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A Note on Convergence of Nets of Multifunctions

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Abstract. The purpose of this paper is to investigate some new types of continuous convergence and quasi-uniform convergence of nets of multifunctions. The main three theorems describe a general types of interrelationships between forms of convergence, the continuity of the limits of nets of multifunctions and the continuity of members of such nets.

1. Introduction and preliminary

Throughout the present paper, (X, π) and (Y, τ) will denote a topological space with no separation properties assume. For a subset A of a topological space (X, π) we denote by $Cl(A)$ and $Int(A)$ the closure and the interior of A, respectively. By a multifunction $F : X \to Y$ we mean a correspondence which assigns to each element x of X a nonempty subset F(x) of Y. The upper and lower inverse images of a set $B \subset Y$ under F are defined by $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, respectively. The image of any set $A \subset X$ under F is defined as $F(A) = \bigcup \{F(x) \subset Y : x \in A\}$.

A multifunction $F : (X, \pi) \to (Y, \tau)$ is said to be upper semi continuous (briefly u.s.c.) (resp. lower semi continuous (briefly l.s.c.)) at a point
$x \in X$ if $x \in Int(F^+(V))$ (resp. $x \in Int(F^-(V))$) for each subset $V \in \tau$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$).

The set of all points at which F is u.s.c. (resp. l.s.c.) is denoted by $C_u(F)$ (resp. $C_l(F)$), [10], [19].

Of course, if a single-valued function $f : (X, \pi) \to (Y, \tau)$ is treated as a multifunction F given by $F(x) = \{f(x)\}$ then, each of the properties $x \in C_u(F)$, $x \in C_l(F)$ is equivalent to the continuity of f at x, i.e., $x \in C(f)$.

A multifunction $F : (X, \pi) \to (Y, \tau)$ is u.s.c. (resp. l.s.c.) if and only if the function $F : (X, \pi) \to (Y, \tau^V)$ (resp. $F : (X, \pi) \to (Y, \tau_V)$) is continuous, where τ^V (resp. τ_V) is the upper (resp. lower) Vietoris topology induced by τ on the family P(Y) of all nonempty subsets of Y (see [9], [15])).

Our basic references for quasi-uniform spaces are [17], [18], [7] and [11]. A quasi-uniformity on Y is a filter $\mathcal{V}$ on $Y \times Y$ which satisfies:

A quasi-uniformity on Y is a filter $\mathcal{V}$ on $Y \times Y$ which satisfies:

(a) $\Delta \subset V$ for all $V \in \mathcal{V}$ and

(b) given $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W \circ W \subset V$, where
$\Delta = \{(y, y) \in Y \times Y : y \in Y\}$ and
$W \circ W = \{(x, y) \in Y \times Y : (x, z), (z, y) \in W \text{ for some } z \in Y\}$.

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The pair $(Y, \mathcal{V})$ is called a quasi-uniform space. A family $\mathcal{B} \subset \mathcal{V}$ is a base for $\mathcal{V}$ if each member of $\mathcal{V}$ contains a member of $\mathcal{B}$. If $\mathcal{V}$ is a quasi-uniformity, then so its conjugate $\mathcal{V}^{-1} = \left\{V^{-1} : V \in \mathcal{V}\right\}$, where $V^{-1} = \{(x, y) \in Y \times Y : (y, x) \in V\}$.

Every quasi-uniformity $\mathcal{V}$ on Y generates the topology $t(\mathcal{V})$ given by the neighborhood base $\{V(y) : V \in \mathcal{V}\}$ for each point $y \in Y$, where $V(y) = \{z \in Y : (y, z) \in V\}$.

Throughout the present paper, the space $(Y, \mathcal{V})$ will always mean the topological space with the topology $t(\mathcal{V})$ unless explicitly stated otherwise.

According to [14], [2] and [12], for a quasi-uniform space $(Y, \mathcal{V})$ and $V \in \mathcal{V}$ we define

$V^+ = \{(A, B) \in P(Y) \times P(Y) : B \subset V(A)\}$ and

$V^- = \left\{(A, B) \in P(Y) \times P(Y) : A \subset V^{-1}(B)\right\}$.

The family $\{V^+ : V \in \mathcal{V}\}$ (resp. $\{V^- : V \in \mathcal{V}\}$) is a base for the upper Hausdorff quasi-uniformity $\mathcal{V}_u$ (resp. lower Hausdorff quasi-uniformity $\mathcal{V}_l$) on $P(Y)$.

So, every quasi-uniformity $\mathcal{V}$ on Y generates the following two pairs of topologies on the family $P(Y)$ of all nonempty subsets of Y: $(t(\mathcal{V}_u), t(\mathcal{V}_l))$ and $((t(\mathcal{V}))^V, (t(\mathcal{V}))_V)$.

Analogously to the u.s.c. and l.s.c. case, one can consider the continuity of multifunctions with respect to $t(\mathcal{V}_u)$ or $t(\mathcal{V}_l)$. We will call them $\mathcal{V}$-upper or $\mathcal{V}$-lower semi-continuity (briefly $\mathcal{V}u.s.c.$ or $\mathcal{V}l.s.c.$, respectively) [20]. The set of all points at which F is $\mathcal{V}.u.s.c.$ (resp. $\mathcal{V}.l.s.c.$), is denoted by $\mathcal{V}C_u(F)$ (resp. $\mathcal{V}C_l(F)$).

As was noted in [2, Lemma 2.4.], the topology $t(\mathcal{V}_u)$ is coarser than $(t(\mathcal{V}))^V$ and, the topology $t(\mathcal{V}_l)$ is finer than $(t(\mathcal{V}))_V$. Therefore, the following property is an immediate consequence of the definition.

Remark 1.1. The inclusions $C_u(F) \subset \mathcal{V}C_u(F)$ and $\mathcal{V}C_l(F) \subset C_l(F)$ hold for any multifunction $F : (X, \pi) \to (Y, \mathcal{V})$.

The $\mathcal{V}$-lower semi-continuity has been studied in [21] under the name H-lsc. Note also that, in the case of single-valued functions, we have the following

Remark 1.2. If we consider a single-valued function $f : (X, \pi) \to (Y, \mathcal{V})$ as a multifunction F given by $F(x) = \{f(x)\}$, then $\mathcal{V}C_u(F) = \mathcal{V}C_l(F) = C_u(F) = C_l(F) = C(f)$.

Now let us consider the general form of convergence for multifunction. Given a topology T on $P(Y)$, a net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions from X to Y is said to be T-convergent at $x \in X$ to a multifunction $F : X \to Y$, if the net $\{F_\sigma(x) : \sigma \in \Sigma\}$ is T-convergent to $F(x)$, [8, p.364].

