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On properties of the operator equation $TT^* = T + T^*$

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Abstract. In this paper, we study properties of the operator equation $TT^* = T+T^*$ which T.T. West observed in [12]. We first investigate the structure of solutions $T \in B(\mathcal{H})$ of such equation. Moreover, we prove that if T is a polynomial root of solutions of that operator equation, then the spectral mapping theorem holds for Weyl and essential approximate point spectra of T and $f(T)$ satisfies a-Weyl's theorem for $f \in H(\sigma(T))$, where $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $B(\mathcal{H})$, $B_0(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators, the ideal of compact operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$, we shall denote $\sigma(T)$, $\sigma_a(T)$, and $\sigma_p(T)$ for the spectrum, approximate point spectrum, and point spectrum of T, respectively. For $T \in B(\mathcal{H})$, the smallest nonnegative integer p such that $ker(T^p) = ker(T^{p+1})$ is called the *ascent* of T. If no such integer exists, we say that T has the infinite ascent. The smallest nonnegative integer q such that $ran(T^q) = ran(T^{q+1})$ is called the *descent* of T. If no such integer exists, we say that T has the infinite descent.

Let $T \in B(\mathcal{H})$ be a solution of the operator equation

$$TT^* = T + T^*. \tag{1}$$

In [12], T. T. West observed that if $T \in B(\mathcal{H})$ is compact and normal, then $\sigma(T)$ is contained in the unit circle with centre at 1 if and only if the operator equation (1) holds. From this paper we are interested in what spectral properties hold for nonnormal solutions of the operator equation (1). In particular, we prove that $T \in B(\mathcal{H})$ is a polynomial root of solutions of the operator equation (1), then the spectral mapping theorem holds for Weyl and essential approximate point specta of T, and from this, $f(T)$ satisfies a-Weyl's theorem for $f \in H(\sigma(T))$, where $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

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2. Preliminaries

An operator $T \in B(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional kernel and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in B(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in B(\mathcal{H})$ is defined by

$$ind(T) := dimker(T) - dimker(T^*).$$

If the dimensions of $ker(T)$ and $ker(T^*)$ are finite, then T is called *Fredholm*. $T \in B(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero, and T is called *Browder* if it is Fredholm with finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{H})$ are defined as follows.

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.
By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in B_0(\mathcal{H})\}$$

is the *essential approximate point spectrum*,

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in B_0(\mathcal{H})\}$$

is the *Browder essential approximate point spectrum*.
If we write $isoK = K \setminus accK$, then we let

$$\pi_{00}(T) := \{\lambda \in iso\sigma(T) : 0 < dimker(T - \lambda) < \infty \},$$

$$\pi^a_{00}(T) := \{\lambda \in iso\sigma_a(T) : 0 < dimker(T - \lambda) < \infty \},$$

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T),$$

and

$$p^a_{00}(T) := \sigma_a(T) \setminus \sigma_{ab}(T).$$

We say that *Weyl's theorem holds for* $T \in B(\mathcal{H})$ if there is equality

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

that *Browder's theorem holds for* $T \in B(\mathcal{H})$ if there is equality

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

that *a-Weyl's theorem holds for* $T \in B(\mathcal{H})$ if there is equality

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi^a_{00}(T),$$

and that *a-Browder's theorem holds for* $T \in B(\mathcal{X})$ if there is equality

$$\sigma_a(T) \setminus \sigma_{ea}(T) = p^a_{00}(T).$$

It is known ([6, 7, 9]) that we have

$$a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem} \Rightarrow \text{Browder's theorem;}$$

$$a\text{-Weyl's theorem} \Rightarrow a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem.}$$

In terms of local spectral theory ([1]) we recall that the *local resolvent set* $\rho_T(x)$ of T at the point $x \in \mathcal{H}$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to \mathcal{H}$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at the point $x \in \mathcal{H}$ is defined as $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspaces of* T by

$$H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\} \text{ for all sets } F \subseteq \mathbb{C}.$$

We say that $T \in B(\mathcal{H})$ has the *single valued extension property at* $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. The operator T is said to have the single valued extension property if T has the single valued extension property at every $\lambda \in \mathbb{C}$.
Evidently, every operator T, as well as its dual T^*, has the single valued extension property at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$, in particular, at every isolated point of $\sigma(T)$. We also have (see [1, Theorem 3.8]) that if $T - \lambda$ has finite ascent, then T has the single valued extension at λ, and dually, if $T - \lambda$ has finite descent, then T^* has the single valued extension property at λ. Moreover, it is well known from [1] that if $T - \lambda$ is semi-Fredholm, then their converses hold.

