 - x_2, \tilde{f}(x_1, z) - \tilde{f}(x_2, z)\rangle \leqslant -L \parallel x_1 - x_2 \parallel^2$$

and

$$\langle y_1 - y_2, \tilde{g}(y_1, z) - \tilde{g}(y_2, z)\rangle \leqslant -L \parallel y_1 - y_2 \parallel^2 .$$

We obtain that any of the two solutions of the RODE (14) satisfy pathwise the differential inequality

$$\frac{d}{dt} \parallel x_1(t) - x_2(t) \parallel^2 \leqslant (-2L + 2a_1 O_t^{(1)}) \parallel x_1(t) - x_2(t) \parallel^2 \tag{15}$$

and

$$\frac{d}{dt} \parallel y_1(t) - y_2(t) \parallel^2 \leqslant (-2L + 2a_2 O_t^{(2)}) \parallel y_1(t) - y_2(t) \parallel^2, \tag{16}$$

and hence we have

$$\parallel x_1(t) - x_2(t) \parallel^2 \leqslant e^{-2t(L - \frac{1}{t}\int_0^t O_\tau^{(1)} d\tau)} \parallel x_1(0) - x_2(0) \parallel^2$$

and

$$\parallel y_1(t) - y_2(t) \parallel^2 \leqslant e^{-2t(L - \frac{1}{t}\int_0^t O_\tau^{(2)} d\tau)} \parallel y_1(0) - y_2(0) \parallel^2 .$$

Thus it follows by Lemma 3.3 that

$$\lim_{t\to\infty} \parallel x_1(t) - x_2(t) \parallel^2 = 0$$

and

$$\lim_{t\to\infty} \parallel y_1(t) - y_2(t) \parallel^2 = 0,$$

which means all solutions converge pathwise to each other.

In order to see what they converge to, we first observe that the RODEs (14) generates a random dynamical system with $\phi(t,\omega,x_0) := x(t,\omega)$, the solution of the RODEs (14) with (deterministic) initial value x_0 at time $t = 0$.Then we need to show that the RODEs (14) is asymptotically dissipative and has a pullback attractor. Omitting ω for brevity, we have pathwise

$$\begin{aligned}
\frac{d}{dt} \parallel x \parallel^2 &= 2\langle x, F(x, O_t^{(1)})\rangle \\
&= 2\langle x, e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1}) + a_1 O_t^{(1)} x\rangle \\
&= 2e^{-2a_1 O_t^{(1)}} \langle e^{a_1 O_t^{(1)}} x, f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1}) - f(-\frac{b_1}{a_1})\rangle \\
&+ 2\langle x, e^{-a_1 O_t^{(1)}} f(-\frac{b_1}{a_1})\rangle + 2a_1 O_t^{(1)} \parallel x \parallel^2 \\
&\leqslant (-L + 2a_1 O_t^{(1)}) \parallel x \parallel^2 + \frac{e^{-2a_1 O_t^{(1)}}}{L} \parallel f(-\frac{b_1}{a_1}) \parallel^2 .
\end{aligned} \tag{17}$$

Integration yields

$$\parallel x(t) \parallel^2 \leqslant \parallel x(t_0) \parallel^2 e^{-L(t-t_0)+2\int_{t_0}^t a_1 O_\tau^{(1)} d\tau} + \frac{\parallel f(-\frac{b_1}{a_1}) \parallel^2}{L} \int_{t_0}^t e^{-2a_1 O_u^{(1)}} e^{-L(t-u)} e^{2\int_{t_0}^t a_1 O_\tau^{(1)} d\tau} du.$$

Moreover, by Lemma 3.3 we have pathwise

$$\lim_{s\to-\infty} \frac{1}{s} \int_s^0 O_\tau^{(1)} d\tau = \lim_{t\to\infty} \frac{1}{t} \int_0^t O_\tau^{(1)} d\tau = 0.$$

Thus we obtain

$$e^{2\int_s^t O_\tau^{(1)} d\tau} \leqslant e^{\frac{L}{2}(t-s)}$$

for $s \leqslant 0$, $t \geqslant 0$ with $| t |, | t_0 |> T_\omega$.

Now we can use pathwise pullback convergence (i.e. with $t_0 \to -\infty$) to show that the closed ball centered at the origin with random radius.

$$R^2(\omega) := 1 + \frac{\parallel f(-\frac{b_1}{a_1}) \parallel^2}{L} \int_{-\infty}^0 e^{-2a_1 O_u^{(1)}} e^{Lu} e^{2\int_u^0 a_1 O_\tau^{(1)} d\tau} du.$$

Similarly, we also have

$$\frac{d}{dt} \parallel y \parallel^2 \leqslant (-L + 2a_2 O_t^{(2)}) \parallel y \parallel^2 + \frac{e^{-2a_2 O_t^{(2)}}}{L} \parallel g(-\frac{b_2}{a_2}) \parallel^2 . \tag{18}$$

Integration yields

$$\parallel y(t) \parallel^2 \leqslant \parallel y(t_0) \parallel^2 e^{-L(t-t_0)+2\int_{t_0}^t a_2 O_\tau^{(2)} d\tau} + \frac{\parallel g(-\frac{b_2}{a_2}) \parallel^2}{L} \int_{t_0}^t e^{-2a_2 O_u^{(2)}} e^{-L(t-u)} e^{2\int_{a_2}^t a_2 O_\tau^{(2)} d\tau} du.$$

Moreover, by Lemma 3.3 we have pathwise

$$\lim_{s\to-\infty} \frac{1}{s} \int_s^0 O_\tau^{(2)} d\tau = \lim_{t\to\infty} \frac{1}{t} \int_0^t O_\tau^{(2)} d\tau = 0.$$

Thus we obtain

$$e^{2\int_s^t O_\tau^{(2)} d\tau} \leqslant e^{\frac{L}{2}(t-s)}$$

for $s \leqslant 0$, $t \geqslant 0$ with $\mid t \mid, \mid t_0 \mid > T_\omega$.

Now we can use pathwise pullback convergence (i.e. with $t_0 \to -\infty$) to show that the closed ball centered at the origin with random radius.

$$R^2(\omega) := 1 + \frac{\parallel g(-\frac{b_2}{a_2}) \parallel^2}{L} \int_{-\infty}^{0} e^{-2a_2 O_u^{(2)}} e^{Lu} e^{2\int_u^0 a_2 O_\tau^{(2)} d\tau} du$$

is a pullback absorbing set for $t > T_\omega$. Theorem 3.1 of RDS then gives us a random attractor $\{A(\omega), \omega \in \Omega\}$. The fact that all trajectories converge to each other forwards in time. The sets in this random attractor are singleton sets, i.e. $A(\omega) = \{a(\omega)\}$. When we transform back to the SDEs have the pathwise singleton set attractor $a(\theta_t(\omega))$, which is a stationary solution the SDEs, since the Ornstein-Uhlenbeck process is stationary.

6. The Asymptotic Behaviour Of the Coupled System

Now, we will show that the stationary solution of coupled synchronized system converges when the parameter ν is large enough. We consider the coupled RODEs system

$$\frac{dx}{dt} = F(x, O_t^{(1)}(\omega)) + \nu(y - x),$$
$$\frac{dy}{dt} = G(y, O_t^{(2)}(\omega)) + \nu(x - y),$$

with

$$\begin{aligned} F(x, O_t^{(1)}) &= e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} x - \tfrac{b_1}{a_1}) + a_1 O_t^{(1)} x, \\ G(y, O_t^{(2)}) &= e^{-a_2 O_t^{(2)}} g(e^{a_2 O_t^{(2)}} y - \tfrac{b_2}{a_2}) + a_2 O_t^{(2)} y. \end{aligned} \tag{19}$$

Using the one-sided Lipschitz conditions on f and g, we obtain similarly to (15) and (16) that

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 \leqslant [-2L - \nu + 2a_1 O_t^{(1)}]\|x_1(t) - x_2(t)\|^2 + \nu\|y_1(t) - y_2(t)\|^2$$

and

$$\frac{d}{dt}\|y_1(t) - y_2(t)\|^2 \leqslant [-2L - \nu + 2a_2 O_t^{(2)}]\|y_1(t) - y_2(t)\|^2 + \nu\|x_1(t) - x_2(t)\|^2,$$

and similarly to (17) and (18) we obtain

$$\frac{d}{dt}\|x\|^2 \leqslant (-L - \nu + 2a_1 O_t^{(1)})\|x\|^2 + \nu\|y\|^2 + \frac{1}{L} e^{-2a_1 O_t^{(1)}} \|f(-\frac{b_1}{a_1})\|^2$$

and

$$\frac{d}{dt}\|y\|^2 \leqslant (-L - \nu + 2a_2 O_t^{(2)})\|y\|^2 + \nu\|x\|^2 + \frac{1}{L} e^{-2a_2 O_t^{(2)}} \|g(-\frac{b_2}{a_2})\|^2.$$

Defining

$$A_\nu(t) = \begin{pmatrix} -2L - \nu + 2a_1 O_t^{(1)} & \nu \\ \nu & -2L - \nu + 2a_2 O_t^{(2)} \end{pmatrix}, \quad t \in \mathbb{R},$$

$$\mathbf{x}(t) = \begin{pmatrix} \|x_1(t) - x_2(t)\|^2 \\ \|y_1(t) - y_2(t)\|^2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Thus we can write the above inequalities as

$$\frac{d}{dt}\mathbf{x} \leqslant A_\nu(t)\mathbf{x}$$

Due to a Gronwall-like inequality, we have

$$\mathbf{x}(x) \leqslant e^{\int_{t_0}^t A_\nu(s)ds}\mathbf{x}(t_0).$$

Let

$$\tilde{A}_\nu(t) = \begin{pmatrix} -L - \nu + 2a_1 O_t^{(1)} & \nu \\ \nu & -L - \nu + 2a_2 O_t^{(2)} \end{pmatrix}, \quad t \in \mathbb{R},$$

$$\mathbf{x}(t) = \begin{pmatrix} \|x(t)\|^2 \\ \|y(t)\|^2 \end{pmatrix}, \quad t \in \mathbb{R}$$

and

$$H(t) = \frac{1}{L}\begin{pmatrix} e^{(-2a_1 O_t^{(1)})}\|f(-\frac{b_1}{a_1})\|^2 \\ e^{(-2a_1 O_t^{(2)})}\|g(-\frac{b_2}{a_2})\|^2 \end{pmatrix},$$

we can write the above inequalities as

$$\frac{d}{dt}\mathbf{x} \leqslant \tilde{A}_\nu(t)\mathbf{x} + H(t).$$

Due to a Gronwall-like inequality, we have

$$\mathbf{x}(x) \leqslant e^{\int_{t_0}^t \tilde{A}_\nu(s)ds}\mathbf{x}(t_0) + \int_{t_0}^t e^{\int_s^t \tilde{A}_\nu(\tau)d\tau}H(s)ds$$

component wise. Now, we need the following simple lemma.