The $(t(\mathcal{V}))^V$-convergence (resp. $(t(\mathcal{V}))_V$-convergence) is known as the upper (resp. lower) pointwise convergence [4,6] defined as follows:

Definition 1.3. A net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions $F_\sigma : (X, \pi) \to (Y, \mathcal{V}), \sigma \in \Sigma$, is said to be upper (resp. lower) pointwise convergent to $F : (X, \pi) \to (Y, \mathcal{V})$ at x, if for every open set $G \subset Y$ with $F(x) \subset G$ (resp. $F(x) \cap G \neq \emptyset$) there exists $\gamma \in \Sigma$ such that $x \in F^+_\sigma(G)$ (resp. $x \in F^-_\sigma(G)$) for all $\sigma \geq \gamma$; briefly $x \in C_u(F, (F_\sigma))$ (resp. $x \in C_l(F, (F_\sigma))$).

Below, we give several characterizations of these types of convergence. Before doing so let us recall some classical definitions.

For a net $\{A_\sigma : \sigma \in \Sigma\}$ of subsets A_σ of a topological space X we denote by lim inf A_σ and lim sup A_σ the sets $\bigcup\{\bigcap\{A_\sigma : \sigma \geq \delta\} : \delta \in \Sigma\}$ and $\bigcap\{\bigcup\{A_\sigma : \sigma \geq \delta\} : \delta \in \Sigma\}$, respectively. We say that $\{A_\sigma : \sigma \in \Sigma\}$ converges to $A \subset X$, denoted by $\lim A_\sigma = A$, if $A = \liminf A_\sigma = \limsup A_\sigma$.

Furthermore, a point $x \in X$ is called a limit point (resp. cluster point) of $\{A_\sigma : \sigma \in \Sigma\}$, denoted by $x \in$ Li A_σ (resp. $x \in$ Ls A_σ) if each neighbourhood of x meets $\{A_\sigma : \sigma \in \Sigma\}$ eventually (resp. frequently).

A net $\{A_\sigma : \sigma \in \Sigma\}$ is said to be K^+-convergent (resp. K^--convergent) to A if Ls $A_\sigma \subset A$ (resp. $A \subset$ Li A_σ).

We say that $\{A_\sigma : \sigma \in \Sigma\}$ topologically convergent to A, denoted by $A =$Lt A_σ, if $A =$ Li $A_\sigma =$ Ls A_σ, (see [1], [10], [16]).

Lemma 1.4. For every multifunctions $F, F_\sigma : (X,\pi) \to (Y,\mathcal{V})$, $\sigma \in \Sigma$, the following conditions are equivalent:

(a) $x \in C_u(F,(F_\sigma))$;

(b) $x \in \liminf F_\sigma^{-1}(\mathcal{A})$ for every $\mathcal{A} \in (t(\mathcal{V}))^V$ such that $F(x) \in \mathcal{A}$.

Lemma 1.5. For every multifunctions $F, F_\sigma : (X,\pi) \to (Y,\mathcal{V})$, $\sigma \in \Sigma$, the following conditions are equivalent:

(a) $x \in C_l(F,(F_\sigma))$;

(b) $x \in \liminf F_\sigma^{-1}(\mathcal{A})$ for every $\mathcal{A} \in (t(\mathcal{V}))_V$ such that $F(x) \in \mathcal{A}$.

(c) $F(x) \subset \mathrm{Li}\, F_\sigma(x)$.

The $t(\mathcal{V}_u)$-convergence (resp. $t(\mathcal{V}_l)$-convergence) is called upper (resp. lower) $\mathcal{V}$-convergence [20].

By $\mathcal{V}C_u(F,(F_\sigma))$ (resp. $\mathcal{V}C_l(F,(F_\sigma))$ we denote the set of all points $x \in X$ at which the net $\{F_\sigma : \sigma \in \Sigma\}$ is $t(\mathcal{V}_u)$-convergent (resp. $t(\mathcal{V}_l)$-convergent) to F.

This type of convergence can be characterized by the following convenient notion:

Given a pair (F,H) of multifunctions from (X,π) to $(Y,\mathcal{V})$ and $V \in \mathcal{V}$, let us denote

$D_u(F,H,V) = \{p \in X : (F(p),H(p)) \in V^+\}$ and

$D_l(F,H,V) = \{p \in X : (F(p),H(p)) \in V^-\}$.

Thus we have the following remark.

Remark 1.6. For every multifunctions $F, F_\sigma : (X,\pi) \to (Y,\mathcal{V})$, $\sigma \in \Sigma$, the following conditions are equivalent:

(a) $x \in \mathcal{V}C_u(F,(F_\sigma))$ (resp. $x \in \mathcal{V}C_l(F,(F_\sigma))$)

(b) $x \in \liminf D_u(F,F_\sigma,V)$ (resp. $x \in \liminf D_l(F,F_\sigma,V)$) for all $V \in \mathcal{V}$.

(c) $x \in \liminf F_\sigma^{-1}(\mathcal{A})$ for every $\mathcal{A} \in t(\mathcal{V}_u)$ (resp. $\mathcal{A} \in t(\mathcal{V}_l)$) such that $F(x) \in \mathcal{A}$.

For the same reasons as in Remark 1.1. we have

Remark 1.7. The inclusions $C_u(F,(F_\sigma)) \subset \mathcal{V}C_u(F,(F_\sigma))$ and $\mathcal{V}C_l(F,(F_\sigma)) \subset C_l(F,(F_\sigma))$ hold for any multifunctions $F, F_\sigma : (X,\pi) \to (Y,\mathcal{V})$, $\sigma \in \Sigma$.

It is easy to show that in the case of the single valued functions $f, f_\sigma : (X,\pi) \to (Y,\mathcal{V})$, $\sigma \in \Sigma$, treated as multifunctions F, F_σ given by $F(x) = \{f(x)\}, F_\sigma(x) = \{f_\sigma(x)\}$, the equality

$D_u(F,F_\sigma,V) = D_u(F,F_\sigma,V) = \{p \in X : f_\sigma(p) \in V(f(x))\}$ holds . So, analogously to Remark 1.2 we have

Remark 1.8. If we consider single-valued functions $f, f_\sigma : (X,\pi) \to (Y,\mathcal{V})$, $\sigma \in \Sigma$, as multifunctions F, F_σ given by $F(x) = \{f(x)\}, F_\sigma(x) = \{f_\sigma(x)\}$, then $\mathcal{V}C_u(F,(F_\sigma)) = \mathcal{V}C_l(F,(F_\sigma)) = C_u(F,(F_\sigma)) = C_l(F,(F_\sigma))$ and, this is the set of all points $x \in X$ at which the net $\{f_\sigma : \sigma \in \Sigma\}$ is pointwise convergent to f.