3. Main results

In this section, we study properties of solutions T satisfying the operator equations $TT^* = T^* + T$. *Throughout this papaer, we denote by $\mathcal{S}(\mathcal{H})$ the collection of all solutions that the operator equation $TT^* = T^* + T$ holds, i.e.,*

$$\mathcal{S}(\mathcal{H}) = \{T \in B(\mathcal{H}) : TT^* = T^* + T\}.$$

We start this section with the following theorem.

Theorem 3.1. An operator T belongs to $\mathcal{S}(\mathcal{H})$ if and only if $T = U^* + I$ where U is an isometry. In particular, if $T \in \mathcal{S}(\mathcal{H})$, the following statements hold.
(i) $T = V \oplus S_+^* + I$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ for some subspace $\mathcal{M}$ of $\mathcal{H}$ that reduces U, where V^* is unitary on $\mathcal{M}$ and S_+ is a unilateral shift on $\mathcal{M}^\perp$.
(ii) $ker(T^* - \lambda) \subseteq ker(T - \mu)$ if $\lambda \neq 1$, where $\mu = \frac{\lambda}{\lambda-1}$ and $ker(T^* - I) = \{0\}$.
(iii) If $\lambda \in \sigma_a(T^*)$ (respectively, $\lambda \in \sigma_p(T^*)$), then $\mu \in \sigma_a(T)$ (respectively, $\mu \in \sigma_p(T)$), where $\mu = \frac{\lambda}{\lambda-1}$. In particular, if $T^* - I$ has dense range, then $\sigma_a(T^*) \subset \sigma_a(T(T - I)^{-1})$ (respectively, $\sigma_p(T^*) \subset \sigma_p(T(T - I)^{-1})$).

Proof. If $T \in \mathcal{S}(\mathcal{H})$, then $TT^* - T - T^* = 0$. So we have that $T(T^* - I) - (T^* - I) = I$ or $(T - I)(T - I)^* = I$. This means that $T^* - I$ is an isometry. Set $U := T^* - I$. Then $T = U^* + I$. Conversely, if $T = U^* + I$, then $TT^* = (U^* + I)(U + I) = T + T^*$, so that $T \in \mathcal{S}(\mathcal{H})$.

(i) Since U is an isometry, in this case, the result follows from [10].

(ii) Suppose that $x \in \ker(T^* - \lambda)$. Then $T^*x = \lambda x$. Since $TT^* = T^* + T$ and $\lambda \neq 1$, we have that $\frac{\lambda}{\lambda-1}T^*x = Tx$. Therefore if $\mu = \frac{\lambda}{\lambda-1}$, then $x \in \ker(T - \mu)$, so that we get that $\ker(T^* - \lambda) \subseteq \ker(T - \mu)$. Since $T^* - I$ is an isometry, it is trivial that $ker(T^* - I) = \{0\}$.

(iii) Suppose that $\lambda \in \sigma_a(T^*)$. Then there exists a sequence $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$ for all positive integer n and $(T^* - \lambda I)x_n \to 0$ as $n \to \infty$. Let $y_n := (T^* - \lambda I)x_n$. Then $T^*x_n = \lambda x_n + y_n$ and $y_n \to 0$. So we have that

$$TT^*x_n = \lambda Tx_n + Ty_n,$$

and

$$Tx_n + \lambda x_n + y_n = \lambda Tx_n + Ty_n.$$

Therefore

$$\{(1-\lambda)T + \lambda I\}x_n = (T - I)y_n,$$

and then it follows from $\lambda \neq 1$ and $\lim_{n\to\infty} y_n = 0$ that $\lim_{n\to\infty}(T - \mu)x_n = 0$ for $\mu = \frac{\lambda}{\lambda-1}$. This means that $\mu \in \sigma_a(T)$. In particular, if $T^* - I$ has dense range, then $T^* - I = U$ is bounded below where U is an isometry, and hence $T^* - I$ is invertible by (i). Thus $1 \notin \sigma(T^*) = \sigma(T)$. If $\lambda \in \sigma_a(T^*)$, then $\lambda \neq 1$ and $\lim_{n\to\infty}(T^* - \lambda I)x_n = 0$. Hence

$$\lim_{n\to\infty}(T + T^* - \lambda T)x_n = \lim_{n\to\infty}(TT^* - \lambda T)x_n = 0.$$

