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Comparison results for Proper Double Splittings of Rectangular Matrices

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Abstract. In this article, we consider two proper double splittings satisfying certain conditions, of a semi-monotone rectangular matrix A and derive new comparison results for the spectral radii of the corresponding iteration matrices. These comparison results are useful to analyse the rate of convergence of the iterative methods (formulated from the double splittings) for solving rectangular linear system $Ax = b$.

1. Introduction

Consider the following linear system,

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, $b \in \mathbb{R}^{n\times 1}$ is a given vector and $x \in \mathbb{R}^{n\times 1}$ is an unknown vector. In order to solve (1), iterative methods of the form

$$x^{i+1} = Hx^i + c, \qquad i = 1, 2, 3... \tag{2}$$

are often employed. The iterative formula (2) is obtained by splitting A into the form $A = U - V$, where U is nonsingular and then setting $H = U^{-1}V$ and $c = U^{-1}b$. Such a splitting is called a single splitting (see, [19]) of A and the matrix H is called an iteration matrix [18]. It is well known (see chapter 7, [6]) that the iterative method (2) converges to the unique solution of (1) (irrespective of the choice of initial vector x°) if and only if $\rho(H) < 1$, where $\rho(H)$ denotes the spectral radius of H, viz., the maximum of the moduli of the eigenvalues of H. Note that standard iterative methods like the Jacobi, Gauss-Seidel and successive over-relaxation methods arise from different choices of real square matrices U and V. A decomposition $A = U - V$ of $A \in \mathbb{R}^{n\times n}$ is called a regular splitting if U^{-1} exists, $U^{-1} \geq 0$ and $V \geq 0$, where the matrix $B \geq 0$ means all the entries of B are nonnegative. The notion of regular splitting was proposed by Varga [18] and it was shown that $\rho(H) < 1$ if and only if A is monotone. Here, matrix A monotone [7] means A^{-1} exists and $A^{-1} \geq 0$. A decomposition $A = U - V$ of $A \in \mathbb{R}^{n\times n}$ is called a weak regular splitting if $U^{-1} \geq 0$ and $U^{-1}V \geq 0$. This was proposed by Ortega and Rheinboldt [14] and again it was shown that $\rho(H) < 1$ if and only if A is monotone. These results show the importance of monotone matrices and the spectral radius $\rho(H)$ of an iteration matrix, in the study of convergence of the iterative methods of form (2). It is well

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known that the convergence of the iterative method (2) is faster whenever $\rho(H)$ is smaller and $\rho(H) < 1$. This leads to the problem of comparison between the spectral radii of the iteration matrices of corresponding iterative methods which are derived from two different splittings $A = U_1 - V_1$ and $A = U_2 - V_2$ of the same matrix A. Results related to this problem are called comparison results for splittings of matrices. So far, various comparison theorems for different kinds of single splittings of matrices have been derived by several authors. For details of these results one could refer to ([3] to [6], [8], [16], [18], [20] and [21]).

Berman and Plemmons [4] then extended the notion of splitting to rectangular matrices and called it as a proper splitting. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper splitting if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$, where $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range space of A and the null space of A, respectively. Analogous to the invertible case, with such a splitting one associates an iterative sequence $x^{i+1} = Hx^i + c$, where (this time) $H = U^{\dagger}V$ (again) called iteration matrix, $c = U^{\dagger}b$ and $U^{\dagger}$ denotes the Moore-Penrose inverse of U (see next section for definition). Once again it is well known that this sequence converges to $A^{\dagger}b$, the least squares solution of minimum norm, of the system $Ax = b$ (irrespective of the initial vector x°) if and only if $\rho(H) < 1$. For details, refer to [6].

Recently, Jena et al. [10] extended the notion of regular and weak regular splittings to rectangular matrices and the respective definitions are given next. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper regular splitting if it is proper splitting such that $U^{\dagger} \geq 0$ and $V \geq 0$. It is called proper weak regular splitting if it is proper splitting such that $U^{\dagger} \geq 0$ and $U^{\dagger}V \geq 0$. Note that Berman and Plemmons [4] proved a convergence theorem for these splittings without specifying the types of matrix decomposition. A matrix $A \in \mathbb{R}^{m\times n}$ is called semi-monotone if $A^{\dagger} \geq 0$. The authors of [10] have considered proper regular splitting of semi monotone matrix A and obtained some comparison results.