A net of functions $f_\sigma : (X,\pi) \to (Y,\mathcal{V})$ is called $\mathcal{V}$-locally uniformly convergent to f at a point $x \in X$, if for every $V \in \mathcal{V}$ there exists $\gamma \in \Sigma$ such that for any $\sigma \geq \gamma$ there exists an open subset U such that $x \in U$ and $f_\sigma(p) \in V(f(p))$ for all $p \in U$, [8].

The above consideration imply that this definition can be formulated by:

$x \in \liminf Int(D_u(F,F_\sigma,V))$ for all $V \in \mathcal{V}$, where $F(x) = \{f(x)\}, F_\sigma(x) = \{f_\sigma(x)\}$, $\sigma \in \Sigma$. So, this type of convergence for multifunctions is defined as follows

Definition 1.9. A net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions from a topological space (X,π) to a quasi-uniform space $(Y,\mathcal{V})$ is said to be upper (resp. lower) quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$ [20], if

$x \in \liminf Int(D_u(F, F_\sigma, V))$ (resp. $x \in \liminf Int(D_l(F, F_\sigma, V))$, for all $V \in \mathcal{V}$;
briefly $x \in \mathcal{V}UC_u(F, (F_\sigma))$ (resp. $x \in \mathcal{V}UC_l(F, (F_\sigma))$).

In [4] and [5], the simultaneous upper and lower quasi-uniformly $\mathcal{V}$-convergence was called almost quasuniformly convergence. The simultaneous upper and lower $\mathcal{V}$-convergence was called in [3] as quasiuniformly convergence. Let us remark also that in [13], the lower and upper $\mathcal{V}$-convergence was treated as some type of the bornological convergence.

As an immediate consequence of Remark 1.6. we have

Remark 1.10. The inclusions $\mathcal{V}UC_u(F, (F_\sigma)) \subset \mathcal{V}C_u(F, (F_\sigma))$ and $\mathcal{V}UC_l(F, (F_\sigma))) \subset \mathcal{V}C_l(F, (F_\sigma))$ hold for any multifunctions $F, F_\sigma : (X, \pi) \to (Y, \mathcal{V})$, $\sigma \in \Sigma$.

The following result is obtained as a corollary in [8]:

Let $\{f_\sigma : \sigma \in \Sigma\}$ be a net of continuous functions at $x_0 \in X$ which is pointwise convergent at x_0 to a function f. Then f is continuous at x_0 if and only if the net $\{f_\sigma : \sigma \in \Sigma\}$ is $\mathcal{V}$-locally uniformly convergent to f at x_0.

The content of this result is related to the three issues:

- the assumption about the type of convergence which provides the continuity of the limit function without the assumptions about the continuity of functions in the net;

- the assumption about the type of convergence which provides the continuity of the limit function under some weak assumptions about the continuity of functions in the net;

- weak assumptions about the continuity of functions in the net which provides
$\mathcal{V}$-locally uniformly convergence if the limit function is continuous.

In this paper we investigate those issues for nets of multifunctions. The result obtained as a corollary for uniform spaces is stronger than the result cited above due to the weakening of the assumption about the continuity of all functions in the net. For this purpose we need some special types of convergence.

Definition 1.11. A net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ is said to be

(a) upper (resp. lower) w-continuously convergent to F at x if
$x \in \liminf Int(F_\sigma^{-1}(\mathcal{A}))$ for every $\mathcal{A} \in (t(\mathcal{V}))^V$ (resp. $\mathcal{A} \in (t(\mathcal{V}))_V$)
such that $F(x) \in \mathcal{A}$; briefly $x \in WCC_u(F, (F_\sigma))$ (resp. $x \in WCC_l(F, (F_\sigma))$);

(b) upper (resp. lower) w-continuously $\mathcal{V}$-convergent to F at a point x, if
$x \in \liminf Int(F_\sigma^{-1}(\mathcal{A}))$ for every $\mathcal{A} \in t(\mathcal{V}_u)$ (resp. $\mathcal{A} \in t(\mathcal{V}_l)$)
such that $F(x) \in \mathcal{A}$; briefly $x \in \mathcal{V}WCC_u(F, (F_\sigma))$ (resp. $x \in \mathcal{V}WCC_l(F, (F_\sigma))$).

The following diagram illustrates the relations among the types of convergence defined so far.
Diagram 1.12.

$$\begin{tikzcd}
 & & C_l(F,(F_\sigma)) \\
 & WCC_l(F,(F_\sigma)) \arrow[ur] & \\
\mathcal{V}UC_l(F,(F_\sigma)) \arrow[rr] & & \mathcal{V}C_l(F,(F_\sigma)) \arrow[uu] \\
 & \mathcal{V}WCC_l(F,(F_\sigma)) \arrow[uu] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}UC_u(F,(F_\sigma)) \arrow[rr] & & \mathcal{V}C_u(F,(F_\sigma)) \\
 & \mathcal{V}WCC_u(F,(F_\sigma)) \arrow[ur] & \\
 & & C_u(F,(F_\sigma)) \arrow[uu] \\
 & WCC_u(F,(F_\sigma)) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

In [8, Definition 6.1 (iii)], the following concept was introduced:

Let $\mathcal{P}$ be a composable subbase for a quasi-uniformity $\mathcal{V}$; a net of functions $f_\sigma : (X, \pi) \to (Y, \mathcal{V})$, $\sigma \in \Sigma$, is $\mathcal{V}$-quasi-locally uniformly convergent at $x \in X$ to a function $f : (X, \pi) \to (Y, \mathcal{V})$ if for every V $\in \mathcal{P}$ and every $\gamma \in \Sigma$ there exist $\sigma \geq \gamma$ and open subset $U \subset X$ such that $f_\sigma(p) \in V(f(p))$ for all $p \in U$, where a subbase $\mathcal{P}$ of $\mathcal{V}$ is composable if for every $V \in \mathcal{P}$ there are W, $U \in \mathcal{P}$ such that $W \circ U \subset V$.

It it easy to see the following equivalent formulation:

$x \in \limsup IntD_u(F, F_\sigma, V)$ for all V $\in \mathcal{P}$, where $F(x) = \{f(x)\}$, $F_\sigma(x) = \{f_\sigma(x)\}$, which exhibits significant similarity with the quasi-uniformly $\mathcal{V}$-convergence. This suggests considering such generalisations for all five types of convergence defined so far.