Since $\lim_{n\to\infty}(T^* - \lambda I)x_n = 0$, we have that

$$\lim_{n\to\infty}[T - \lambda(T - I)]x_n = \lim_{n\to\infty}(T + \lambda I - \lambda T)x_n = 0.$$

Since $T - I$ is invertible, we get that $\lim_{n\to\infty}(T(T - I)^{-1} - \lambda)x_n = 0$, so that $\lambda \in \sigma_a(T(T - I)^{-1})$. Thus it follows that $\sigma_a(T^*) \subset \sigma_a(T(T - I)^{-1})$. Similarly, it follows that $\sigma_p(T^*) \subset \sigma_p(T(T - I)^{-1})$. □

We next give some examples for operators in $\mathcal{S}(\mathcal{H})$.

Example 3.2. Let $T = \begin{pmatrix} I & U^* \\ U^* & I \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$, where U is the unilateral shift. Then $T = \begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix}^* + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and it follows from Theorem 3.1 that $TT^* = T + T^*$, that is, $T \in \mathcal{S}(\mathcal{H})$.

Also, we let $T = \begin{pmatrix} U^{*2} + I & 0 \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$, where U is the unilateral shift. Then $T = \begin{pmatrix} U^2 & 0 \\ 0 & -I \end{pmatrix}^* + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, and hence $T \in \mathcal{S}(\mathcal{H})$ from Theorem 3.1. □

Recall that $T \in B(\mathcal{H})$ is said to be *subnormal* if T has a normal extension, that is, there exists a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ can be embedded in $\mathcal{K}$ and there exists a normal operator N of the following form

$$N = \begin{pmatrix} T & S \\ 0 & R \end{pmatrix}$$

for some bounded operators $S : \mathcal{H}^\perp \to \mathcal{H}$ and $R : \mathcal{H}^\perp \to \mathcal{H}^\perp$, and T is called *hyponormal* if $TT^* \leq T^*T$. We now consider the following corollary in terms of the localized single valued extension property for the solution $T \in \mathcal{S}(\mathcal{H})$.

Corollary 3.3. If $T \in \mathcal{S}(\mathcal{H})$, then T^* has the single valued extension property. However, it is not necessary that T has the single valued extension property.

Proof. Since $T \in \mathcal{S}(\mathcal{H})$, it follows from Theorem 3.1 that $T = U^* + I$ where U is an isometry. So $T^* = U + I$. Since U is an isometry, it has the unitary extension, so that it is subnormal, which implies that it is hyponormal. Thus $U + I$ is also hyponormal. Therefore T^* has the single valued extension property. On the other hand, since U^* is a coisometry, it is not necessary that U^* has the single valued extension property from [5]. Hence $T = U^* + I$ does not have the single valued extension property in general. □

Corollary 3.4. If $T \in \mathcal{S}(\mathcal{H})$, then the following statements hold.
(i) $\sigma_{T\oplus T^*}(x_1 \oplus x_2) = \sigma_T(x_1) \cup \sigma_{T^*}(x_2)$.
(ii) $\sigma(T) = \sigma_s(T) = \sigma_a(T) = \sigma_{se}(T)$.
(iii) If Q is quasinilpotent, then $T^* + Q^*$ has the single valued extension property.
(iv) If F is a close subset in $\mathbb{C}$ such that $\mathcal{H}_{T^*}(F)$ is closed, then $\sigma(T^*|\mathcal{H}_{T^*}(F)) \subseteq F \cap \sigma(T^*)$.
(v) If F_1 and F_2 are two closed and disjoint subsets of $\mathbb{C}$, then $\mathcal{H}_{T^*}(F_1 \cup F_2) = \mathcal{H}_{T^*}(F_1) \oplus \mathcal{H}_{T^*}(F_2)$.