Lemma 6.1. *We have*

$$\|e^{\int_0^t A_\nu(\tau)d\tau}\boldsymbol{x}\| \leqslant e^{-Lt}\|\boldsymbol{x}\|, \ \ \boldsymbol{x} \in \mathbb{R}^2$$

for $t \geqslant T_w$ and all $\nu \geqslant 1$.

Proof. *First note that the matrix $\int_0^t A_\nu(\tau)d\tau$ is symmetric. Thus, the exists of a orthonormal basis of eigenvectors $u_{\nu,t}^{(1)}, u_{\nu,t}^{(2)}$ with eigenvalues $\lambda_{\nu,t}^{(1)}, \lambda_{\nu,t}^{(2)}$, and we have*

$$e^{\int_0^t A_\nu(\tau)d\tau}\boldsymbol{x} = e^{\lambda_{\nu,t}^{(1)}}c_{\boldsymbol{x},\nu,t}^{(1)}u_{\nu,t}^{(1)} + e^{\lambda_{\nu,t}^{(2)}}c_{\boldsymbol{x},\nu,t}^{(2)}u_{\nu,t}^{(2)},$$

where

$$c_{\boldsymbol{x},\nu,t}^{(1)}u_{\nu,t}^{(1)} + c_{\boldsymbol{x},\nu,t}^{(2)}u_{\nu,t}^{(2)} = \boldsymbol{x}.$$

Since $u_{\nu,t}^{(1)}$ and $u_{\nu,t}^{(2)}$ are orthogonal, we obtain

$$\|e^{\int_0^t A_\nu(\tau)d\tau}\boldsymbol{x}\|^2 \quad = e^{2\lambda_{\nu,t}^{(1)}}\|c_{\boldsymbol{x},\nu,t}^{(1)}u_{\nu,t}^{(1)}\|^2 + e^{2\lambda_{\nu,t}^{(2)}}\|c_{\boldsymbol{x},\nu,t}^{(2)}u_{\nu,t}^{(2)}\|^2 \leqslant e^{2max\{\lambda_{\nu,t}^{(1)}+\lambda_{\nu,t}^{(2)}\}}\|\boldsymbol{x}\|^2. \tag{20}$$

The eigenvalues of $\int_0^t A_\nu(\tau)d\tau$ *are given by*

$$\lambda_{\nu,t}^{(1/2)} = -(2L+\nu)t + \int_0^t (a_1 O_\tau^{(1)} + a_2 O_\tau^{(2)})d\tau \pm \sqrt{(\int_0^t (a_1 O_\tau^{(1)} - a_2 O_\tau^{(2)})d\tau)^2 + \nu^2 t^2},$$

hence it follows by Lemma 3.3 that

$$\lambda_{\nu,t}^{(1/2)} \leqslant -Lt \tag{21}$$

for $|t| > T_w$ *and all* $\nu \geqslant 1$. $\square$

Analogously to Lemma 6.1 we can show

Lemma 6.2. *Let* $t_0 \leqslant 0$ *and* $t \geqslant 0$. *We have*

$$||e^{\int_{t_0}^t \tilde{A}_\nu(\tau)d\tau}\boldsymbol{x}|| \leqslant e^{-\frac{L}{2}(t-t_0)}||\boldsymbol{x}||, \quad \boldsymbol{x} \in \mathbb{R}^2,$$

for $|t_0|$, $|t| \geqslant T_w$ *and all* $\nu \geqslant 1$.

Now set

$$C_\nu(w) := \frac{1}{L}\int_{-\infty}^0 e^{\int_u^0 \tilde{A}_\nu(\tau)d\tau} \begin{pmatrix} e^{(-2a_1 O_t^{(1)})}||f(-\frac{b_1}{a_1})||^2 \\ e^{(-2a_2 O_t^{(2)})}||g(-\frac{b_2}{a_2})||^2 \end{pmatrix} du$$

and define

$$R_\nu^2(w) = 1 + ||C_\nu(w)||^2.$$

Then by pullback techniques and Lemma 6.2, we see that the random balls $B_\nu(w)$ *in* $\mathbb{R}^{2d}$ *centered on the origin and with radius* $R_\nu(w)$ *are pullback absorbing. Moreover note that*

$$\frac{d}{d\nu}||C_\nu(w)||^2 = 2\left\langle \tfrac{d}{d\nu}C_\nu(w), C_\nu(w) \right\rangle = 2\left\langle \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} C_\nu(w), C_\nu(w) \right\rangle \leqslant 0$$

and consequently $R_\nu(w) \leqslant R_1(w)$ *for* $\nu \geqslant 1$. *Hence the random dynamical system generated by the coupled RODE (9) has a random attractor* $A_\nu(w)$ *in* $B_\nu(w)$ *for each* w. *But we know that all solutions converge to each other pathwise forwards in time. Thus the* $A_\nu(w)$ *are singleton sets, say* $A_\nu(w) = (\bar{x}_\nu(w), \bar{y}_\nu(w))$.

Let us now estimate the difference of the components of the coupled system. We have pathwise

$$\begin{aligned}
\frac{d}{dt}|x-y|^2 &= 2\langle x-y, \frac{dx}{dt} - \frac{dy}{dt}\rangle \\
&= 2\langle x-y, e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}}x - \frac{b_1}{a_1}) - e^{-a_2 O_t^{(2)}} g(e^{a_2 O_t^{(2)}}y - \frac{b_2}{a_2})\rangle \\
&\quad + 2\langle x-y, a_1 x O_t^{(1)} - a_2 y O_t^{(2)}\rangle + 2\langle x-y, 2\nu(y-x)\rangle \\
&\leqslant -4\nu||x-y||^2 + 2||x-y||(e^{-a_1 O_t^{(1)}}||f(e^{a_1 O_t^{(1)}}x - \frac{b_1}{a_1})|| \\
&\quad + e^{-a_2 O_t^{(2)}}||g(e^{a_2 O_t^{(2)}}x - \frac{b_2}{a_2})|| + ||a_1 x O_t^{(1)} - a_2 y O_t^{(2)}||) \\
&\leqslant -\nu||x-y||^2 + \frac{1}{\nu}e^{-\alpha_1 O_t^{(1)}}||f(e^{a_1 O_t^{(1)}}x - \frac{b_1}{a_1})||^2 \\
&\quad + \frac{1}{\nu}e^{-\alpha_2 O_t^{(2)}}||g(e^{a_2 O_t^{(2)}}y - \frac{b_2}{a_2})||^2 + \frac{1}{\nu}|a_1 O_t^{(1)}|^2||x||^2 \\
&\quad + \frac{1}{\nu}|a_2 O_t^{(2)}|^2||y||^2.
\end{aligned}$$

Hence labelling the solutions now with ν to indicate this dependence, we have

$$\frac{d}{dt}\|x_\nu - y_\nu\|^2 \leqslant -\nu\|x_\nu - y_\nu\|^2 + \frac{1}{\nu}M^\nu_{T_1,T_2,w}$$

with

$$\begin{aligned} M^\nu_{T_1,T_2,w} &= \sup_{t\in[T_1,T_2]}(e^{-a_1O_t^{(1)}}\|f(e^{a_1O_t^{(1)}}x - \frac{b_1}{a_1})\|^2 + |a_1O_t^{(1)}|^2\|x\|^2) \\ &\quad + \sup_{t\in[T_1,T_2]}(e^{-a_2O_t^{(2)}}\|g(e^{a_2O_t^{(2)}}y - \frac{b_2}{a_2})\|^2 + |a_2O_t^{(2)}|^2\|y\|^2) \end{aligned}$$

We can restrict ourselves without loss of generality to solutions in the compact absorbing balls $B_\nu(w)$, which are all contained in the common compact ball $B_1(w)$ for $\nu \geqslant 1$. Hence $M^\nu_{T_1,T_2,w}$ is uniformly bounded in ν and we have

$$\frac{d}{dt}\|x_\nu - y_\nu\|^2 \leqslant -\nu\|x_\nu - y_\nu\|^2 + \frac{1}{\nu}M^\nu_{T_1,T_2,w}$$

with

$$M_{T_1,T_2,w} = \sup_{\nu\geqslant 1} M^\nu_{T_1,T_2,w},$$

from which we conclude that

$$\|x_\nu(t) - y_\nu(t)\|^2 \to 0, \quad \nu \to \infty,$$

uniformly in $t \in [T_1, T_2]$ for any bounded T_1 and T_2.