Definition 1.13. A net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ is said to be

(a) upper (resp. lower) quasi-uniformly $\mathcal{V}$-subconvergent to F at a point $x \in X$, if
$x \in \limsup Int(D_u(F, F_\sigma, V))$ (resp. $x \in \limsup Int(D_l(F, F_\sigma, V))$ for all $V \in \mathcal{V}$;
briefly $x \in \mathcal{V}US_u(F, (F_\sigma))$ (resp. $x \in \mathcal{V}US_l(F, (F_\sigma))$);

(b) upper (resp. lower) $\mathcal{V}$-subconvergent to F at a point $x \in X$, if
$x \in \limsup D_u(F, F_\sigma, V)$ (resp. $x \in \limsup D_l(F, F_\sigma, V)$) for all $V \in \mathcal{V}$;
briefly $x \in \mathcal{V}S_u(F, (F_\sigma))$ (resp. $x \in \mathcal{V}S_l(F, (F_\sigma))$;

(c) upper (resp. lower) pointwise subconvergent to F at a point $x \in X$, if
$x \in \limsup F_\sigma^{-1}(\mathcal{A})$ for every $\mathcal{A} \in (t(\mathcal{V}))^V$ (resp. $\mathcal{A} \in (t(\mathcal{V}))_V$) such that $F(x) \in \mathcal{A}$
briefly $x \in S_u(F, (F_\sigma))$ (resp. $x \in S_l(F, (F_\sigma))$;

(d) upper (resp. lower) w-continuously subconvergent to F at x if
$x \in \limsup Int(F_\sigma^{-1}(\mathcal{A}))$ for every $\mathcal{A} \in (t(\mathcal{V}))^V$ (resp. $\mathcal{A} \in (t(\mathcal{V}))_V$)
such that $F(x) \in \mathcal{A}$; briefly $x \in WCS_u(F, (F_\sigma))$ (resp. $x \in WCS_l(F, (F_\sigma))$);

(e) upper (resp. lower) w-continuously $\mathcal{V}$-subconvergent to F at a point x, if
$x \in \limsup Int(F_\sigma^{-1}(\mathcal{A}))$ for every $\mathcal{A} \in t(\mathcal{V}_u)$ (resp. $\mathcal{A} \in t(\mathcal{V}_l)$)
such that $F(x) \in \mathcal{A}$; briefly $x \in \mathcal{V}WCS_u(F, (F_\sigma))$ (resp. $x \in \mathcal{V}WCS_l(F, (F_\sigma))$).

The relationships between those types of convergence are analogous to those presented in the above diagram, i.e.,

Diagram 1.14.

$$\begin{tikzcd}
 & & S_l(F,(F_\sigma)) \\
 & WCS_l(F,(F_\sigma)) \arrow[ur] & \\
\mathcal{V}US_l(F,(F_\sigma)) \arrow[rr] & & \mathcal{V}S_l(F,(F_\sigma)) \arrow[uu] \\
 & \mathcal{V}WCS_l(F,(F_\sigma)) \arrow[uu] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}US_u(F,(F_\sigma)) \arrow[rr] & & \mathcal{V}S_u(F,(F_\sigma)) \\
 & \mathcal{V}WCS_u(F,(F_\sigma)) \arrow[ur] & \\
 & & S_u(F,(F_\sigma)) \arrow[uu] \\
 & WCS_u(F,(F_\sigma)) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

As an immediate consequence of the definition and Lemmas 1.4, 1.5 and 1.6 we have

Remark 1.15. For any multifunctions $F, F_\sigma : (X, \pi) \to (Y, \mathcal{V})$, $\sigma \in \Sigma$, the following inclusions hold:
(a) $\mathcal{V}UC_u(F,(F_\sigma)) \subset \mathcal{V}US_u(F,(F_\sigma))$ (resp. $\mathcal{V}UC_l(F,(F_\sigma)) \subset \mathcal{V}US(F,(F_\sigma))$;
(b) $\mathcal{V}C_u(F,(F_\sigma)) \subset \mathcal{V}S_u(F,(F_\sigma))$ (resp. $\mathcal{V}C_l(F,(F_\sigma)) \subset \mathcal{V}S_l(F,(F_\sigma))$;
(c) $C_u(F,(F_\sigma)) \subset S_u(F,(F_\sigma))$ (resp. $C_l(F,(F_\sigma)) \subset S_l(F,(F_\sigma))$);
(d) $WCC_u(F,(F_\sigma)) \subset WCS_u(F,(F_\sigma))$ (resp. $WCC_l(F,(F_\sigma)) \subset WCS_l(F,(F_\sigma))$) and
(e) $\mathcal{V}WCC_u(F,(F_\sigma)) \subset \mathcal{V}WCS_u(F,(F_\sigma))$ (resp.
$\mathcal{V}WCC_l(F,(F_\sigma)) \subset \mathcal{V}WCS(F,(F_\sigma))$).

2. Main results

We begin with a theorem that shows which types of convergences guarantee some kind of continuity.

Theorem 2.1. For every multifunctions F, $F_\sigma : (X, \pi) \to (Y, \mathcal{V}), \sigma \in \Sigma$, the following hold:

(a) $WCC_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_u(F,(F_\sigma)) \subset C_l(F)$ and
$WCS_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma)) \subset C_l(F)$;

(b) $\mathcal{V}WCC_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_u(F,(F_\sigma)) \subset \mathcal{V}C_l(F)$ and
$\mathcal{V}WCS_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma)) \subset \mathcal{V}C_l(F)$;

(c) $\mathcal{V}WCC_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_l(F,(F_\sigma)) \subset \mathcal{V}C_u(F)$ and
$\mathcal{V}WCS_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_l(F,(F_\sigma)) \subset \mathcal{V}C_u(F)$.