Proof. Since $T \in \mathcal{S}(\mathcal{H})$, it follows from Corollary 3.3 that T^* has the single valued extension property. Hence these statements are shown by [1]. In particular, if Q is quasinilpotent then so is Q^*, hence this implies from [1, Corollary 2.12] that (iii) is proved. □

In the following corollary, we consider operators in $\mathcal{S}(\mathcal{H})$ on a finite dimensional space.

Corollary 3.5. If $T \in \mathcal{S}(\mathbb{C}^n)$, then it is normal. In particular, if $n = 2$, then T can be written by

$$T = \begin{pmatrix} a^* + 1 & -be^{-i\theta} \\ b^* & ae^{-i\theta} + 1 \end{pmatrix}, \ |a|^2 + |b|^2 = 1. \tag{2}$$

Proof. Since $T \in \mathcal{S}(\mathbb{C}^n)$. However, since $T = U^* + I$ where U is isometry on $\mathbb{C}^n$. But, U is unitary on finite dimensional spaces, it follows from Theorem 3.1 that T is normal. It is known that the general unitary matrix form on $\mathbb{C}^2$ as follows.

$$U = \begin{pmatrix} a & b \\ -b^*e^{i\theta} & a^*e^{i\theta} \end{pmatrix}, \ |a|^2 + |b|^2 = 1.$$

Thus if $T \in \mathcal{S}(\mathbb{C}^2)$, then it follows from $T = U^* + I$ that (2) is satisfied. □

Remark 3.6. The converse of Corollary 3.5 does not hold. For example, $T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \in B(\mathbb{C}^2)$, then it is a compact and normal operator. However, $\sigma(T) = \{\frac{1}{2}, \frac{1}{3}\} = \sigma(T^*)$ is not contained in the unit circle with centre 1, hence $T \notin \mathcal{S}(\mathbb{C}^2)$ from Theorem 4.2 of T. T. West (see [12]). On the other hand, $T \in \mathcal{S}(\mathbb{C}^n)$ if and only if $\sigma(T)$ lies on the unit circle with centre 1.

The following example shows that $\mathcal{S}(\mathcal{H})$ contains non-hyponormal operators. Moreover, there are no inclusions between the two collections, $\mathcal{S}(\mathcal{H})$ and the set of hyponormal operators.

Example 3.7. Let $T := U^* + I$, where U is the unilateral shift and I is the identity operator in $B(\mathcal{H})$. Since U is a non-unitary isometry, it follows from Theorem 3.1 that $T \in \mathcal{S}(\mathcal{H})$. However, $T^*T - TT^* = UU^* - I < 0$, and hence T is not hyponormal. However, T^* is hyponormal. Furthermore, we can have many examples which T is hyponormal but is not contained in $\mathcal{S}(\mathcal{H})$. For a simple example, if T is the identity operator in $B(\mathcal{H})$, then $T \notin \mathcal{S}(\mathcal{H})$ and $TT^* = T^*T$.

We say that $T \in B(\mathcal{H})$ is *quasinormal* if T commutes with T^*T, that is, $T(T^*T) = (T^*T)T$, 2-*normal* if T is unitarily equivalent to a 2×2 operator matrix whose entries are commuting normal operators, and *binormal* if T^*T commutes TT^*. We next characterize operators in $\mathcal{S}(\mathcal{H})$ which are hyponormal, quasinormal, and binormal.

Theorem 3.8. If $T \in \mathcal{S}(\mathcal{H})$, then the following statements hold.
(i) T is quasinormal if and only if

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \ker|T|^\perp \oplus \ker|T|,$$

where $A = U^2 + I$ and U is unitary defined on $\ker|T|^\perp$.
(ii) T is binormal if and only if T is hyponormal.