7. The Synchronized Solution as $\nu \to \infty$

Now we can prove the solution of "averaged" RODEs is the attracting stationary solution.

Theorem 7.1. *$(\bar{x}_{\nu_n}(t,\omega), \bar{y}_{\nu_n}(t,\omega)) \to (\bar{z}(t,\omega), \bar{z}(t,\omega))$ pathwise uniformly on bounded time intervals $[T_1, T_2]$ of $\mathbb{R}$ for any sequence $\nu_n \to \infty$, where $\bar{z}$ is the attracting stationary solution of the "averaged" RODEs*

$$\frac{dz}{dt} = \frac{1}{2}[e^{-a_1O_t^{(1)}}f(e^{a_1O_t^{(1)}}z - \frac{b_1}{a_1}) + e^{-a_2O_t^{(2)}}g(e^{a_2O_t^{(2)}}z - \frac{b_2}{a_2}) + (a_1O_t^{(1)} + a_2O_t^{(2)})z]. \tag{22}$$

The equivalent stochastic differential equation is given by

$$dZ_t = \frac{1}{2}[e^{-\eta_t}f(e^{\eta_t}Z_t - \frac{b_1}{a_1}) + e^{\eta_t}g(e^{-\eta_t}Z_t - \frac{b_2}{a_2})]dt + \frac{1}{2}a_1Z_t \diamond dL_t^\alpha + \frac{1}{2}a_2Z_t \circ dW_t \tag{23}$$

with

$$\eta_t = \frac{1}{2}(a_1O_t^{(1)} - a_2O_t^{(2)}).$$

Proof. Define

$$\bar{z}_\nu(\omega) := \frac{1}{2}(\bar{x}_\nu(\omega) + \bar{y}_\nu(\omega))$$

and observe that $\bar{z}_\nu(t,\omega) = \bar{z}_\nu(\theta_{t\omega})$ satisfies the RODEs

$$\frac{d\bar{z}}{dt} = \frac{1}{2}[e^{-a_1O_t^{(1)}}f(e^{a_1O_t^{(1)}}\bar{z} - \frac{b_1}{a_1} + e^{-a_2O_t^{(2)}}g(e^{a_2O_t^{(2)}}\bar{z} - \frac{b_2}{a_2}) + a_1\bar{z}O_t^{(1)} + a_2\bar{z}O_t^{(2)}].$$

Thus

$$\sup_{t\in[T_1,T_2]}|\frac{d}{dt}\bar{z}_\nu(t,\omega)| \leqslant M_{T_1,T_2,\omega} < \infty,$$

by continuity and the fact that these solutions belong to the common compact ball $B_1(\omega)$. We can use the Ascoli theorem to conclude that there is a subsequence $\nu_{n_j} \to \infty$ such that $\bar{z}_{n_j}(t,\omega) \to \bar{z}(t,\omega)$ as $n_j \to \infty$. Now

$$\begin{aligned} \bar{z}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega) &= \frac{1}{2}(\bar{x}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega)) \to 0, \\ \bar{z}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega) &= \frac{1}{2}(\bar{y}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega)) \to 0, \end{aligned}$$

as $\nu_{n_j} \to \infty$, see the previous section, so

$$\begin{aligned} \bar{x}_{\nu_{n_j}}(t,\omega) &= 2\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{y}_{n_j}(t,\omega) \to \bar{z}(t,\omega), \\ \bar{y}_{\nu_{n_j}}(t,\omega) &= 2\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{x}_{n_j}(t,\omega) \to \bar{z}(t,\omega), \end{aligned}$$

as $\bar{\nu}_{n_j} \to \infty$. Moreover, using the integral equation representation

$$\begin{aligned} \bar{z}_\nu(t,\omega) &= \bar{z}(T_1,\omega) + \frac{1}{2}\int_{T_1}^t e^{-a_1 O_s^{(1)}(\omega)} f(e^{a_1 O_s^{(1)}} \bar{x}_\nu(s,\omega) - \frac{b_1}{a_1})ds \\ &+ \frac{1}{2}\int_{T_1}^t e^{-a_2 O_s^{(2)}(\omega)} g(e^{a_2 O_s^{(2)}} \bar{y}_\nu(s,\omega) - \frac{b_2}{a_2})ds + \frac{1}{2}\int_{T_1}^t a_1 \bar{x}_\nu(s,\omega) O_s^{(1)} + a_2 \bar{y}_\nu(s,\omega) O_s^{(2)} ds. \end{aligned}$$

It follows that the ν_{n_j} subsequence converges pathwise to

$$\begin{aligned} \bar{z}(t,\omega) &= \bar{z}(T_1,\omega) + \frac{1}{2}\int_{T_1}^t e^{-a_1 O_s^{(1)}(\omega)} f(e^{a_1 O_s^{(1)}} \bar{z}(s,\omega) - \frac{b_1}{a_1})ds \\ &+ \frac{1}{2}\int_{T_1}^t e^{-a_2 O_s^{(2)}(\omega)} g(e^{a_2 O_s^{(2)}} \bar{z}(s,\omega) - \frac{b_2}{a_2})ds + \frac{1}{2}\int_{T_1}^t (a_1 O_s^{(1)} + a_2 O_s^{(2)}) \bar{z}(s,\omega) ds \end{aligned}$$

on the interval $[T_1, T_2]$, so $\bar{z}(t,\omega)$ is a solution of the RODEs (22) for all $t \in \mathbb{R}$. By the same techniques as in the previous sections, it has a random attractor consisting of a singleton set formed by a single stationary stochastic process which thus must be equal to $\bar{z}(t,\omega)$. Finally, we note that pathwise all possible subsequences here have the same limit, so by Lemma 3.2, every full sequence $\bar{z}_\nu(t,\omega)$ actually converges to $\bar{z}(t,\omega)$ for the whole sequence $\nu_n \to \infty$. □

Corollary 7.2. $(\bar{x}_\nu(t,\omega), \bar{y}_\nu(t,\omega)) \to (\bar{z}(t,\omega), \bar{z}(t,\omega))$ *as* $\nu \to \infty$ *pathwise on any bounded time interval* $[T_1, T_2]$ *of* $\mathbb{R}$.

Example 7.3 (Example and simulation of synchronization). *Now, we will consider two stochastic differential equations*

$$dX_t = -X_t dt + (0.95X_t + 0.05) \diamond dL_t^{0.75}$$

and

$$dY_t = -2Y_t dt + (0.25Y_t + 0.5) \circ dW_t.$$

The corresponding RODE are

$$\frac{dx}{dt} = -(x - \frac{0.05}{0.95} e^{-0.95 O_t^{(1)}}) + 0.95 x O_t^{(1)}$$

and

$$\frac{dy}{dt} = -2(y - \frac{0.5}{0.25} e^{-0.25 O_t^{(2)}}) + 0.25 y O_t^{(2)},$$

where

$$O_t^{(1)} = e^{-t}\int_{-\infty}^{t} e^{u} dL_u^{0.75}, \quad O_t^{(2)} = e^{-t}\int_{-\infty}^{t} e^{u} dW_u.$$

The averaged RODE is

$$\frac{dz}{dt} = z(-\frac{3}{2} + \frac{1}{2}(0.95O_t^{(1)} + 0.25O_t^{(2)})) + \frac{1}{2}(\frac{0.05}{0.95}e^{-0.95O_t^{(1)}} + \frac{0.5}{0.25}e^{-0.25O_t^{(2)}})$$

and the equivalent stochastic differential equation is

$$dZ_t = -\frac{3}{2}Z_t dt + \frac{3}{2}(\frac{0.05}{0.95}e^{-\eta_t} + \frac{0.5}{0.25}e^{\eta_t})dt + \frac{1}{2}0.95Z_t \diamond dL_t^{0.75} + \frac{1}{2}0.25Z_t \circ dW_t$$

with the explicit solution

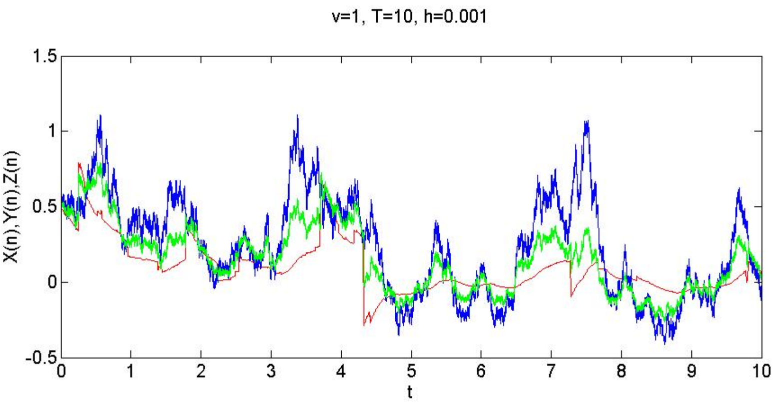
$$\begin{aligned} z(t) &= e^{-\frac{3}{2}(t-t_0)+\frac{1}{2}\int_{t_0}^{t}(0.95O_\tau^{(1)}+0.25O_\tau^{(2)}))d\tau} z_0 \\ &+\frac{1}{2}\int_{t_0}^{t} e^{-\frac{3}{2}(t-t_0)+\int_{t_0}^{t}(0.95O_\tau^{(1)}+0.25O_\tau^{(2)}))d\tau}(\frac{0.05}{0.95}e^{-0.95O_u^{(1)}} + \frac{0.5}{0.25}e^{-0.25O_u^{(2)}})du. \end{aligned}$$

The pullback limit as $t_0 \to -\infty$ gives a stationary solution

$$\tilde{z}(t) = \frac{1}{2}\int_{-\infty}^{t} e^{-\frac{3}{2}(t-s)+\int_{-\infty}^{t}(0.95O_\tau^{(1)}+0.25O_\tau^{(2)}))d\tau}(\frac{0.05}{0.95}e^{-0.95O_u^{(1)}} + \frac{0.5}{0.25}e^{-0.25O_u^{(2)}})du$$

and attracts all other solutions pathwise.