Now, we turn our focus on to the comparison results for double splittings that are available in the literature. A decomposition $A = P - R + S$, where P is nonsingular, is called a double splitting of $A \in \mathbb{R}^{n\times n}$. This notion was introduced by *Woźnicki* [19]. With such a splitting, the following iterative scheme was formulated for solving (1):

$$x^{i+1} = P^{-1}Rx^i - P^{-1}Sx^{i-1} + P^{-l}b, \qquad i = 1, 2, 3... \tag{3}$$

Following the idea of Golub and Varga [9] , *Woźnicki* wrote equation (3) in the following equivalent form:

$$\begin{pmatrix} x^{i+1} \\ x^i \end{pmatrix} = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^i \\ x^{i-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ 0 \end{pmatrix},$$

where I is the identity matrix. Then, it was shown that the iterative method (3) converges to the unique solution of (1) for all initial vectors x^0 , x^1 if and only if the spectral radius of the iteration matrix

$$W = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix}$$

is less than one, that is $p(W) < 1$.

Based on this idea, in recent years, several comparison theorems have been proved for double splittings of matrices. We briefly review few of them here. First, let us recall the definitions of regular and weak regular double splittings. A decomposition $A = P - R + S$ is called regular double splitting if $P^{-1} \geq 0$, $R \geq 0$ and $-S \geq 0$; it is called weak regular double splitting if $P^{-1} \geq 0$, $P^{-1}R \geq 0$ and $-P^{-1}S \geq 0$ [15]. Shen and Huang [15] have considered regular and weak regular double splittings of a monotone matrix or Hermitian positive definite matrix and obtained some comparison theorems. Miao and Zheng [12] have obtained comparison theorem for the spectral radii of matrices arising from double splitting of different monotone matrices. Song and Song [17] have studied convergence and comparison theorems for nonnegative double splittings of a real square nonsingular matrices. Li and Wu [11] have obtained some comparison theorems for double splittings of a matrix. Jena et al. [10] and Mishra [13] have introduced the notions of double proper regular splittings and double proper weak regular splittings and derived some comparison theorems. Recently, Alekha kumar and Mishra [1] have considered proper nonnegative double splittings of nonnegative matrix and derived certain comparison theorems.

The main need to study theory of double splittings is due to the fact that some times we cannot ensure the convergence of single splittings using the known results. This problem can be partially settled by studying convergence theory of double splittings. It is important to note that Theorem 2.8 which says that convergent double splitting is also convergent single splitting. Study of convergence theory of proper double splittings further extends the case of double splittings for nonsingular matrix introduced by *Woźnicki* [19]. Standard iterative methods like Jacobi, Gauss-Seidel and SOR etc. can also be obtained by choosing particular matrices in the double splitting of A, and is shown in *Woźnicki* [19].

In this article we generalize the comparison results of Shen and Huang [15] from square nonsingular matrices to rectangular matrices and from classical inverses to Moore-Penrose inverses. Infact, we consider two double splittings $A = P_1 - R_1 + S_1$ and $A = P_2 - R_2 + S_2$ of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$ and derive two comparison theorems for the spectral radii of the corresponding iteration matrices. In section 2, we introduce notations and preliminary results. We present main results in section 3.

2. Notations, Definitions and Preliminaries

In this section, we fix notations and collect basic definitions and preliminary results which will be used in the sequel. Let $\mathbb{R}^{m\times n}$ denote the set of all real matrices with m rows and n columns. For $A \in \mathbb{R}^{m\times n}$, the transpose of A is denoted by A^t; and the matrix $X \in \mathbb{R}^{n\times m}$ satisfying $AXA = A$, $XAX = X$, $(AX)^t = AX$ and $(XA)^t = XA$ is called the Moore-Penrose inverse of A. It always exists and unique, and is denoted by $A^\dagger$. If A is invertible then $A^\dagger = A^{-1}$. Let L and M be complementary subspaces of a real Euclidean space $\mathbb{R}^n$. Then the projection of $\mathbb{R}^n$ on L along M is denoted by $P_{L,M}$. If, in addition, L and M are orthogonal then it is called an orthogonal projection and it is denoted simply by P_L. The following well known properties (see, [2]) of $A^\dagger$, will be used in this manuscript: $\mathcal{R}(A^t) = \mathcal{R}(A^\dagger)$, $\mathcal{N}(A^t) = \mathcal{N}(A^\dagger)$, $AA^\dagger = P_{\mathcal{R}(A)}$, $A^\dagger A = P_{\mathcal{R}(A^t)}$. In particular, if $x \in \mathcal{R}(A^t)$ then $x = A^\dagger Ax$.