Proof. (a): Let $x \in WCC_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_u(F,(F_\sigma))$ (resp. $x \in WCS_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma))$), let G be an open subset of Y such that $F(x) \cap G \neq \emptyset$ and let us take $W, V \in \mathcal{V}$ such that $V(y) \subset G$ for some $y \in F(x)$, and $W \circ W \subset V$. Since $x \in WCC_l(F,(F_\sigma))$ (resp. $x \in \mathcal{V}^{-1}UC_u(F,(F_\sigma))$), there exists $\gamma \in \Sigma$ such that
$x \in Int(F_\sigma^-(W(y))$ (1)
(resp. $x \in Int(D_l(F_\sigma, F, W))$) (2)
for all $\sigma \geq \gamma$ and, because $x \in \mathcal{V}^{-1}US_u(F,(F_\sigma))$ (resp. $x \in WCS_l(F,(F_\sigma))$ we may find $\delta \geq \gamma$ such that each of the statements (1) and (2) implies that $x \in U = Int(F_\delta^-(W(y)) \cap Int(D_l(F_\sigma, F, W))$. Therefore, for every $p \in U$ we have $F_\delta(p) \cap W(y) \neq \emptyset$ and $F_\delta(p) \subset W^{-1}(F(p))$. So, $(y, a) \in W$ and $(a, b) \in W$ for some $a \in F_\delta(p)$ and $b \in F(p)$ which shows that $b \in F(p) \cap V(y)$ and consequently, that $x \in Int(F^-(G))$. This finishes the proof of (a).
(b): Let $x \in \mathcal{V}WCC_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_u\,\{F,(F_\sigma)\}$ (resp. $x \in \mathcal{V}WCS_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma))$, let $V \in \mathcal{V}$ be established and let $W \in \mathcal{V}$ be such that $W \circ W \subset V$. Since $x \in \mathcal{V}WCC_l(F,(F_\sigma))$ (res.$x \in \mathcal{V}^{-1}UC_u(F,(F_\sigma))$), there exists $\gamma \in \Sigma$ such that
$x \in Int(F_\sigma^{-1}(W^-(F(x))))$ (3)
(resp. $x \in Int(D_l(F_\sigma, F, W))$) (4)
for all $\sigma \geq \gamma$ and, since $x \in \mathcal{V}^{-1}US_u(F,(F_\sigma))$ (resp. $x \in \mathcal{V}WCS_l(F,(F_\sigma))$), there exists $\delta \geq \gamma$ such that each of the statements (3) and (4) implies that $x \in U = Int(F_\delta^{-1}(W^-(F(x)))) \cap Int(D_l(F_\delta, F, W))$. If $p \in U$, then $F_\delta(p) \subset W^{-1}(F(p))$ and $(F(x), F_\delta(p)) \in W^-$ which implies $F(x) \subset W^-(F_\delta(p))$ and consequently, for every $a \in F(x)$ we have $(a, b) \in W$ and $(b, c) \in W$ for some $b \in F_\delta(p)$ and $c \in F(p)$. So, $F(x) \subset V^{-1}(F(p))$ which means that $F(p) \in V^-(F(x))$ for any $p \in U$, and proves that $x \in Int(F^{-1}(V^-(F(x))))$. This shows that $x \in \mathcal{V}C_l(F)$.
(c): Let $x \in \mathcal{V}WCC_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_l(F,(F_\sigma))$ (resp. $x \in \mathcal{V}WCS_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_l(F,(F_\sigma))$, let $V \in \mathcal{V}$ and $W \in \mathcal{V}$ be as above. Since $x \in \mathcal{V}WCC_u(F,(F_\sigma))$ (resp. $x \in \mathcal{V}^{-1}UC_l(F,(F_\sigma))$), there exists $\gamma \in \Sigma$ such that
$x \in Int(F_\sigma^{-1}((W^+)(F(x))))$ (5)
(resp. $x \in Int(D_l(F, F_\sigma, W^{-1}))$) (6)
for all $\sigma \geq \gamma$ and, $x \in Int(D_l(F, F_\sigma, W^{-1}))$ (resp. $x \in Int(F_\sigma^{-1}((W^+)(F(x)))))$ for some $\delta \geq \gamma$ because $x \in \mathcal{V}^{-1}US_l(F,(F_\sigma))$ (resp. $x \in \mathcal{V}WCS_u(F,(F_\sigma))$). So, each of the statements (5) and (6) implies that
$x \in U = Int(F_\delta^{-1}((W^+)(F(x)))) \cap Int(D_l(F, F_\delta, W^{-1}))$. Consequently, for every $p \in U$ we have $F_\delta(p) \in (W^+)(F(x))$, which means $F_\delta(p) \subset W(F(x))$, and $((F(p), F_\delta(p)) \in (W^{-1})^-$, which means $F(p) \subset W(F_\delta(p))$. Therefore, for

every $a \in F(p)$ we have $(b, a) \in W$ and $(c, b) \in W$ for some $b \in F_\delta(p)$ and $c \in F(x)$. So, $(c, a) \in V$ which gives that $F(p) \subset V(F(x))$. So, $F(p) \in (V^+)(F(x)))$ for any $p \in U$ which shows that $x \in \mathcal{V}C_u(F)$ and finishes the proof.

We now show that the assumption concerning the continuity of multifunctions in the net, in the form
$x \in \liminf C_l(F_\sigma)$ (resp. $x \in \liminf C_u(F_\sigma)$),
$x \in \liminf \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \liminf \mathcal{V}C_u(F_\sigma)$) or
$x \in \limsup C_l(F_\sigma)$ (resp. $x \in \limsup C_u(F_\sigma)$),
$x \in \limsup \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \limsup \mathcal{V}C_u(F_\sigma)$) implies the identity or inclusions of different types of convergence listed in diagrams 1.12 and 1.14.

Theorem 2.2. For every multifunctions F, $F_\sigma : (X, \pi) \to (Y, \mathcal{V}), \sigma \in \Sigma$, the following hold:

(a) $C_l(F, (F_\sigma)) \cap \liminf C_l(F_\sigma) \subset WCC_l(F, (F_\sigma))$ and
$C_u(F, (F_\sigma)) \cap \liminf C_u(F_\sigma) \subset WCC_u(F, (F_\sigma))$;

(b) $S_l F, (F_\sigma) \cap \liminf C_l(F_\sigma) \subset WCS_l(F, (F_\sigma))$
$C_l(F, (F_\sigma)) \cap \limsup C_l(F_\sigma)] \subset WCS_l(F, (F_\sigma)$,
$S_u(F, (F_\sigma)) \cap \liminf C_u(F_\sigma) \subset WCS_u(F, (F_\sigma))$ and
$C_u(F, (F_\sigma)) \cap \limsup C_u(F_\sigma)] \subset WCS_u(F, (F_\sigma))$;

(c) $\mathcal{V}C_l(F, (F_\sigma)) \cap \liminf \mathcal{V}C_l((F_\sigma) \subset \mathcal{V}WCC_l(F, (F_\sigma))$ and
$\mathcal{V}C_u(F, (F_\sigma)) \cap \liminf \mathcal{V}C_u(F_\sigma) \subset \mathcal{V}WCC_u(F, (F_\sigma))$;