Proof. (i) Suppose that $T = U|T| \in \mathcal{S}(\mathcal{H})$. Then $U|T| + |T|U^* = U|T|^2U^*$. If T is quasinormal, then it follows from the Fuglede-Putnam theorem that $U^*|T| = |T|U^*$, and hence we have $U|T| + U^*|T| = UU^*|T|^2$, so that $(U+U^*-UU^*|T|)|T| = 0$. Since T is quasinormal, we have $(U+U^*-UU^*|T|)|T|U = (U+U^*-UU^*|T|)U|T| = 0$. On $\overline{\mathrm{ran}|T|}$, $U^2+U^*U-UU^*U|T| = 0$. Since $U^*U = I$, $U^2+I = T$ on $\overline{\mathrm{ran}|T|}$. Since $(\ker|T|)^{\perp} = (\ker(U))^{\perp} = \overline{\mathrm{ran}|T|}$, U is an isometry on $\overline{\mathrm{ran}|T|}$. Since T is quasinormal, it is hyponormal, so that $T+T^* \leq T^*T$. Since $(T-I)^*(T-I) \geq I$, $T-I$ is bounded below. Moreover, Since $T \in \mathcal{S}(\mathcal{H})$, we have that $(T-I)(T-I)^* = I$. Hence $T-I$ is right invertible. Then $T-I$ is surjective. Hence $T-I = U^*$ is invertible. This means that U is unitary on $\overline{\mathrm{ran}|T|}$. Therefore we can write T as follows,

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \ker|T|^{\perp} \oplus \ker|T|,$$

where $A = U^2 + I$ and U is unitary defined on $\ker|T|^{\perp}$.

(ii) If $T \in \mathcal{S}(\mathcal{H})$, then $T = U^* + I$ by Theorem 3.1, where U is an isometry. Since $(T^*T)(TT^*) = (TT^*)(T^*T)$ and $U^*U = I$, we get that $(U+I)(U^*+I)(U^*+I)(U+I) = (U^*+I)(U+I)(U+I)(U^*+I)$, so that $UU^* = I$. Thus U is a unitary and $T = U^* + I$. This implies that T is normal, so that it is hyponormal.

Conversely, Since $T \in \mathcal{S}(\mathcal{H})$ and hyponormal, it follows from the proof of (i) that $T-I$ is invertible. Moreover, since $T \in \mathcal{S}(\mathcal{H})$, we get from Theorem 3.1 that $T = U^*+I$, where U is an isometry. Hence $T^* = U+I$ is hyponormal. Since T is hyponormal by hypothesis, T is normal. Therefore it follows that T is binormal. □

Corollary 3.9. If $T \in \mathcal{S}(\mathcal{H})$ and is binormal, then the following statements hold.
(i) T has a nontrivial invariant subspace and
(ii) T^n is hyponormal for $n \geq 1$.

Proof. (i) By Theorem 3.8, T is binormal and hyponormal. So it follows from [2, Theorem 2] that T has a nontrivial invariant subspace.

(ii) It is obvious from [3, Theorem 3]. □

Let $A := \begin{pmatrix} I & iI \\ iI & I \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$. Then $AA^* = A^*A = A + A^*$. We now let $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ and define T by

$$T := \frac{1}{2}\begin{pmatrix} A & A^* \\ A & A^* \end{pmatrix} \in B(\mathcal{K} \oplus \mathcal{K}).$$

Then $TT^* = T + T^*$ and T is 2-normal. So there are mutually commuting normal operators N_1, N_2, and N_3 in $B(\mathcal{K})$ such that T is unitary equivalent to an upper triangular operator matrix $S := \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$. Then $S = UTU^*$ for some unitary operator $U \in B(\mathcal{K} \oplus \mathcal{K})$. Since $T \in \mathcal{S}(\mathcal{K} \oplus \mathcal{K})$, we get that $SS^* = S + S^*$. Hence $S \in \mathcal{S}(\mathcal{K} \oplus \mathcal{K})$. Moreover, S has the single valued extension property, so does S^* by Corollary 3.3. Since S^* is also unitary equivalent to T^*, we have that T^* has the single valued extension property.

In general, 2-normal operators do not belong to $\mathcal{S}(\mathcal{H})$. The following proposition state some conditions for which 2-normal operators are included in $\mathcal{S}(\mathcal{H})$.