A matrix $A \in \mathbb{R}^{m\times n}$ is nonnegative, if all the entries of A are nonnegative, this is denoted $A \geq 0$. The same notation and nomenclature are also used for vectors. For $A, B \in \mathbb{R}^{m\times n}$, we write $B \geq A$ if $B - A \geq 0$.

We now present some results connecting nonnegativity of a matrix and its spectral radius.

Lemma 2.1. *(Theorem 2.1.11, [6]) Let $A \in \mathbb{R}^{n\times n}$ and $A \geq 0$. Then $\alpha x \leq Ax$, $x \geq 0 \Rightarrow \alpha \leq \rho(A)$ and $Ax \leq \beta x$, $x > 0 \Rightarrow \rho(A) \leq \beta$.*

Theorem 2.2. *(Theorem 3.16, [18]) Let $B \in \mathbb{R}^{n\times n}$ and $B \geq 0$. Then $\rho(B) < 1$ if and only if $(I - B)^{-1}$ exists and $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \geq 0$.*

The next theorem is a part of the Perron-Frobenius theorem.

Theorem 2.3. *(Theorem 2.20, [18]) Let $A \in \mathbb{R}^{n\times n}$ and $A \geq 0$. Then*
(i) A has a nonnegative real eigenvalue equal to the spectral radius.
(ii) There exists a nonnegative real eigenvector for its spectral radius.
(iii) If A is irreducible there exists a positive real eigenvector for its spectral radius.

Lemma 2.4. *(Lemma 2.2, [15]) Let $A = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix} \geq 0$ and $\rho(B + C) < 1$. Then, $\rho(A) < 1$.*

As we mentioned in the introduction, a decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper splitting if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$. The next two results are on proper splittings. The first one was used in the proof of Theorem 3 in [4]. However, its proof is easy.

Theorem 2.5. *Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Then $AA^\dagger = UU^\dagger$ and $A^\dagger A = U^\dagger U$.*

Proof. Since $AA^\dagger = P_{\mathcal{R}(A)} = P_{\mathcal{R}(U)} = UU^\dagger$, we get $AA^\dagger = UU^\dagger$. Similarly other part can be proved. □

Theorem 2.6. *(Theorem 3, [4]) Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$ such that $U^\dagger \geq 0$ and $U^\dagger V \geq 0$. Then the following are equivalent:*
(i) $A^\dagger \geq 0$.
(ii) $A^\dagger V \geq 0$.
(iii) $\rho(U^\dagger V) < 1$.

Note that, a proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$, satisfying the conditions $U^\dagger \geq 0$ and $U^\dagger V \geq 0$ is named as proper weak regular splitting by Jena et al. [10].

We now turn to results on double splittings. For $A \in \mathbb{R}^{m\times n}$, a decomposition $A = P - R + S$ is called a *double splitting* of A. A double splitting $A = P - R + S$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper double splitting* if $\mathcal{R}(A) = \mathcal{R}(P)$ and $\mathcal{N}(A) = \mathcal{N}(P)$. Again, consider the following rectangular linear system

$$Ax = b, \tag{4}$$

where $A \in \mathbb{R}^{m\times n}$ (this time A need not be nonsingular), $b \in \mathbb{R}^{m\times 1}$ is a given vector and $x \in \mathbb{R}^{n\times 1}$ is an unknown vector. Similar to the nonsingular case, if we use proper double splitting $A = P - R + S$ to solve (4), it leads to the following iterative scheme:

$$x^{k+1} = P^\dagger R x^k - P^\dagger S x^{k-1} + P^\dagger b, \text{ where } k = 1, 2, ... \tag{5}$$

Motivated by *Woźnicki's* [19] idea, equation (5) can be written as

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} P^\dagger R & -P^\dagger S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^\dagger b \\ 0 \end{pmatrix}.$$

If we denote, $X^{k+1} = \begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix}$, $W = \begin{pmatrix} P^\dagger R & -P^\dagger S \\ I & 0 \end{pmatrix}$, $X^k = \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix}$ and $B = \begin{pmatrix} P^\dagger b \\ 0 \end{pmatrix}$, then we get

$$X^{k+1} = WX^k + B, k = 1, 2... \tag{6}$$

Then, it can be shown that the iterative method (6) converges to to the unique least squares solution of minimum norm, of (4) if and only if $\rho(W) < 1$.