Figure 1 shows the trajectories of the numerical solution of the system with different values of ν. It shows that as ν increases the trajectories approach to each other faster.



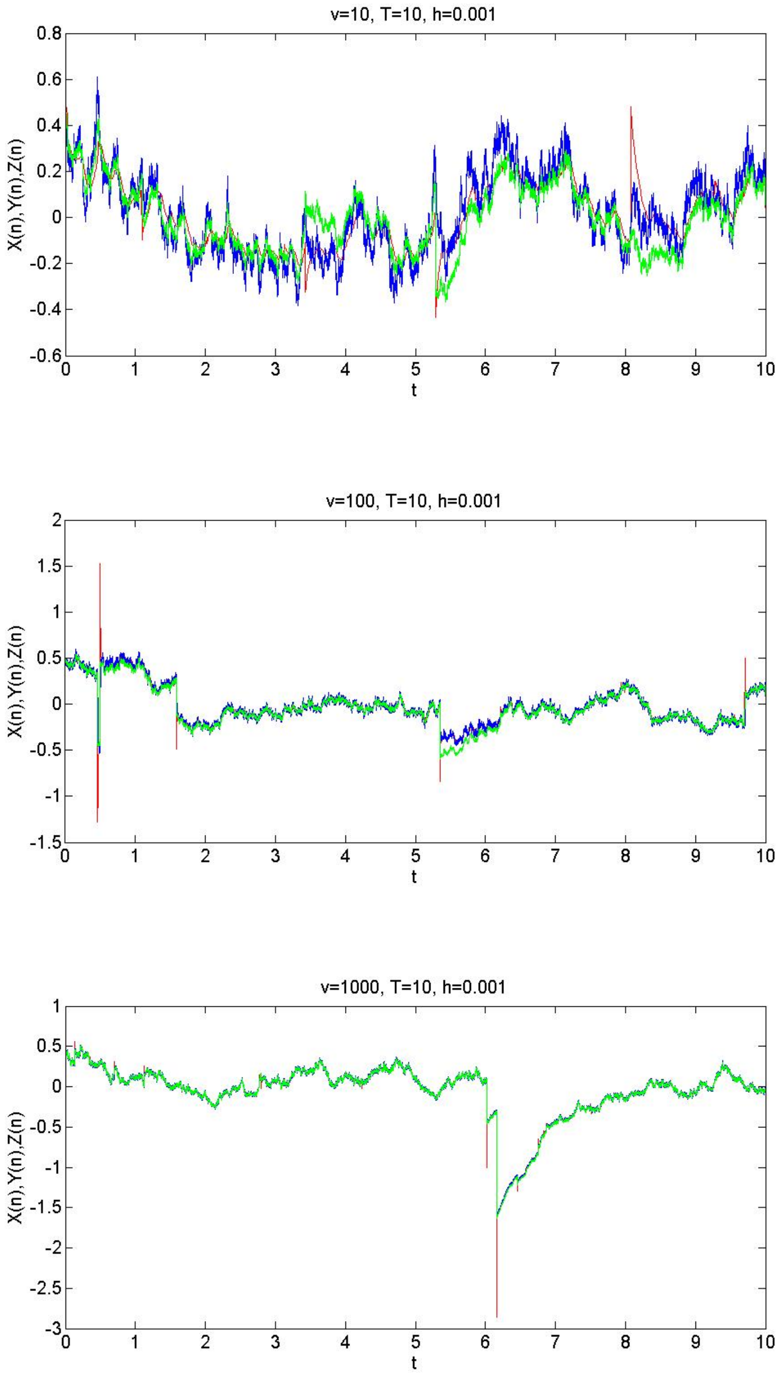


Figure 1: A trajectories of the coupled system $dX_t = -X_t dt + \nu(Y_t - X_t)dt + (0.95X_t + 0.05) \diamond dL_t^{(0.75)}$, $dY_t = -2Y_t dt + \nu(X_t - Y_t)dt + (0.25Y_t + 0.5) \circ dW_t$ and the corresponding trajectories of the averaged system $dZ_t = -3/2(Z_t) + (0.95Z_t + 0.05) \diamond dL_t^{(0.75)} + (0.25Z_t + 0.5) \circ dW_t$ for four values of ν

In [2, 7, 8, 22, 23] the noises that had been used are of the same type, while in reality no one can control the type of the noises which will affect the system, for that reason our paper is dealing with the effect of different types of noise which is closest to reality

8. SYNCHRONIZATION WHEN $a_1 \neq 0$ and $a_2 = 0$

We consider Stochastic differential equations in $\mathbb{R}^d$

$$\begin{aligned} dX_t &= f(X_t)dt + (a_1 X_t + b_1) \diamond dL_t^{(\alpha)}, \\ dY_t &= g(Y_t)dt + b_2 dW_t. \end{aligned} \tag{24}$$

Here L_t^{α} is a two sided scalar α-stable process and W_t is a two-sided scalar Wiener process independent of L_t^{α}. b_1 and b_2 are constant vectors in $\mathbb{R}^d$, a_1 constants in $\mathbb{R}$. The functions f and g are sufficiently regular to ensure the existence and uniqueness of local solution, and additionally satisfy one-side dissipative Lipschitz condition (2). Using the transformation

$$x(t,\omega) = e^{(-a_1 O_t^{(1)}(\omega))}(X_t(\omega) + \frac{b_1}{a_1})$$

and

$$y(t,\omega) = Y_t(\omega) - b_2 O_t^{(2)},$$

where

$$O_t^{(1)} = e^{-t}\int_{-\infty}^{t} e^u dL_u^{(\alpha)}, \quad O_t^{(2)} = e^{-t}\int_{-\infty}^{t} e^u dW_u, \quad t \in R$$

are two stationary Ornstein-Uhlenbeck processes.

We will start to transform it to the pathwise random ordinary differential equation (RODE)

$$\begin{aligned} \tfrac{dx}{dt} &= F(x, O_t^{(1)}) := e^{(-a_1 O_t^{(1)})} f(e^{(a_1 O_t^{(1)})} x - \tfrac{b_1}{a_1}) + a_1 O_t^{(1)} x, \\ \tfrac{dy}{dt} &= G(y, O_t^{(2)}) := g(y + b_2 O_t^{(2)}) + b_2 O_t^{(2)}. \end{aligned} \tag{25}$$

We will show in the next section that each of the stochastic systems in (24) a pathwise asymptotically stable and has random attractor which consists of a single stationary stochastic process. Then we will study their behavior after synchronization by linear cross coupling, i.e. we will consider the coupled RODE

$$\begin{aligned} \frac{dx}{dt} &= F(x, O_t^{(1)}(w)) + \nu(y - x), \\ \frac{dy}{dt} &= G(y, O_t^{(2)}(w)) + \nu(x - y), \end{aligned}$$

we will also show above system has a pathwise asymptotically stable and has random attractor consist of a single stationary stochastic process $(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega))$. In particular, $(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega)) \to (\bar{z}(\omega), \bar{z}(\omega))$ as $\nu \to \infty$ where $\bar{z}(\omega)$ is the pathwise asymptotically stable solution of the averaged RODE

$$\frac{dz}{dt} = \frac{1}{2}[F(z, O_t^{(1)}) + G(z, O_t^{(2)})],$$

that is

$$\frac{dz}{dt} = \frac{1}{2}[e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} z - \frac{b_1}{a_1}) + g(z + b_2 O_t^{(2)}) + a_1 O_t^{(1)} z + b_2 O_t^{(2)}]. \tag{26}$$

The equivalent SDE is given by

$$\begin{aligned} dZ_t = \; & \tfrac{1}{2}[f(Z_t - \tfrac{1}{2} b_2 O_t^{(2)} e^{a_1 O_t^{(1)}}) + e^{a_1 O_t^{(1)}} g(e^{-\frac{1}{2} a_1 O_t^{(1)}}(Z_t + \tfrac{b_1}{a_1}) + \tfrac{1}{2} b_2 O_t^{(2)})]dt \\ & + \tfrac{1}{2}[(a_1 Z_{t-} + b_1) \diamond dL_t^{(\alpha)} + b_2 e^{a_1 O_t^{(1)}} dW_t], \end{aligned}$$

where

$$z_t = e^{-\frac{1}{2} a_1 O_t^{(1)}}(Z_t + \frac{b_1}{a_1}) - \frac{1}{2} b_2 O_t^{(2)}.$$

In terms of the original system of the SDE (24), the coupled SDE have the form

$$\begin{aligned} dX_t &= [f(X_t) + \nu(e^{a_1 O_t^{(1)}} Y_t - X_t) + \nu(\tfrac{b_1}{a_1} - b_2 O_t^{(2)} e^{a_1 O_t^{(1)}})]dt + (a_1 X_t + b_1) \diamond dL_t^{\alpha}, \\ dY_t &= [g(Y_t) + \nu(e^{-a_1 O^{(1)}} X_t - Y_t) + \nu(b_2 O_t^{(2)} - \tfrac{b_1}{a_1} e^{-a_1 O_t^{(1)}})]dt + b_2 dW_t. \end{aligned} \tag{27}$$

Then this system has a unique stationary stochastic solution $(\bar{X}_t^{\nu}, \bar{Y}_t^{\nu})$, which is pathwise globally asymptotically stable with

$$(\bar{X}_t^{\nu}(\omega), \bar{Y}_t^{\nu}(\omega)) \to (\bar{z}_t(\omega) e^{a_1 O_t^{(1)}} - \frac{b_1}{a_1}, \bar{z}_t(\omega) + b_2 O_t^{(2)}), as \quad \nu \to \infty,$$

where $\bar{z}_t(\omega)$ stationary solution of (26).