Proposition 3.10. Suppose that T is a 2-normal operator and is unitary equivalent to the operator matrix $\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$, where N_1, N_2, and N_3 are mutually commuting normal operators. Then the following statements are valid.
(i) If N_1, N_2, and N_3 are in $\mathcal{S}(\mathcal{H})$ and $N_2^* = -N_2$, then $T \in \mathcal{S}(\mathcal{H})$.
(ii) If N_i are compact for $i = 1, 2, 3$, N_1 and N_3 are in $\mathcal{S}(\mathcal{H})$, and $\sigma(N_1) \cap \sigma(N_3)$ has no interior points, then $T \in \mathcal{S}(\mathcal{H})$.

Proof. (i) We note that a skew symmetric operator in $\mathcal{S}(\mathcal{H})$ is only the zero. Since $N_2 = 0$, we have that $\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$ is contained in $\mathcal{S}(\mathcal{H})$. Since T is unitary equivalent to such operator matrix, it follows that $T \in \mathcal{S}(\mathcal{H})$. The second implication is satisfied by the similar method.

(ii) If $\sigma(N_1) \cap \sigma(N_3)$ has no interior points, then it follows from [8] that

$$\sigma(N_1) \cup \sigma(N_2) = \sigma \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}.$$

Since N_1 and N_3 are in $\mathcal{S}(\mathcal{H})$, it follows from [12] that $\sigma \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$ is contained in the unit circle with centre at 1. So this means that $\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix} \in \mathcal{S}(\mathcal{H})$. Therefore $T \in \mathcal{S}(\mathcal{H})$. $\square$

For operators A, B we let $[A, B] = AB - BA$ and $\theta := \{T \in B(\mathcal{H}) : [T^*T, T + T^*] = 0\}$ (see [4]). And we recall that T is *isoloid* (respectively, *a-isoloid*) if every isolated point of $\sigma(T)$ (respectively, $\sigma_a(T)$) is an eigenvalue. Also we say that T is *polaroid* (respectively, *a-polaroid*) if every isolated point of $\sigma(T)$ (respectively, $\sigma_a(T)$) is a pole of the resolvent of T. It is well known that a-polaroid $\Longrightarrow$ polaroid $\Longrightarrow$ isoloid and a-polaroid $\Longrightarrow$ a-isoloid $\Longrightarrow$ isoloid. We say that T is *normaloid* if $\|T\| = r(T)$ for the spectral radius $r(T)$ of $T \in B(\mathcal{H})$ and *convexoid* if the closure of the numerical range $W(T)$ coincides with the convex hull of its spectrum. Then we have the following lemma.

Lemma 3.11. If $T \in \mathcal{S}(\mathcal{H})$, then the following statements hold.
(i) If T is invertible, then both T and T^{-1} are normal.
(ii) T is normaloid.
(iii) If $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.
(iv) $\pi_{00}(T) = \emptyset$ and T is isoloid.
(v) $\sigma(T) = \sigma_w(T)$ and Weyl's theorem holds for T.

Proof. (i) If T is invertible, there exists an inverse S of T such that $TS = ST = I$. Since $T \in \mathcal{S}(\mathcal{H})$, we have that $TT^* = T + T^*$, or $T^* = I + ST^*$. This implies that $(I - S)T^* = I$, so that $I - S = S^*$. Thus $SS^* = S^*S$ and this means that both T and T^{-1} are normal.

(ii) If $T \in \mathcal{S}(\mathcal{H})$, then $T^* \in \theta$ and it follows from [4, Theorem 2] that $\|T^*\| = r(T^*)$, where $r(T)$ denote the spectral radius of T. This means that T^* is normaloid. Hence T is also normaloid.

(iii) If $\lambda = 0$, then T is normaloid from (ii). Since $\|T\| = r(T) = 0$, that is, $T = 0$. Now, we suppose that $\lambda \neq 0$. Since T is invertible, by part (i), T^{-1} is normal. Hence T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$. So $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows from [11, Lemma 3] that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda I$.

(iv) Assume that $\lambda \in \pi_{00}(T)$. Then $\lambda \in iso\sigma(T)$ and $0 < dimker(T) < \infty$. Using the spectral projection $P := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

However, $T_1 \in \mathcal{S}(\mathcal{H})$, and hence $T_1 = \lambda I$ by part (iii). Therefore $T - \lambda I = 0 \oplus (T_2 - \lambda I)$, so $ker(T - \lambda I) = \mathcal{H} \oplus \{0\}$. It follows that $dimker(T - \lambda I) = \infty$. This is a contradiction. This means that $\pi_{00}(T) = \emptyset$. Moreover, if $\lambda \in iso\sigma(T)$, then λ is an eigenvalue of T, and so T is isoloid.