Next, we introduce some subclasses of proper double splittings.

Definition 2.7. *Let $A \in \mathbb{R}^{m\times n}$. A proper double splitting $A = P - R + S$ is called*
(i) regular proper double splitting if $P^\dagger \geq 0$, $R \geq 0$ and $-S \geq 0$.
(ii) weak regular proper double splitting if $P^\dagger \geq 0$, $P^\dagger R \geq 0$ and $-P^\dagger S \geq 0$.

Note that the authors of [10] called regular proper double splittings and weak regular proper double splittings as double proper regular splittings and double proper weak regular splittings, respectively. However, we feel that the present usage is more appropriate and hence we continue the same nomenclature throughout this manuscript.

The next result gives the relation between the spectral radius of the iteration matrices associated with a single splitting and a double splitting.

Theorem 2.8. *(Theorem 4.3, [13]) Let $A = P - R + S$ be a weak regular proper double splitting of $A \in \mathbb{R}^{m\times n}$. Then $\rho(W) < 1$ if and only if $\rho(U^\dagger V) < 1$, where $U = P$ and $V = R - S$.*

We conclude this section with a convergence theorem for a proper double splitting of a monotone matrix.

Theorem 2.9. *(Theorem 3.6, [10]) Let $A \in \mathbb{R}^{m\times n}$ such that $A^\dagger \geq 0$. Let $A = P - R + S$ be a weak regular proper double splitting. Then, $\rho(W) < 1$.*

3. Main Results

In this section, the main results of this article are presented. These results extend the results of Shen and Huang [15] from nonsingular matrices to rectangular matrices and from classical inverses to Moore-Penrose inverses.

Let $A \in \mathbb{R}^{m\times n}$. Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double proper splittings of A. Set $W_1 = \begin{pmatrix} P_1^\dagger R_1 & -P_1^\dagger S_1 \\ I & 0 \end{pmatrix}$ and $W_2 = \begin{pmatrix} P_2^\dagger R_2 & -P_2^\dagger S_2 \\ I & 0 \end{pmatrix}$.

The next result gives the comparison between $\rho(W_1)$ and $\rho(W_2)$. As mentioned earlier, this comparison is useful to analyse the rate of convergence of the iterative methods formulated from these double splittings, for solving linear system $Ax = b$.

Theorem 3.1. *Let $A \in \mathbb{R}^{m\times n}$ be such that $A^\dagger \geq 0$. Let $A = P_1 - R_1 + S_1$ be a regular proper double splitting such that $P_1P_1^\dagger \geq 0$ and let $A = P_2 - R_2 + S_2$ be a weak regular proper double splitting. If $P_1^\dagger \geq P_2^\dagger$ and any one of the following conditions,*
(i) $P_1^\dagger R_1 \geq P_2^\dagger R_2$
(ii) $P_1^\dagger S_1 \geq P_2^\dagger S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. Since $A = P_1 - R_1 + S_1$ is a regular proper double splitting of A, by Theorem 2.9, we get $\rho(W_1) < 1$. Similarly, $\rho(W_2) < 1$. It remains to show that $\rho(W_1) \leq \rho(W_2)$.

Assume that $\rho(W_1) = 0$. Then the conclusion follows, obviously. So, without loss of generality assume that $\rho(W_1) \neq 0$. Since $A = P_1 - R_1 + S_1$ is a regular proper double splitting, we have $W_1 = \begin{pmatrix} P_1^\dagger R_1 & -P_1^\dagger S_1 \\ I & 0 \end{pmatrix} \geq 0$. Then, by the Perron-Frobenius theorem, there exists a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n}$, $x \geq 0$ and $x \neq 0$ such that $W_1 x = \rho(W_1)x$. This implies that

$$P_1^\dagger R_1 x_1 - P_1^\dagger S_1 x_2 = \rho(W_1)x_1. \tag{7}$$
$$x_1 = \rho(W_1)x_2. \tag{8}$$