8.1. *The uncoupled system when $a_1 \neq 0$ and $a_2 = 0$*

In this section, we will prove the uncoupled equations SDE(24) has unique stochastic stationary solutions, which are

$$\begin{aligned} dX_t &= f(X_t)dt + (a_1 X_t + b_1) \diamond dL_t^{\alpha}, \\ dY_t &= g(Y_t)dt + b_2 dW_t, \end{aligned} \tag{28}$$

where f, g are continuously differential, satisfy the one-sided dissipative Lipschitz conditions (2). Its solution paths are generally not differentiable. Thus we rewrite

$$\begin{aligned} dX_t &= [f(X_t) + (a_1 X_t + b_1) O_t^{(1)}]dt + (a_1 X_t + b_1) \diamond dO_t^{(1)}, \\ dY_t &= [g(Y_t) + b_2 O_t^{(2)}]dt + b_2 dO_t^{(2)}, \end{aligned} \tag{29}$$

where $O^{(1)}$ and $O^{(2)}$, $t \in \mathbb{R}$, is the stationary solution of

$$dO_t^{(1)} = -O_t^{(1)} dt + dL_t^{\alpha},$$

and

$$dO_t^{(2)} = -O_t^{(2)} dt + dW_t.$$

That is

$$O_t^{(1)} = e^{-t} \int_{-\infty}^{t} e^{u} dL_u^{\alpha}, \quad O_t^{(2)} = e^{-t} \int_{-\infty}^{t} e^{u} dW_u, \quad t \in \mathbb{R}.$$

Then we transform (29) to the pathwise random ordinary differential equation

$$\begin{aligned} \tfrac{dx}{dt} &= F(x, O_t^{(1)}) := e^{(-a_1 O_t^{(1)})} f(e^{(a_1 O_t^{(1)})} x - \tfrac{b_1}{a_1}) + a_1 O_t^{(1)} x, \\ \tfrac{dy}{dt} &= G(y, O_t^{(2)}) := g(y + b_2 O_t^{(2)}) + b_2 O_t^{(2)}. \end{aligned} \tag{30}$$

The vector-field function

$$\tilde{f}(x, z) = e^{-a_1 z} f(e^{a_1 z} x - \frac{b_1}{a_1}),$$

and

$$\tilde{g}(x, z) = g(y + z)$$

in the system (30) satisfies a one-sided Lipschitz condition in its first variable uniformly in the second with the same constant as the original drift coefficient f, g, since we have

$$\langle x_1 - x_2, \tilde{f}(x_1, z) - \tilde{f}(x_2, z) \rangle \leqslant -L \parallel x_1 - x_2 \parallel^2$$

and

$$\langle y_1 - y_2, \tilde{g}(y_1, z) - \tilde{g}(y_2, z)\rangle \leqslant -L \parallel y_1 - y_2 \parallel^2 .$$

we obtain that any of the two solutions of the RODE (30) satisfy pathwise the differential inequality

$$\frac{d}{dt} \parallel x_1(t) - x_2(t) \parallel^2 \leqslant (-2L + 2O_t^{(1)}) \parallel x_1(t) - x_2(t) \parallel^2 \tag{31}$$

and

$$\frac{d}{dt} \parallel y_1(t) - y_2(t) \parallel^2 \leqslant -2L \parallel y_1(t) - y_2(t) \parallel^2, \tag{32}$$

and hence we have

$$\parallel x_1(t) - x_2(t) \parallel^2 \leqslant e^{-2t(L - \frac{1}{t}\int_0^t O_\tau^{(1)} d\tau)} \parallel x_1(0) - x_2(0) \parallel^2$$

and

$$\parallel y_1(t) - y_2(t) \parallel^2 \leqslant e^{-2tL} \parallel y_1(0) - y_2(0) \parallel^2 .$$

Thus it follows by Lemma 3.3 that

$$\lim_{t\to\infty} \parallel x_1(t) - x_2(t) \parallel^2 = 0$$

and

$$\lim_{t\to\infty} \parallel y_1(t) - y_2(t) \parallel^2 = 0,$$

which means all solutions converge pathwise to each other.

In order to see what they converge to, we first observe that the RODEs (30) generates a random dynamical system with $\phi(t, \omega, x_0) := x(t, \omega)$, the solution of the RODEs (30) with (deterministic) initial value x_0 at time $t = 0$.Then we need to show that the RODEs (30) is asymptotically dissipative and has a pullback attractor. Omitting ω for brevity, we have pathwise.

$$\begin{aligned}
\frac{d}{dt} \parallel x \parallel^2 &= 2\langle x, F(x, O_t^{(1)})\rangle \\
&= 2\langle x, e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1}) + a_1 O_t^{(1)} x\rangle \\
&= 2e^{-2a_1 O_t^{(1)}} \langle e^{a_1 O_t^{(1)}} x, f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1}) - f(-\frac{b_1}{a_1})\rangle \\
&+ 2\langle x, e^{-a_1 O_t^{(1)}} f(-\frac{b_1}{a_1})\rangle + 2a_1 \parallel x \parallel^2 O_t^{(1)}\rangle \\
&\leqslant (-L + 2a_1 O_t^{(1)}) \parallel x \parallel^2 + \frac{1}{L} e^{-2a_1 O_t^{(1)}} \parallel f(-\frac{b_1}{a_1}) \parallel^2 .
\end{aligned} \tag{33}$$

Integration yields

$$\begin{aligned}
\parallel x(t) \parallel^2 &\leqslant \parallel x(t_0) \parallel^2 e^{-L(t-t_0)+2\int_{t_0}^t a_1 O_\tau^{(1)} d\tau} \\
&+ \frac{\parallel f(-\frac{b_1}{a_1}) \parallel^2}{L} \int_{t_0}^t e^{-2a_1 O_u^{(1)}} e^{-L(t-u)} e^{2\int_{t_0}^t a_1 O_\tau^{(1)} d\tau} du.
\end{aligned}$$

Moreover, by Lemma 3.3 we have pathwise

$$\lim_{s\to-\infty} \frac{1}{s} \int_s^0 O_\tau^{(1)} d\tau = \lim_{t\to\infty} \frac{1}{t} \int_0^t O_\tau^{(1)} d\tau = 0.$$

Thus we obtain

$$e^{2\int_s^t O_\tau^{(1)} d\tau} \leqslant e^{\frac{L}{2}(t-s)}$$

for $s \leqslant 0$, $t \geqslant 0$ with $| t |, | t_0 | > T_\omega$.

Now we can use pathwise pullback convergence (i.e. with $t_0 \to -\infty$) to show that the closed ball centered at the origin with random radius.

$$R^2(\omega) := 1 + \frac{\| f(-\frac{b_1}{a_1}) \|^2}{L} \int_{-\infty}^{0} e^{-2a_1 O_u^{(1)}} e^{Lu} e^{2\int_u^0 a_1 O_\tau^{(1)} d\tau} du.$$

and

$$\begin{aligned} \frac{d}{dt} \| y \|^2 &= 2\langle y, G(y, O_t^{(2)})\rangle \\ &= 2\langle y, g(y + b_2 O_t^{(2)}) + b_2 O_t^{(2)}\rangle \\ &\leqslant -L \| y \|^2 + \frac{1}{L} \| g(b_2 O_t^{(2)}) + b_2 O_t^{(2)}) \|^2 . \end{aligned} \tag{34}$$

Integration yields

$$\| y(t) \|^2 \leqslant \| y(t_0) \|^2 e^{-L(t-t_0)} + \frac{1}{L} \int_{t_0}^{t} e^{-L(t-u)} \| g(b_2 O_u^{(2)}) + b_2 O_u^{(2)} \|^2 du.$$

Now we can use pathwise pullback can vergence (i.e. with $t_0 \to -\infty$) to show that the closed ball centered at the origin with random radius

$$R^2(\omega) := 1 + \frac{1}{L} \int_{\infty}^{0} e^{-L(t-u)} \| g(b_2 O_u^{(2)}) + b_2 O_u^{(2)} \|^2 du,$$

is a pullback absorbing set for $t > T_\omega$. The Theorem 3.1 of RDS then gives us a random attractor $\{A(\omega), \omega \in \Omega\}$. The fact that all trajectories converge to each other forwards in time says. The sets in this random attractor are singleton sets, i.e. $A(\omega) = \{a(\omega)\}$. When we transform back to the SDEs have the pathwise singleton set attractor $a(\theta_{t\omega})$, which is a stationary solution the SDEs, since the Ornstein-Uhlenbeck process is stationary.