(v) Suppose that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T - \lambda$ is Weyl but not invertible. But since T^* has the single valued extension property, $T - \lambda$ is Browder, so that $\alpha(T - \lambda) < \infty$ and λ is a pole of the resolvent of T. Thus $\lambda \in \mathrm{iso}\sigma(T)$. If $T - \lambda$ is injective, then it is invertible, however, this is contradiction. So $\lambda \in \pi_{00}(T)$. Hence $\sigma(T) \setminus \sigma_w(T) \subseteq \pi_{00}(T)$. However, it follows from (iv) of this lemma that $\sigma(T) = \sigma_w(T)$ and Weyl's theorem holds for T. $\square$

Remark 3.12. In fact, if $T \in \mathcal{S}(\mathcal{H})$, then T^* has the single valued extension property, so that all of Weyl's theorem, Browder's theorem, a-Weyl's theorem, and a-Browder's theorem hold for T by (v) of Lemma 3.11.

Recall that T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$ if $p(T) \in \mathcal{S}(\mathcal{H})$ for some nonconstant polynomial p. In general, if $T \in \mathcal{S}(\mathcal{H})$, then it is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$. Indeed, if $T \in \mathcal{S}(\mathcal{H})$, then there exist nonconstant polynomials $p(z) = z$ and $q(z) = \overline{p(\overline{z})}$ such that $p(T)p(T)^* = p(T) + p(T)^*$ by the functional calculus, so that $q(T) \in \mathcal{S}(\mathcal{H})$. However, the converse does not hold. For example, if $T = \begin{pmatrix} 0 & \sqrt{2}I \\ \sqrt{2}I & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$, then $T^2 = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} \in \mathcal{S}(\mathcal{H} \oplus \mathcal{H})$, but $T \notin \mathcal{S}(\mathcal{H} \oplus \mathcal{H})$. We now have the following corollary for a polynomial root of operators in $\mathcal{S}(\mathcal{H})$.

Corollary 3.13. If T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$ with $\sigma(T) = \{0\}$, then T is nilpotent.

Proof. Suppose that $p(T) \in \mathcal{S}(\mathcal{H})$ for some nonconstant polynomial p. Since $\sigma(p(T)) = p(\sigma(T)) = \{p(0)\}$, we get that $p(T) - p(0)$ is quasinilpotent. It follows from (iii) of Lemma 3.11 that for a nonzero constant c and an integer $m \geq 1$,

$$cT^m(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n) = p(T) - p(0) = 0.$$

Since $T - \lambda_i$ is invertible for every $\lambda_i \neq 0$ for $1 \leq i \leq n$, we have $T^m = 0$. Therefore T is nilpotent. □

Recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$. We next consider the spectral mapping theorem for Weyl spectrum and essential approximate point spectrum of T, when T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$.

Theorem 3.14. If T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$, then we have that for $f \in H(\sigma(T))$,

$$\sigma_w(f(T)) = f(\sigma_w(T)), \tag{3}$$

and Weyl's theorem holds for $f(T)$.

Proof. Let $\lambda \in \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \cdots (T - \alpha_n)g(T), \tag{4}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n$ are complex numbers and $g(T)$ is invertible. Since the operators in the right side of (4) commute, every $T - \alpha_i$ is Fredholm. Since $p(T) \in \mathcal{S}(\mathcal{H})$ for some nonconstant complex polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, we get that $p(T)^*$ has the single valued extension property. Let $q(z) = \overline{p(\overline{z})}$, and $q(T^*) = p(T)^*$. So $q(T^*)$ has the single valued extension property, equivalently, so does T^*. So $ind(T - \alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$. This implies that

$$0 \leq \sum ind(T - \alpha_i) = ind(f(T) - \lambda) = 0,$$

so that $T - \alpha_i$ is Weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_w(T))$ and this means that $f(\sigma_w(T)) \subset \sigma_w(f(T))$. Since the converse inclusion holds with no other restriction on T, it follows that (3) holds. This implies from (iv) and (v) of Lemma 3.11 that

$$\begin{aligned} \sigma(f(T)) \setminus \pi_{00}(f(T)) &= f(\sigma(T) \setminus \pi_{00}(T)) \\ &= f(\sigma_w(T)) = \sigma_w(f(T)). \end{aligned}$$