Upon pre multiplying equation (7) by P_1 and using equation (8), we get

$$[\rho(W_1)]^2 P_1 x_1 = \rho(W_1)P_1P_1^\dagger R_1 x_1 - P_1P_1^\dagger S_1 x_1. \tag{9}$$

We have $P_1P_1^\dagger \geq 0$, $R_1 \geq 0$, $-S_1 \geq 0$ and $x_1 \geq 0$. Therefore, by (9), $[\rho(W_1)]^2 P_1 x_1 \geq 0$. Now, again from (9),

$$\begin{aligned} 0 &= [\rho(W_1)]^2 P_1 x_1 - \rho(W_1)P_1P_1^\dagger R_1 x_1 + P_1P_1^\dagger S_1 x_1 \\ &\leq \rho(W_1)P_1 x_1 - \rho(W_1)P_1P_1^\dagger R_1 x_1 + \rho(W_1)P_1P_1^\dagger S_1 x_1 \\ &= \rho(W_1)[P_1 x_1 - P_1P_1^\dagger(R_1 - S_1)x_1] \\ &= \rho(W_1)[P_1 x_1 - R_1 x_1 + S_1 x_1] \\ &= \rho(W_1)Ax_1, \end{aligned}$$

where we have used the facts that $0 < \rho(W_1) < 1$ and $\mathcal{R}(R_1 - S_1) \subseteq \mathcal{R}(P_1)$. This proves that $Ax_1 \geq 0$.
Also, by using equations (7) and (8), we get

$$\begin{aligned} W_2 x - \rho(W_1)x &= \begin{pmatrix} P_2^\dagger R_2 x_1 - P_2^\dagger S_2 x_2 - \rho(W_1)x_1 \\ x_1 - \rho(W_1)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \nabla \\ 0 \end{pmatrix}, \end{aligned}$$

where $\nabla = (P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1$.
Case(i) Let us assume that $P_1^\dagger R_1 \geq P_2^\dagger R_2$. Since $0 < \rho(W_1) < 1$, we get $(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 \geq \frac{1}{\rho(W_1)}(P_2^\dagger R_2 - P_1^\dagger R_1)x_1$. Then

$$\begin{aligned}
\nabla =&(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1\\
\geq& \frac{1}{\rho(W_1)}(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1\\
=& \frac{1}{\rho(W_1)}[(P_2^\dagger(R_2 - S_2)x_1 - P_1^\dagger(R_1 - S_1)x_1]\\
=& \frac{1}{\rho(W_1)}[P_2^\dagger P_2 - P_2^\dagger A - P_1^\dagger P_1 + P_1^\dagger A]x_1\\
=& \frac{1}{\rho(W_1)}(P_1^\dagger - P_2^\dagger)Ax_1,
\end{aligned} \tag{10}$$

where we have used the fact that $P_1^\dagger P_1 = P_2^\dagger P_2$. Since $Ax_1 \geq 0$ and $P_1^\dagger \geq P_2^\dagger$, from the above inequality, we get $\nabla \geq 0$. Then, $W_2 x - \rho(W_1)x = \begin{pmatrix}\nabla\\0\end{pmatrix} \geq 0$. This implies that $\rho(W_1)x \leq W_2 x$. So, by Lemma 2.1, $\rho(W_1) \leq \rho(W_2)$. This proves that $\rho(W_1) \leq \rho(W_2) < 1$.
Case(ii) Assume that $P_1^\dagger S_1 \geq P_2^\dagger S_2$. Since $0 < \rho(W_1) < 1$ and $Ax_1 \geq 0$, again we get

$$\begin{aligned}
\nabla =&(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1\\
\geq& (P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1\\
=& (P_1^\dagger - P_2^\dagger)Ax_1 \geq 0.
\end{aligned}$$

This implies that $W_2 x - \rho(W_1)x = \begin{pmatrix}\nabla\\0\end{pmatrix} \geq 0$. So, again by Lemma 2.1, we get $\rho(W_1) \leq \rho(W_2)$. This proves that $\rho(W_1) \leq \rho(W_2) < 1$. □

The following example shows that the converse of Theorem 3.1 is not true.