8.2. Asymptotic behaviour of coupled synchronized system

Now, we will show that the stationary solution of coupled synchronized system converge when the parameter ν is large enough. Now we consider the coupled RODEs system

$$\begin{aligned} \frac{dx}{dt} &= F(x, O_t^{(1)}(\omega)) + \nu(y - x), \\ \frac{dy}{dt} &= G(y, O_t^{(2)}(\omega)) + \nu(x - y), \end{aligned}$$

with

$$F(x, O_t^{(1)}(\omega)) = e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1}) + \nu(y - x) + a_1 O_t^{(1)} x,$$

$$G(y, O_t^{(2)}(\omega)) = g(y + b_2 O_t^{(2)}(\omega)) + \nu(x - y) + b_2 O_t^{(2)}.$$

Using the one-sided Lipschitz conditions on f and g, we obtain similarly to (31) and(32) that

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 \leqslant [-2L - \nu + 2a_1 O_t^{(1)}]\|x_1(t) - x_2(t)\|^2 + \nu\|y_1(t) - y_2(t)\|^2,$$

and

$$\frac{d}{dt}\|y_1(t)-y_2(t)\|^2 \leqslant [-2L-\nu]\|y_1(t)-y_2(t)\|^2+\nu\|x_1(t)-x_2(t)\|^2.$$

and similarly to (33), (34) we obtain

$$\frac{d}{dt}\|x(t)\|^2 \leqslant (-2L-\nu+2a_1O_t^{(1)})\|x(t)\|^2+\nu\|y(t)\|^2+\frac{1}{L}e^{(-2a_1O_t^{(1)})}\|f(-\frac{b_1}{a_1})\|^2,$$

and

$$\frac{d}{dt}\parallel y(t)\parallel^2 \leqslant (-L-v)\parallel y(t)\parallel^2+\nu\parallel x(t)\parallel^2+\frac{1}{L}(\parallel g(b_2)\parallel^2+|b_2O_t^{(2)}|^2)$$

Defining

$$A_\nu(t)=\begin{pmatrix} -2L-\nu+2a_1O_t^{(1)} & \nu \\ \nu & -2L-\nu \end{pmatrix},\quad t\in\mathbb{R}$$

and

$$\mathbf{x}(t)=\begin{pmatrix} \|x_1(t)-x_2(t)\|^2 \\ \|y_1(t)-y_2(t)\|^2 \end{pmatrix},\quad t\in\mathbb{R}.$$

Thus we can write the above inequalities as

$$\frac{d}{dt}\mathbf{x}\leqslant A_\nu(t)\mathbf{x}$$

Due to a Gronwall-like inequality, we have

$$\mathbf{x}(x)\leqslant e^{\int_{t_0}^t A_\nu(s)ds}\mathbf{x}(t_0).$$

Let

$$\tilde{A}_\nu(t)=\begin{pmatrix} -L-\nu+2a_1O_t^{(1)} & \nu \\ \nu & -L-\nu \end{pmatrix},\quad t\in\mathbb{R},$$

$$\mathbf{x}(t)=\begin{pmatrix} \|x(t)\|^2 \\ \|y(t)\|^2 \end{pmatrix},\quad t\in\mathbb{R}$$

and

$$H(t)=\frac{1}{L}\begin{pmatrix} \frac{1}{L}e^{(-2O_t^{(1)})}\|f(-\frac{b_1}{a_1})\|^2 \\ \|g(b_2)\|^2+|b_2O_t^{(2)}|^2 \end{pmatrix},$$

we can write the above inequalities as

$$\frac{d}{dt}\mathbf{x}\leqslant \tilde{A}_\nu(t)\mathbf{x}+H(t).$$

Due to a Gronwall-like inequality, we have

$$\mathbf{x}(x)\leqslant e^{\int_{t_0}^t \tilde{A}_\nu(s)ds}\mathbf{x}(t_0)+\int_{t_0}^t e^{\int_s^t \tilde{A}_\nu(\tau)d\tau}H(s)ds$$

component wise. Now, we need the following simple lemma.

Lemma 8.1. *We have*

$$\|e^{\int_0^t A_\nu(\tau)d\tau}\boldsymbol{x}\| \leqslant e^{-Lt}\|\boldsymbol{x}\|, \quad \boldsymbol{x} \in \mathbb{R}^2$$

for $t \geqslant T_w$ and all $\nu \geqslant 1$.

Proof. *First note that the matrix $\int_0^t A_\nu(\tau)d\tau$ is symmetric. Thus, the exists of a orthonormal basis of eigenvectors $u_{\nu,t}^{(1)}, u_{\nu,t}^{(2)}$ with eigenvalues $\lambda_{\nu,t}^{(1)}, \lambda_{\nu,t}^{(2)}$, and we have*

$$e^{\int_0^t A_\nu(\tau)d\tau}\boldsymbol{x} = e^{\lambda_{\nu,t}^{(1)}}c_{\boldsymbol{x},\nu,t}^{(1)}u_{\nu,t}^{(1)} + e^{\lambda_{\nu,t}^{(2)}}c_{\boldsymbol{x},\nu,t}^{(2)}u_{\nu,t}^{(2)},$$

where

$$c_{\boldsymbol{x},\nu,t}^{(1)}u_{\nu,t}^{(1)} + c_{\boldsymbol{x},\nu,t}^{(2)}u_{\nu,t}^{(2)} = \boldsymbol{x}.$$

Since $u_{\nu,t}^{(1)}$ and $u_{\nu,t}^{(2)}$ are orthogonal, we obtain

$$\begin{aligned} \|e^{\int_0^t A_\nu(\tau)d\tau}\boldsymbol{x}\|^2 &= e^{2\lambda_{\nu,t}^{(1)}}\|c_{\boldsymbol{x},\nu,t}^{(1)}u_{\nu,t}^{(1)}\|^2 + e^{2\lambda_{\nu,t}^{(2)}}\|c_{\boldsymbol{x},\nu,t}^{(2)}u_{\nu,t}^{(2)}\|^2 \\ &\leqslant e^{2max\{\lambda_{\nu,t}^{(1)}+\lambda_{\nu,t}^{(2)}\}}\|\boldsymbol{x}\|^2. \end{aligned} \tag{35}$$

The eigenvalues of $\int_0^t A_\nu(\tau)d\tau$ are given by

$$\lambda_{\nu,t}^{(1/2)} = -(2L+\nu)t + \int_0^t a_1 O^{(1)}d\tau \pm \sqrt{(\int_0^t a_1 O_\tau^{(1)}d\tau)^2 + \nu^2 t^2},$$

hence it follows by Lemma 3.3 that

$$\lambda_{\nu,t}^{(1/2)} \leqslant -Lt \tag{36}$$

for $|t| > T_w$ and all $\nu \geqslant 1$. □

Analogously to Lemma 8.1 we can show

Lemma 8.2. *Let $t_0 \leqslant 0$ and $t \geqslant 0$. We have*

$$\|e^{\int_{t_0}^t \tilde{A}_\nu(\tau)d\tau}\boldsymbol{x}\| \leqslant e^{-\frac{L}{2}(t-t_0)}\|\boldsymbol{x}\|, \quad \boldsymbol{x} \in \mathbb{R}^2,$$

for $|t_0|$, $|t| \geqslant T_w$ and all $\nu \geqslant 1$.

Now set

$$C_\nu(w) := \frac{1}{L}\int_{-\infty}^0 e^{\int_u^0 \tilde{A}_\nu(\tau)d\tau}\begin{pmatrix} e^{(-2O_t^{(1)})}\|f(\frac{-b_1}{a_1})\|^2 \\ \|g(b_2 O_t^{(2)})\|^2 + |b_2 O_t^{(2)}|^2 \end{pmatrix}du$$

and define

$$R_\nu^2(w) = 1 + \|C_\nu(w)\|^2.$$

Then by pullback techniques and Lemma 8.2, we see that the random balls $B_\nu(w)$ in $\mathbb{R}^{2d}$ centered on the origin and with radius $R_\nu(w)$ are pullback absorbing. Moreover note that

$$\frac{d}{d\nu}\|C_\nu(w)\|^2 = 2\left\langle \tfrac{d}{d\nu}C_\nu(w), C_\nu(w) \right\rangle = 2\left\langle \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} C_\nu(w), C_\nu(w) \right\rangle \leqslant 0$$

and consequently $R_\nu(w) \leqslant R_1(w)$ for $\nu \geqslant 1$. Hence the random dynamical system generated by the coupled RODE (24) has a random attractor $A_\nu(w)$ in $B_\nu(w)$ for each w. But we know that all solutions converge to each other pathwise forwards in time. Thus the $A_\nu(w)$ are singleton sets, say $A_\nu(w) = (\bar{x}_\nu(w), \bar{y}_\nu(w))$.

Let us now estimate the difference of the components of the coupled system. We have pathwise

$$\begin{aligned}
\frac{d}{dt}|x-y|^2 &= 2\langle x-y, \frac{dx}{dt} - \frac{dy}{dt}\rangle \\
&= 2\langle x-y, e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} x - \frac{b_1-1)}{a_1}) - g(y + b_2 O_t^{(2)})\rangle \\
&\quad +2\langle x-y, a_1 x O_t^{(1)}\rangle + 2\langle x-y, -b_2 O_t^{(2)}\rangle + 2\langle x-y, 2\nu(y-x)\rangle \\
&\leqslant -4\nu\|x-y\|^2 + 2\|x-y\|(e^{-a_1 O_t^{(1)}}\|f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1})\| + \|g(y+b_2 O_t^{(2)})\| \\
&\quad +\|a_1 x O_t^{(1)}\| + \|b_2 O_t^{(2)}\|) \\
&\leqslant -\nu\|x-y\|^2 + \frac{1}{\nu} e^{-\alpha_1 O_t^{(1)}}\|f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1})\|^2 + \frac{1}{\nu}\|g(y+b_2 O_t^{(2)})\|^2 \\
&\quad +\frac{1}{\nu}|b_2 O_t^{(2)}|^2 + \frac{1}{\nu}|a_1 O_t^{(1)}|^2\|x\|^2
\end{aligned}$$

Hence labelling the solutions now with ν to indicate this dependence, we have

$$\frac{d}{dt}\|x_\nu - y_\nu\|^2 \leqslant -\nu\|x_\nu - y_\nu\|^2 + \frac{1}{\nu} M^\nu_{T_1,T_2,w}$$

with

$$\begin{aligned}
M^\nu_{T_1,T_2,w} &= \sup_{t\in[T_1,T_2]} (e^{-a_1 O_t^{(1)}}\|f(e^{a_1 O_t^{(1)}} x - \frac{b_1}{a_1})\|^2 + |a_1 O_t^{(1)}|^2\|x\|^2) \\
&\quad + \sup_{t\in[T_1,T_2]} \|g(y+b_2 O_t^{(2)})\|^2 + \sup_{t\in[T_1,T_2]} (|b_2 O_t^{(2)}|^2).
\end{aligned}$$

We can restrict ourselves without loss of generality to solutions in the compact absorbing balls $B_\nu(w)$, which are all contained in the common compact ball $B_1(w)$ for $\nu \geqslant 1$. Hence $M^\nu_{T_1,T_2,w}$ is uniformly bounded in ν and we have

$$\frac{d}{dt}\|x_\nu - y_\nu\|^2 \leqslant -\nu\|x_\nu - y_\nu\|^2 + \frac{1}{\nu} M^\nu_{T_1,T_2,w}$$

with

$$M_{T_1,T_2,w} = \sup_{\nu\geqslant 1} M^\nu_{T_1,T_2,w},$$

from which we conclude that

$$\|x_\nu(t) - y_\nu(t)\|^2 \to 0, \quad \nu \to \infty,$$

uniformly in $t \in [T_1, T_2]$ for any bounded T_1 and T_2.