(d) $\mathcal{V}S_l(F, (F_\sigma)) \cap \liminf \mathcal{V}C_l(F_\sigma) \subset \mathcal{V}WCS_l(F, (F_\sigma))$,
$\mathcal{V}C_l(F, (F_\sigma)) \cap \limsup \mathcal{V}C_l(F_\sigma) \subset \mathcal{V}WCS_l(F, (F_\sigma))$,
$\mathcal{V}S_u(F, (F_\sigma)) \cap \liminf \mathcal{V}C_u(F_\sigma) \subset \mathcal{V}WCS_u(F, (F_\sigma))$ and
$\mathcal{V}C_u(F, (F_\sigma) \cap \limsup \mathcal{V}C_u(F_\sigma)] \subset \mathcal{V}WCS_u(F, (F_\sigma))$;

Proof. (a): Let $x \in C_l(F, (F_\sigma)) \cap \liminf C_l(F_\sigma)$ (resp. $x \in C_u(F, (F_\sigma)) \cap \liminf C_u(F_\sigma)$) and let G be an open subset of Y such that $F(x) \cap G \neq \emptyset$ (resp. $F(x) \subset G$). Then there exists $\gamma \in \Sigma$ such that $x \in F_\sigma^-(G)))$ and $x \in C_l(F_\sigma)$ (resp. $x \in F_\sigma^+(G)))$ and $x \in C_u(F_\sigma)$ for all $\sigma \geq \gamma$. So, $x \in Int(F_\sigma^-(G)))$ (resp. $x \in Int(F_\sigma^+(G)))$ for all $\sigma \geq \gamma$ and consequently, $x \in WCC_l(F, (F_\sigma))$ (resp. $x \in WCC_u(F, (F_\sigma)))$.

(b): Let $x \in S_l(F, (F_\sigma)) \cap \liminf C_l(F_\sigma)$ (resp. $x \in S_u(F, (F_\sigma)) \cap \liminf C_u(F_\sigma)$), let G be an open subset of Y such that $F(x) \cap G \neq \emptyset$ (resp. $F(x) \subset G$) and let $\gamma \in \Sigma$ be established. Since $x \in \liminf C_l(F_\sigma)$ (resp. $x \in \liminf C_u(F_\sigma)$), there exists $\xi \in \Sigma$ such that $x \in C_l(F_\sigma)$ (resp. $x \in C_u(F_\sigma)$) for all $\sigma \geq \xi$. Let us take $\delta \in \Sigma$ such that $\delta \geq \gamma$ and $\delta \geq \xi$. Then, because of the assumption that $x \in S_l(F, (F_\sigma))$ (resp. $x \in S_u(F, (F_\sigma)))$, there exists $\sigma \geq \delta$ such that $x \in F_\sigma^-(G)$ and $x \in C_l(F_\sigma)$ (resp. $x \in F_\sigma^+(G)$ and $x \in C_u(F_\sigma)$). Consequently, $x \in Int(F_\sigma^-(G))$ (resp. $x \in Int(F_\sigma^+(G)))$ which shows that $x \in WCS_l(F, (F_\sigma))$ (resp.$x \in WCS_u \{F, (F_\sigma)\}$).

Now suppose that $x \in C_l(F, (F_\sigma)) \cap \limsup C_l(F_\sigma)$
(resp. $x \in C_u(F, (F_\sigma)) \cap \limsup C_u(F_\sigma)$) and, let $G \subset Y$ and $\gamma \in \Sigma$ be as above. So, there exists $\xi \in \Sigma$ such that $x \in F_\sigma^-(G)$ (resp. $x \in F_\sigma^+(G)$) for all $\sigma \geq \xi$. If we take $\delta \in \Sigma$ such that $\delta \geq \gamma$ and $\delta \geq \xi$, then there exists $\sigma \geq \delta$ such that $x \in C_l(F_\sigma)$ (resp. $x \in C_u(F_\sigma)$). This implies that $x \in Int(F_\sigma^-(G))$ (resp. $x \in Int(F_\sigma^+(G)))$. So, $x \in WCS_l(F, (F_\sigma))$
(resp. $x \in WCS_u(F, (F_\sigma))))$.

(c): Let $x \in \mathcal{V}C_l(F, (F_\sigma) \cap \liminf \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \mathcal{V}C_u(F, (F_\sigma)) \cap \liminf \mathcal{V}C_u(F_\sigma)$), let $V \in \mathcal{V}$ be established and let $W \in \mathcal{V}$ be such that $W \circ W \subset V$. Then there exists $\gamma \in \Sigma$ such that $x \in D_l(F, F_\sigma, W)$ and $x \in \mathcal{V}C_l(F_\sigma)$ (resp. $x \in D_u(F, F_\sigma, W)$ and $x \in \mathcal{V}C_u(F_\sigma)$) for all $\sigma \geq \gamma$. Therefore, $F(x) \subset W^{-1}(F_\sigma(x))$ and $x \in Int(F_\sigma^{-1}(W^-(F_\sigma(x))))$ (resp. $F_\sigma(x) \subset W(F(x))$ and $x \in Int(F_\sigma^{-1}(W^+(F_\sigma(x))))$ for each $\sigma \geq \gamma$. Let us take a certain $\sigma \geq \gamma$, and let $p \in Int(F_\sigma^{-1}(W^-(F_\sigma(x))))$ (resp. $p \in Int(F_\sigma^{-1}(W^+(F_\sigma(x)))))$. Then $F_\sigma(p) \in W^-(F_\sigma(x)))$ (resp. $F_\sigma(p) \in W^+(F_\sigma(x)))$ and consequently, $F_\sigma(x) \subset W^{-1}(F_\sigma(p))$ (resp. $F_\sigma(p) \subset W(F_\sigma(x)))$. So, for all $a \in F(x)$ (resp. $a \in F_\sigma(p)$) there exist $b \in F_\sigma(x)$ and $c \in F_\sigma(p)$ (resp. $b \in F_\sigma(x)$ and $c \in F(x)$) such that $(a, b) \in W$, and $(b, c) \in W$ (resp. $(b, a) \in W$, and $(c, b) \in W$). This gives $(a, c) \in V$ (resp. $(c, a) \in V$) and consequently, $a \in V^{-1}(F_\sigma(p))$ (resp. $a \in V(F(x)))$. It shows that $F(x) \subset V^{-1}F_\sigma(p)$ (resp. $F_\sigma(p) \subset V(F(x)))$, so that $F_\sigma(p) \in V^-(F(x))$ (resp. $F_\sigma(p) \in V^{+(}F(x))$. So, $x \in Int(F_\sigma^{-1}(V^-(F(x))))$ (resp. $x \in Int(F_\sigma^{-1}(V^+(F(x)))))$ and the proof of (c) is complete.