Example 3.2. *Let* $A = \begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\end{pmatrix}$. *Let* $P_1 = \begin{pmatrix}2 & 2 & 0\\0 & 0 & 2\end{pmatrix}$, $R_1 = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$, $S_1 = \begin{pmatrix}0 & -1 & 0\\0 & 0 & 0\end{pmatrix}$, $P_2 = \begin{pmatrix}2 & 2 & 0\\0 & 0 & 1\end{pmatrix}$, $R_2 = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$ *and* $S_2 = \begin{pmatrix}-1 & 0 & 0\\0 & 0 & 0\end{pmatrix}$. *Then* $P_1^\dagger = \frac{1}{4}\begin{pmatrix}1 & 0\\1 & 0\\0 & 2\end{pmatrix}$, $P_1^\dagger R_1 = \frac{1}{4}\begin{pmatrix}1 & 0 & 0\\1 & 0 & 0\\0 & 0 & 2\end{pmatrix}$, $P_1^\dagger S_1 = \frac{1}{4}\begin{pmatrix}0 & -1 & 0\\0 & -1 & 0\\0 & 0 & 0\end{pmatrix}$, $P_2^\dagger = \frac{1}{4}\begin{pmatrix}1 & 0\\1 & 0\\0 & 4\end{pmatrix}$, $P_2^\dagger R_2 = \frac{1}{4}\begin{pmatrix}0 & 1 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$ *and* $P_2^\dagger S_2 = \frac{1}{2}\begin{pmatrix}-1 & 0 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{pmatrix}$. *It is easy to verify that* $A = P_1 - R_1 + S_1$ *is a regular proper double splitting and* $A = P_2 - R_2 + S_2$ *is a weak regular proper double splitting. Also,* $0.5000 = \rho(W_1) \leq \rho(W_2) = 0.5000 < 1$. *However, the conditions* $P_1^\dagger \geq P_2^\dagger$, $P_1^\dagger R_1 \geq P_2^\dagger R_2$ *and* $P_1^\dagger S_1 \geq P_2^\dagger S_2$ *do not hold.*

Corollary 3.3. *(Theorem 3.1, [15]) Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a regular double splitting and* $A = P_2 - R_2 + S_2$ *be a weak regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and any one of the following conditions,*
(i) $P_1^{-1}R_1 \geq P_2^{-1}R_2$
(ii) $P_1^{-1}S_1 \geq P_2^{-1}S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$, *where* $W_1 = \begin{pmatrix}P_1^{-1}R_1 & -P_1^{-1}S_1\\I & 0\end{pmatrix}$ *and* $W_2 = \begin{pmatrix}P_2^{-1}R_2 & -P_2^{-1}S_2\\I & 0\end{pmatrix}$.

Corollary 3.4. *Let $A^{-1} \geq 0$. Let $A = P_1 - R_1 + S_1$ be a regular double splitting and $A = P_2 - R_2 + S_2$ be a weak regular double splitting. If $P_1^{-1} \geq P_2^{-1}$ and $R_1 \geq R_2$ hold, then $\rho(W_1) \leq \rho(W_2) < 1$.*

The conclusion of Theorem 3.1 can also be achieved by replacing a regular proper double splitting $A = P_1 - R_1 + S_1$ with a weak regular proper double splitting; and a weak regular proper double splitting $A = P_2 - R_2 + S_2$ with a regular proper double splitting, in Theorem 3.1. The following is the exact statement of this result.

Theorem 3.5. *Let $A \in \mathbb{R}^{m\times n}$ such that $e = (1, 1, ..., 1)^t \in \mathcal{R}(A)$ and $A^\dagger \geq 0$. Let $A = P_1 - R_1 + S_1$ be a weak regular proper double splitting and let $A = P_2 - R_2 + S_2$ be a regular proper double splitting such that $P_2^\dagger$ has no zero row and $P_2P_2^\dagger \geq 0$. If $P_1^\dagger \geq P_2^\dagger$ and any one of the following conditions,*
(i) $P_1^\dagger R_1 \geq P_2^\dagger R_2$
(ii) $P_1^\dagger S_1 \geq P_2^\dagger S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. Since $A = P_1 - R_1 + S_1$ is a weak regular proper double splitting of A, by Theorem 2.9, we get $\rho(W_1) < 1$. Similarly, $\rho(W_2) < 1$. It remains to show that $\rho(W_1) \leq \rho(W_2)$.