8.3. The synchronized solution as $\nu \to \infty$

Now we can prove the solution of "averaged" RODEs is the attracting stationary solution.

Theorem 8.3. *$(\bar{x}_{\nu_n}(t,\omega), \bar{y}_{\nu_n}(t,\omega)) \to (\bar{z}(t,\omega), \bar{z}(t,\omega))$ pathwise uniformly on bounded time intervals $[T_1, T_2]$ of $\mathbb{R}$ for any sequence $\nu_n \to \infty$, where $\bar{z}_t(\omega)$ is the attracting stationary solution of the "averaged" RODEs*

$$\frac{dz}{dt} = \frac{1}{2}[e^{-a_1 O_t^{(1)}} f(e^{a_1 O_t^{(1)}} z - \frac{b_1}{a_1}) + g(z + b_2 O_t^{(2)}) + a_1 O_t^{(1)} z + b_2 O_t^{(2)}]. \tag{37}$$

The equivalent SDEs is given by

$$\begin{aligned}
dZ_t = &\ \frac{1}{2}[f(Z_t - \frac{1}{2} b_2 O_t^{(2)} e^{a_1 O_t^{(1)}}) + e^{a_1 O_t^{(1)}} g(e^{-\frac{1}{2} a_1 O_t^{(1)}}(Z_t + \frac{b_1}{a_1}) + \frac{1}{2} b_2 O_t^{(2)})]dt \\
&+\frac{1}{2}[(a_1 Z_{t-} + b_1) \diamond dL_t^{(\alpha)} + b_2 e^{a_1 O_t^{(1)}} dW_t].
\end{aligned} \tag{38}$$

Proof. Define

$$\bar{z}_\nu(\omega) := \frac{1}{2}(\bar{x}_\nu(\omega) + \bar{y}_\nu(\omega))$$

and obseve that $\bar{z}_\nu(t,\omega) = \bar{z}_\nu(\theta_{t\omega})$ satisfies the RODEs

$$\frac{d\bar{z}}{dt} = \frac{1}{2}[e^{(-a_1 O_t^{(1)})} f(e^{(a_1 O_t^{(1)})}\bar{z} - \frac{b_1}{a_1}) + g(\bar{z}_t + b_2 O_t^{(2)}) + a_1 O_t^{(1)}\bar{z} + b_2 O_t^{(2)}].$$

Thus

$$\sup_{t\in[T_1,T_2]} |\frac{d}{dt}\bar{z}_\nu(t,\omega)| \leqslant M_{T_1,T_2,\omega} < \infty,$$

by continuity and the fact that these solutions belong to the common compact ball $B_1(\omega)$. We can use the Ascoli theorem to conclude that there is a subsequence $\nu_{n_j} \to \infty$ such that $\bar{z}_{n_j}(t,\omega) \to \bar{z}(t,\omega)$ as $n_j \to \infty$. Now

$$\begin{aligned}
\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega) &= \frac{1}{2}(\bar{x}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega)) \to 0,\\
\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega) &= \frac{1}{2}(\bar{y}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega)) \to 0,
\end{aligned}$$

as $\nu_{n_j} \to \infty$, see the previous section, so

$$\begin{aligned}
\bar{x}_{\nu_{n_j}}(t,\omega) &= 2\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{y}_{n_j}(t,\omega) \to \bar{z}(t,\omega),\\
\bar{y}_{\nu_{n_j}}(t,\omega) &= 2\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{x}_{n_j}(t,\omega) \to \bar{z}(t,\omega),
\end{aligned}$$

as $\bar{\nu}_{n_j} \to \infty$. Moreover, using the integral equation representation

$$\begin{aligned}
\bar{z}_\nu(t,\omega) &= \bar{z}(T_1,\omega) + \frac{1}{2}\int_{T_1}^t e^{-a_1 O_s^{(1)}(\omega)} f(e^{a_1 O_s^{(1)}}\bar{x}_\nu(s,\omega) - \frac{b_1}{a_1})ds + g(\bar{y}_\nu(s,\omega) + b_2 O_t^{(2)})ds\\
&\quad + \frac{1}{2}\int_{T_1}^t a_1\bar{x}_\nu(s,\omega)O_s^{(1)}ds + \frac{1}{2}\int_{T_1}^t b_2 O_s^{(2)}ds.
\end{aligned}$$

It follows that the ν_{n_j} subsequence converges pathwise to

$$\begin{aligned}
\bar{z}_\nu(t,\omega) &= \bar{z}(T_1,\omega) + \frac{1}{2}\int_{T_1}^t e^{-a_1 O_s^{(1)}(\omega)} f(e^{a_1 O_s^{(1)}}\bar{z}_\nu(s,\omega) - \frac{b_1}{a_1})ds + g(\bar{z}_\nu(s,\omega) + b_2 O_t^{(2)})ds\\
&\quad + \frac{1}{2}\int_{T_1}^t a_1\bar{z}_\nu(s,\omega)O_s^{(1)}ds + \frac{1}{2}\int_{T_1}^t b_2 O_s^{(2)}ds.
\end{aligned}$$

on the interval $[T_1, T_2]$, so $\bar{z}(t,\omega)$ is a solution of the RODEs(37) for all $t \in \mathbb{R}$. By the same techniques as in the previous sections, it has a random attractor consisting of a singleton set formed by a single stationary stochastic process which thus must be equal to $\bar{z}(t,\omega)$. Finally, we note that pathwise all possible subsequences here have the same limit, so by Lemma 3.2, every full sequence $\bar{z}_\nu(t,\omega)$ actually converges to $\bar{z}(t,\omega)$ for the whole sequence $\nu_n \to \infty$. □

Corollary 8.4. *$(\bar{x}_\nu(t,\omega), \bar{y}_\nu(t,\omega)) \to (\bar{z}(t,\omega), \bar{z}(t,\omega))$ as $\nu \to \infty$ pathwise on any bounded time interval $[T_1, T_2]$ of $\mathbb{R}$.*

8.4. Example and simulation of synchronization

Example 8.5. *Now, we will consider two stochastic differential equations*

$$dX_t = -X_t dt + (0.95X_t + 0.05) \diamond dL_t^{0.75}$$

and

$$dY_t = -2Y_t dt + 0.5dW_t.$$

The corresponding RODE are

$$\frac{dx}{dt} = -(x - \frac{0.05}{0.95}e^{-0.95O_t^{(1)}}) + 0.95xO_t^{(1)}$$

and

$$\frac{dy}{dt} = -2y - 0.5O_t^{(2)},$$

where

$$O_t^{(1)} = e^{-t}\int_{-\infty}^{t} e^u dL_u^{0.75}, \quad O_t^{(2)} = e^{-t}\int_{-\infty}^{t} e^u dW_u.$$

The averaged RODE is

$$\frac{dz}{dt} = z(-\frac{3}{2} + \frac{1}{2}0.95O_t^{(1)}) - \frac{1}{2}0.5O_t^{(2)} + \frac{1}{2}(\frac{0.05}{0.95}e^{-0.95O_t^{(1)}})$$

and the equivalent stochastic differential equation is

$$\begin{aligned} dZ_t = \quad & -\tfrac{1}{2}(Z_t + \tfrac{0.05}{0.95} - \tfrac{1}{2}e^{\eta_t^{(1)}}0.5\eta_t^{(2)})dt + \tfrac{1}{2}(\tfrac{0.05}{0.95}e^{-\eta_t^{(1)}}) + \tfrac{1}{2}(0.95Z_t + 0.05 - \tfrac{1}{2}0.24\eta_t^{(2)}e^{-\eta_t^{(1)}}) \diamond dL_t^{0.75} \\ & + \tfrac{1}{2}e^{\eta_t^{(1)}}0.5dW_t \end{aligned}$$

with the explicit solution

$$z(t) = e^{-\frac{3}{2}(t-t_0)+\frac{1}{2}\int_{t_0}^{t}0.95O_\tau^{(1)}d\tau}z_0 + \frac{1}{2}\int_{t_0}^{t} e^{-\frac{3}{2}(t-t_0)+\frac{1}{2}\int_{t_0}^{t}0.95O_\tau^{(1)}d\tau}(-0.5O_u^{(2)} + \frac{0.05}{0.95}e^{-0.95O_u^{(1)}})du.$$

The pullback limit as $t_0 \to -\infty$ gives a stationary solution

$$\tilde{z}(t) = \frac{1}{2}\int_{-\infty}^{t} e^{-\frac{3}{2}(t-t_0)+\frac{1}{2}\int_{-\infty}^{t}(0.95O_\tau^{(1)}d\tau}(-0.5O_u^{(2)} + \frac{0.05}{0.95}e^{-0.95O_u^{(1)}})du.$$

and attracts all other solutions pathwise.
Figure 2 shows the trajectories of the numerical solution of the system with different values of ν. It shows that as ν increases the trajectories approach to each other faster.