(d): Let $x \in \mathcal{V}S_l(F,(F_\sigma)) \cap \liminf \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \mathcal{V}S_u(F,(F_\sigma) \cap \liminf \mathcal{V}C_u(F_\sigma))$, let $V \in \mathcal{V}$ and $\gamma \in \Sigma$ be established, and let $W \in \mathcal{V}$ be such that $W \circ W \subset V$. Then there exists $\xi \in \Sigma$ such that $x \in \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \mathcal{V}C_u(F_\sigma)$) for all $\sigma \geq \xi$. Let us take $\delta \in \Sigma$ such that $\delta \geq \gamma$ and $\delta \geq \xi$. Then, because $x \in \mathcal{V}S_l(F,(F_\sigma)))$ (resp. $x \in \mathcal{V}S_u(F,(F_\sigma))$ we may find $\sigma \geq \delta$ such that $x \in D_l(F,F_\sigma,W)$ (resp. $x \in D_u(F,F_\sigma,W)$) and $x \in Int(F_\sigma^{-1}(W^-(F_\sigma(x))))$ (resp. $x \in Int(F_\sigma^{-1}(W^+(F_\sigma(x)))))$. This implies that $F(x) \subset W^{-1}(F_\sigma(x))$ (resp. $F_\sigma(x) \subset W(F(x))$). Let $p \in Int(F_\sigma^{-1}(W^-(F_\sigma(x))))$ (resp. $p \in Int(F_\sigma^{-1}(W^+(F_\sigma(x))))$, then $(F_\sigma(x),F_\sigma(p)) \in W^-$ (resp. $(F_\sigma(x),F_\sigma(p)) \in W^+$). Therefore, $F_\sigma(x) \subset W^{-1}(F_\sigma(p))$ (resp. $F_\sigma(p) \subset W(F_\sigma(x))$) and, since $F(x) \subset W^{-1}(F_\sigma(x))$ (resp. $F_\sigma(x) \subset W(F(x))$), for every $a \in F(x)$ (resp. $a \in F_\sigma(p)$) there exist $b \in F_\sigma(x)$ (resp. $b \in F_\sigma(x)$) and $c \in F_\sigma(p)$ (resp. $c \in F(x)$) such that $(a,b) \in W$ and $(b,c) \in W$ (resp. $(b,a) \in W$, and $(c,b) \in W$). So, $a \in V^{-1}(c)$ (resp. $a \in V(c)$) and consequently, $F(x) \subset V^{-1}F_\sigma(p))$ (resp. $F_\sigma(p) \subset V(F(x))$) which means that $F_\sigma(p) \in V^-F(x))$ (resp. $F_\sigma(p) \in V^+((F(x)))$ and proves that $x \in Int(F_\sigma^{-1}(V^-(F(x))))$ (resp. $x \in Int(F_\sigma^{-1}(V^+(F(x))))$. So, $x \in \mathcal{V} - WCSC_l\{F,(F_\sigma)\}$ (resp. $x \in \mathcal{V} - WCSC_u\{F,(F_\sigma)\}$).

Now suppose that $x \in \mathcal{V}C_l(F,(F_\sigma)) \cap \limsup \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \mathcal{V}C_u\{F,(F_\sigma)\} \cap \limsup \mathcal{V}C_u(F_\sigma)$ and, let $V,W \in \mathcal{V}$ and $\gamma \in \Sigma$ be as above. Since $x \in \mathcal{V}C_l(F,(F_\sigma)$ (resp. $x \in \mathcal{V}C_u(F,(F_\sigma)))$, there exists $\xi \in \Sigma$ such that $x \in D_l(F,F_\sigma,W)$ (resp. $x \in D_u(F,F_\sigma,W)$) for all $\sigma \geq \xi$. Let $\delta \in \Sigma$ be such that $\delta \geq \gamma$ and $\delta \geq \xi$. Because $x \in \limsup \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \limsup \mathcal{V}C_u(F_\sigma)$), there exists $\sigma \geq \delta$ such that $x \in \mathcal{V}C_l(F_\sigma)$ (resp. $x \in \mathcal{V}C_u(F_\sigma)$). So, $x \in D_l(F,F_\sigma,W) \cap Int(F_\sigma^{-1}(W^-(F_\sigma(x))))$ (resp. $x \in D_u(F,F_\sigma,W) \cap Int(F_\sigma^{-1}(W^+(F_\sigma(x)))))$ and, the rest of the proof is quite analogous to the above.

Corollary 2.3. For every multifunctions F, $F_\sigma : (X,\pi) \to (Y,\mathcal{V}), \sigma \in \Sigma$ we have:

(a) On the set lim inf $C_l(F_\sigma)$ the following hold:
(i) $C_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_u(F,(F_\sigma)) \subset C_l(F)$ and
(ii) $S_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma)) \subset C_l(F)$.

(b) On the set lim inf $\mathcal{V}C_l(F_\sigma)$ the following hold:
(i) $\mathcal{V}C_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_u(F,(F_\sigma)) \subset \mathcal{V}C_l(F)$ and
(ii) $\mathcal{V}S_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma)) \subset \mathcal{V}C_l(F)$.

(c) On the set lim sup $\mathcal{V}C_l(F_\sigma)$ the following hold:
$\mathcal{V}C_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_u(F,(F_\sigma)) \subset \mathcal{V}C_l(F)$.

(d) On the set lim inf $\mathcal{V}C_u(F_\sigma)$ the following hold:
(i) $\mathcal{V}C_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}US_l(F,(F_\sigma)) \subset \mathcal{V}C_u(F)$ and
(ii) $\mathcal{V}S_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_l(F,(F_\sigma)) \subset \mathcal{V}C_u(F)$.