Therefore $f(T)$ satisfies Weyl's theorem for $f \in H(\sigma(T))$. □

Theorem 3.15. If T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$, then we have that for $f \in H(\sigma(T))$,

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)), \tag{5}$$

and a-Browder's theorem holds for $f(T)$.

Proof. Let $\lambda \in \sigma_{ea}(f(T))$. Then $f(T) - \lambda$ is upper semi-Fredholm and $ind(f(T) - \lambda) \leq 0$. And we have that

$$f(T) - \lambda = c(T - \mu_1)(T - \mu_2) \cdots (T - \mu_n)g(T), \tag{6}$$

where $c, \mu_1, \mu_2, \cdots, \mu_n$ are complex numbers and $g(T)$ is invertible. Then (5) is proved by the similar way to the proof of Theorem 3.14. Now, since T satisfies a-Weyl's theorem from Remark 3.12, we get that $\sigma_{ea}(T) = \sigma_{ab}(T)$. This implies that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)),$$

and hence $f(T)$ obeys a-Browder's theorem. □

From this, the following theorem says that $f(T)$ satisfies a-Weyl's theorem whenever T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$.

Theorem 3.16. If T is a polynomial root of operators in $\mathcal{S}(\mathcal{H})$, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. Suppose that $p(T) \in \mathcal{S}(\mathcal{H})$ for some nonconstant polynomial p. We first claim that a-Weyl's theorem holds for T. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T - \lambda$ is upper semi-Fredholm and $ind(T - \lambda) \leq 0$. We let $q(T^*) = p(T)^*$ for some nonconstant polynomial q. Since $p(T)^*$ has the single valued extension property, $q(T^*)$ also has. It follows from [1] that T^* has the single valued extension property. Since $T - \lambda$ is a semi Fredholm operator, it has finite descent and $ind(T - \lambda) \geq 0$. Since $dimker(T^* - \overline{\lambda}) = dimker(T - \lambda) < \infty$ and $T - \lambda$ has finite descent, it has also finite ascent, so that $\lambda \in iso\sigma_a(T)$. This implies that $\lambda \in \pi_{00}^a(T)$. Conversely, let $\lambda \in \pi_{00}^a(T)$. Since T^* has the single valued extension property, $\lambda \in iso\sigma(T)$. Using the spectral projection $P = \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$ and $p(T_1) \in \mathcal{S}(\mathcal{H})$, we have that $p(T_1) = p(\lambda)$. Thus it follows that for a nonzero constant c and an integer $n \geq 1$,

$$c(T_1 - \lambda)(T_1 - \alpha_1)(T_1 - \alpha_2) \cdots (T_1 - \alpha_n) = p(T_1) - p(\lambda) = 0.$$

But, $T_1 - \alpha_i$ is invertible for $i = 1, 2, \cdots, n$, we get that $T_1 = \lambda I$. Since $T_2 - \lambda$ is invertible, we have that $ker(T - \lambda) = \mathcal{H} \oplus \{0\}$, so that $dimker(T - \lambda) = \infty$. This is a contradiction. Thus $\pi_{00}^a(T) = \emptyset$. Thus $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. So a-Weyl's theorem holds for T. We know that if $\lambda \in iso\sigma_a(T)$, then T is a-polaroid from the preceding proof. This implies that T is a-isoloid, and then we have that

$$f(\sigma_a(T) \setminus \pi_{00}^a(T)) = \sigma_a(f(T)) \setminus \pi_{00}^a(f(T)) \text{ for every } f \in H(\sigma(T)).$$

Moreover, it follows from Theorem 3.15 that

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) = f(\sigma(T) \setminus \pi_{00}^a(T)) = \sigma_a(f(T)) \setminus \pi_{00}^a(f(T)),$$

which means that a-Weyl's theorem holds for $f(T)$. □

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