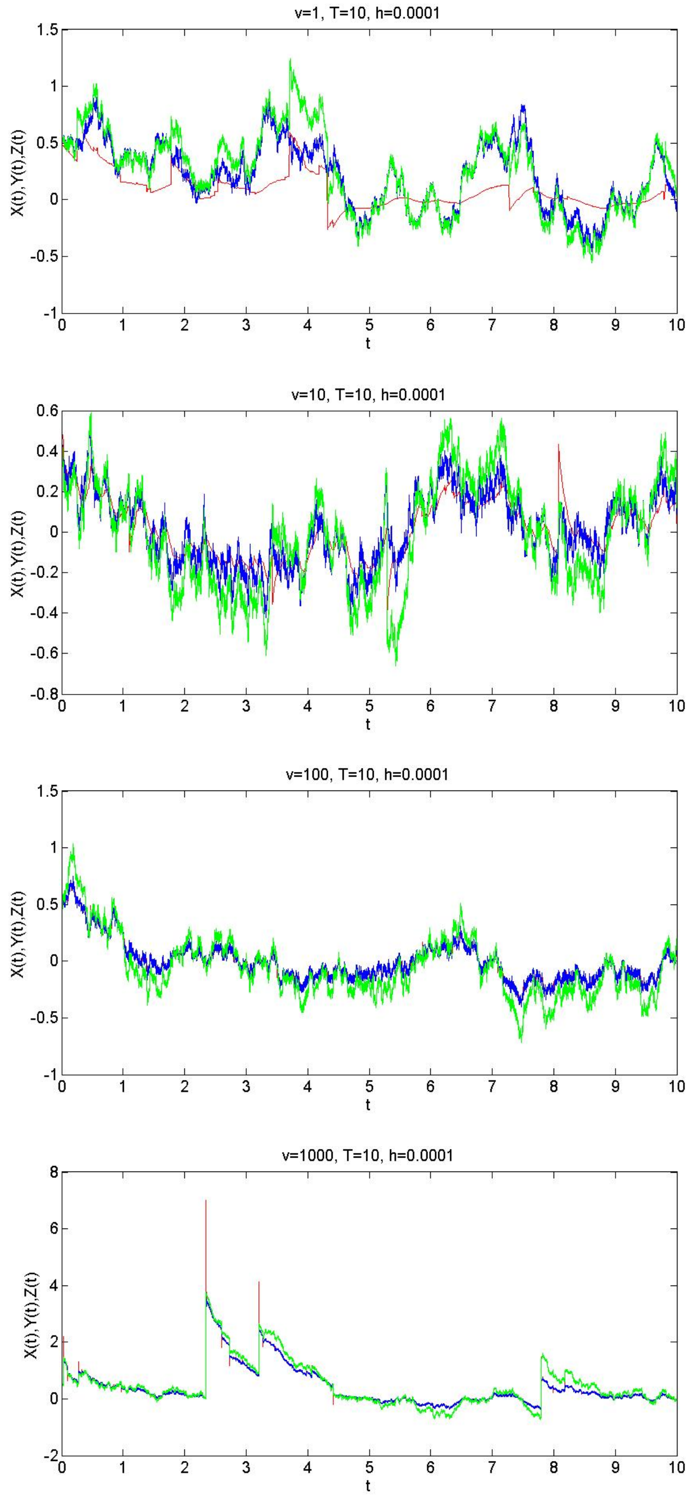


Figure 2: A trajectories of the coupled system $dX_t = -X_t dt + \nu(Y_t - X_t)dt + (0.95X_t + 0.05) \diamond dL_t^{(0.75)}$, $dY_t = -2Y_t dt + \nu(X_t - Y_t)dt + 0.5dW_t$ and the corresponding trajectories of the averaged system $dZ_t = -3/2(Z_t) + (0.95Z_t + 0.05) \diamond dL_t^{(0.75)} + 0.5dW_t$ for four values of ν

Acknowledgements

The authors acknowledge the support provided by NSFs of China (No.11271013, 11471340, 10901065) and the Fundamental Research Funds for the Central Universities, HUST: 2016YXMS003, 2014TS066.

References

[1] V.Afraimovich and H.Rodrigues, Uniform dissipativeness and synchronization of nonautonomous equations. In international Conference on Differential Equations (Lisboa 1995), Word Scientific Publishling, River Edge, NJ.,1998,3-17.
[2] S. AL-azzawi, J. Liu and X. Liu, Convergence rate of synchronization of systems with additive noise, Discrete and cantinuous dynamical systems, Series B.22(2017),227-245.
[3] D. Applebaum, Lévy Processes and Stochastic Calculus, Cambridge University Press, Cambridge, UK, 2004.
[4] L. Arnold, Random Dynamical Systems, Springer-Verlag, Heidel berg, 1998.
[5] P. Billingsley, Convergence of Probability Measure, Wiley, New York, 1968.
[6] H. Crauel, A. Debussche and F. Flandoli, Random atrractor, J. Dynam.Diff.Eq.9(1997),307-341.
[7] T. Caraballo and P. E. Kloeden, The persistence of synchronization under environmental noise, Proc. Roy. Soc. Lond. A.461(2005),2257-2267.
[8] T. Caraballo, P. E. Kloeden and A. Neuenkirch, Synchronization of systems with multiplicative noise, Stoch. Dyn,2008,139-154 .
[9] T. Caraballo, P. E. Kloeden and B.SchmalfuB, Exponentially stable stationary solutions for stochastic evolution equations and their pertubation, Appl. Math. Optim.50(2004),183-207.
[10] A.Carvalho, H.Rodrigues and T. Dlotko, Upper semicontinuity of attractors and synchronization. J. Math. Anal. Appl.220(1998),13-41.
[11] M. Errami, F. Russo and P. Vallois, Itô formula for $C^{1,\lambda}$ functions of a *càdlàg* process and related calculus, Probab. Theory Related Fields. 122(2002),191-221.
[12] T. Fujiwara and H. Kunita, Canonical SDEs based on semimartingales with spatial parameters. Part I.Stochastic flows of diffeomorphisms, Kyushuj. Math.53(1999),265-300.
[13] P. Imkeller and B. SchmalfuB, The conjugacy of stochastic and random differential equations and the existence of global attractors, J. Dyn. Diff. Eqns.13(2001),215-249.
[14] A. Janicki and A. Weron, Simulation and chaotic behavior of α-stable stochastic processes. New York: Marcel Dekker, 1994.
[15] P. E. Kloeden, Synchronization of nonautonomous dynamical systems, Elect. J. Diff. Eqns.39(2003),1-10.
[16] P. E. Kloeden, Nonautonomous attractors of switching systems, Dyn. Syst.21(2006),209-230.
[17] P. E. Kloeden and R. Pavani, Dissipative synchronization of nonautonomous and random system, CAMM-Mitt.32(1)(2009),80-92.
[18] P. E.Kloeden, C.Potzsche and M.Rasmussen. Discrete-time nonautonomous dynamical system. Stability and bifurcation theory for non-autonomous differential equations. Springe Berlin Heidelberg,2013,35-102.
[19] H. Kunita, Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms, in: Real and Stochastic Analysis,in: Trends Math, Birkh auser, Boston,2004,305-373.
[20] T. G. Kurtz, E. Pardoux and P. Protter, Stratonovich stochastic differential equations driven by general semimartingales, Ann. Inst H. Poincaré probab. statist.23(1995),351-377.
[21] V. Lakshmikantham and S. Leela, Differential and Integral Inequalities, Vol.1, Acadamic press, New York, 1969.
[22] X. Liu, J. Duan, J. Liu and P. E. Kloeden, Synchronization of dissipative dynamical systems driven by non-Gaussian Lévy noise, Int. J. Stoch. Anal.Art.ID 502803,2010.
[23] X. Liu, J. Duan, J. Liu and P. E. Kloeden, Synchronization of systems of Marcus canonical equations driven by α-stable noises,Nonlinear Analysis: Real World Applications.11(2011),3437-3445.
[24] S. Marcus, Modelling and approximation of stochastic differential equations driven by semimaringales, Stochastics.4(1981),223-245.
[25] I. Nourdin and T. Simon, On the absolute continuity of Lévy Processes with drift, Ann. Probab.34(3)(2006),1035-1051.
[26] B. Oksendal, Stochastic Differential Equations, sixth ed. Springer-Verlag, New York, 2003.
[27] S. Peszat and J.Zabczyk, Stochastic Partial Differential Equations with Lévy Processes, Cambridge University Press, Cambridge, UK, 2007.
[28] J. Robinson, Infinite-dimensional Dynamical Systems, Cambridge University Press, Cambridege, UK, 2001.
[29] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.
[30] G. Samorodnitsky and M. Taqqu, Stable Non-Gaussian Random Processes, Chapman & Hall, New York, 1994.
[31] D. Schertzer, M. Larcheveque, J. Duan, V.Yanovsky and S.Lovejoy, Fractional Fokker-Planck equation for nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises, J.Math. Phys.42(2000),200-212.
[32] R. Situ, Theory of Stochastic Differential Equations with Jumps and Applications, Springer, 2005.
[33] S. Strogatz and Sync, The emerging science of spontaneous order, Hyperion Press, New York, 2003.