(e) On the set lim sup $\mathcal{V}C_u(F_\sigma)$ the following hold:
$\mathcal{V}C_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}UC_l(F,(F_\sigma)) \subset \mathcal{V}C_u(F)$;

In the case of a uniform structure $\mathcal{V}$ and the single valued functions
$f, f_\sigma : (X,\pi) \to (Y,\mathcal{V}), \sigma \in \Sigma$, treated as multifunctions F, F_σ given by $F(x) = \{f(x)\}$,
$F_\sigma(x) = \{f_\sigma(x)\}$ we have

(i) $S_l(F,(F_\sigma)) = S_u(F,(F_\sigma)) = \mathcal{V}S_l(F,(F_\sigma)) = \mathcal{V}S_u(F,(F_\sigma))$ and it consists of the points of the far weaker property than the pointwise convergence (see Remark 1.8);

(ii) $\mathcal{V}^{-1}UC_u(F,(F_\sigma)) = \mathcal{V}^{-1}UC_l(F,(F_\sigma)) = \mathcal{V}UC_u(F,(F_\sigma)) = \mathcal{V}UC_l(F,(F_\sigma))$ and it consists of the points of the $\mathcal{V}$-locally uniformly convergence [12];

(iii) $\mathcal{V}^{-1}US_u(F,(F_\sigma)) = \mathcal{V}^{-1}US_l(F,(F_\sigma)) = \mathcal{V}US_u(F,(F_\sigma)) = \mathcal{V}US_l(F,(F_\sigma))$ and it consists of the points of the far weaker property than the points of the $\mathcal{V}$-locally uniformly convergence.

We will use the same name (Definition for single valued functions, i.e. $x \in S_l(F,(F_\sigma))$ means the pointwise subconvergence, $x \in US_l(F,(F_\sigma))$ means the quasi-uniformly $\mathcal{V}$-subconvergence. The relationships between them are as follows:

$$\begin{tikzcd}
\text{quasi-uniformly } \mathcal{V}\text{-subconvergence} \arrow[r] & \text{pointwise subconvergence} \\
\mathcal{V}\text{-locally uniformly convergence} \arrow[r] \arrow[u] & \text{pointwise convergence} \arrow[u]
\end{tikzcd}$$

Corollary 2.4. Let (f_σ) be a net of functions from a topological space (X,π) to a uniform space $(Y,\mathcal{V})$ and let $f_\sigma : (X,\pi) \to (Y,\mathcal{V})$. If $x \in \liminf C(f_\sigma)$, (f_σ) is pointwise convergent and quasi-uniformly $\mathcal{V}$-subconvergent at x to f, then f is continuous at x.

Finally we show how the continuity of the limit function affects types of convergence.

Theorem 2.5. For every multifunctions $F, F_\sigma : (X,\pi) \to (Y,\mathcal{V}), \sigma \in \Sigma$, the following hold:

(a) $\mathcal{V}WCC_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_u(F) \subset \mathcal{V}UC_l(F,(F_\sigma))$ and
$\mathcal{V}WCC_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_l(F) \subset \mathcal{V}UC_u(F,(F_\sigma))$;

(b) $\mathcal{V}WCS_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_u(F) \subset \mathcal{V}US_l(F,(F_\sigma))$ and
$\mathcal{V}WCS_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_l(F) \subset \mathcal{V}US_u(F,(F_\sigma))$.

Proof. (a): Let $x \in \mathcal{V}WCC_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_u(F,(F_\sigma))$ (resp. $x \in \mathcal{V}WCC_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_l(F,(F_\sigma))$) and let us take $V, W \in \mathcal{V}$ such that $W \circ W \subset V$. Then there exists $\gamma \in \Sigma$ such that $x \in U = Int(F_\sigma^{-1}(W^-(F(x)))) \cap Int(F^{-1}((W^-)^{-1}(F(x)))$ (resp. $x \in U^* = Int(F_\sigma^{-1}(W^+(F(x)))) \cap Int(F^{-1}((W^+)^{-1}(F(x)))))$ for each $\sigma \geq \gamma$. So, for every $p \in U$ (resp. $p \in U^*$) we have $F_\sigma(p) \in W^-(F(x))$ and $F(p) \in (W^-)^{-1}(F(x))$ (resp. $F_\sigma(p) \in W^+(F(x))$ and $F(p) \in (W^+)^{-1}(F(x))$). This means that $F(x) \subset W^{-1}(F_\sigma(p))$ and $F(p) \subset W^{-1}(F(x))$ (resp. $F_\sigma(p) \subset W(F(x))$ and $F(x) \subset W(F(p))$). Therefore, for every $a \in F(p)$ (resp. $a \in F_\sigma(p)$) there exist $b \in F(x)$ and $c \in F_\sigma(p)$ (resp. $b \in F(x)$ and $c \in F(p)$) such that $(a,b),(b,c) \in W$ (resp. $(b,a),(c,b) \in W$) and consequently, $F(p) \subset V^{-1}(F_\sigma(p))$ (resp. $F_\sigma(p) \subset V(F(p))$). This proves that $U \subset D_l(F,F_\sigma,V)$ (resp. $U^* \subset D_u(F,F_\sigma,V)$) and finishes the proof of (a).

The proof of (b) is quite analogous to the one above.

Corollary 2.6. For every multifunctions F, $F_\sigma : (X,\pi) \to (Y,\mathcal{V}), \sigma \in \Sigma$ we have:

(a) On the set $\liminf \mathcal{V}C_l(F_\sigma)$ the following hold:
(i) $\mathcal{V}C_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_u(F) \subset \mathcal{V}UC_l(F,(F_\sigma))$ and
(ii) $\mathcal{V}S_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_u(F) \subset \mathcal{V}US_l(F,(F_\sigma))$.

(b) On the set $\limsup \mathcal{V}C_l(F_\sigma)$ the following hold:
$\mathcal{V}C_l(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_u(F) \subset \mathcal{V}US_l(F,(F_\sigma))$

(c) On the set $\liminf \mathcal{V}C_u(F_\sigma)$ the following hold:
(i) $\mathcal{V}C_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_l(F) \subset \mathcal{V}UC_u(F,(F_\sigma))$ and
(ii) $\mathcal{V}S_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_l(F) \subset \mathcal{V}US_u(F,(F_\sigma))$.

(d) On the set $\limsup \mathcal{V}C_u(F_\sigma)$ the following hold:
$\mathcal{V}C_u(F,(F_\sigma)) \cap \mathcal{V}^{-1}C_l(F,(F_\sigma)) \subset \mathcal{V}US_u(F,(F_\sigma))$.

The following consequence of Corollary 2.4 and the above corollary improves the classical result cited in [8, Corollary 6.4].

Corollary 2.7. Let $\{f_\sigma : \sigma \in \Sigma\}$ be a net of functions such that $x_0 \in \liminf C(f_\sigma)$ which is pointwise convergent at x_0 to a function f. Then f is continuous at x_0 if and only if the net $\{f_\sigma : \sigma \in \Sigma\}$ is $\mathcal{V}$-locally uniformly convergent to f at x_0.

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