Let J be an $m \times n$ matrix in which each entry is equal to 1. For given $\epsilon > 0$, set $A_\epsilon = A - \epsilon J$, $R_1(\epsilon) = R_1 + \frac{1}{2}\epsilon J$, $S_1(\epsilon) = S_1 - \frac{1}{2}\epsilon J$, $R_2(\epsilon) = R_2 + \frac{1}{2}\epsilon J$, $S_2(\epsilon) = S_2 - \frac{1}{2}\epsilon J$, $W_1(\epsilon) = \begin{pmatrix} P_1^\dagger R_1(\epsilon) & -P_1^\dagger S_1(\epsilon) \\ I & 0 \end{pmatrix}$ and $W_2(\epsilon) = \begin{pmatrix} P_2^\dagger R_2(\epsilon) & -P_2^\dagger S_2(\epsilon) \\ I & 0 \end{pmatrix}$. We have, $e = (1, 1, ..., 1)^t \in \mathcal{R}(A)$. So, there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $J = AB$. Then $A_\epsilon = A - \epsilon J = (A - \epsilon AB) = (A - \epsilon AA^\dagger AB) = (A - \epsilon AA^\dagger J) = A(I - \epsilon A^\dagger J)$. Now, choose the above ϵ such that $\rho(\epsilon A^\dagger J) < 1$ and $\mathcal{N}(A_\epsilon) = \mathcal{N}(A)$. Since $\rho(\epsilon A^\dagger J) < 1$, $I - \epsilon A^\dagger J$ is invertible and hence $\mathcal{R}(A_\epsilon) = \mathcal{R}(A)$. Then $A_\epsilon = A - \epsilon J$ becomes a proper splitting and thus we can conclude that $A_\epsilon = P_1 - R_1(\epsilon) + S_1(\epsilon)$ is a weak regular proper double splitting and $A_\epsilon = P_2 - R_2(\epsilon) + S_2(\epsilon)$ is a regular proper double splitting.

For the same ϵ, define $X = (I - \epsilon A^\dagger J)^{-1}A^\dagger$, we shall prove that X is the Moore-Penrose inverse of A_ϵ. Let $x \in \mathcal{R}(A_\epsilon^t)$. Then

$$\begin{aligned} XA_\epsilon x &= (I - \epsilon A^\dagger J)^{-1}A^\dagger(A - \epsilon AA^\dagger J)x \\ &= (I - \epsilon A^\dagger J)^{-1}(A^\dagger Ax - \epsilon A^\dagger AA^\dagger Jx) \\ &= (I - \epsilon A^\dagger J)^{-1}(x - \epsilon A^\dagger Jx) \\ &= x \end{aligned}$$

and for $y \in \mathcal{N}(A_\epsilon^t)$, we get

$$Xy = (I - \epsilon A^\dagger J)^{-1}A^\dagger y = 0$$

Hence, by the definition, $A_\epsilon^\dagger = X = (I - \epsilon A^\dagger J)^{-1}A^\dagger$. Also, $A_\epsilon^\dagger = (I + \epsilon A^\dagger J + \epsilon(A^\dagger J)^2 + ...)A^\dagger \geq 0$. Then $\rho(P_2^\dagger(R_2(\epsilon) - S_2(\epsilon))) < 1$. So, by Lemma 2.4, $\rho(W_2(\epsilon)) < 1$.

Clearly, $P_2^\dagger R_2(\epsilon) > 0$ and $-P_2^\dagger S_2(\epsilon) > 0$. So, $W_2(\epsilon) \geq 0$ and irreducible. Then, by the Perron-Frobenius theorem, there exists a vector $x(\epsilon) = \begin{pmatrix} x_1(\epsilon) \\ x_2(\epsilon) \end{pmatrix} \in \mathbb{R}^{2n}$, $x(\epsilon) > 0$ such that $W_2(\epsilon)x(\epsilon) = \rho(W_2(\epsilon))x(\epsilon)$. This implies,

$$P_2^\dagger R_2(\epsilon)x_1(\epsilon) - P_2^\dagger S_2(\epsilon)x_2(\epsilon) = \rho(W_2(\epsilon))x_1(\epsilon) \tag{11}$$

$$x_1(\epsilon) = \rho(W_2(\epsilon))x_2(\epsilon). \tag{12}$$

If $\rho(W_2(\epsilon)) = 0$ then from equations (11) and (12), $x(\epsilon) = 0$. This is a contradiction. So, $0 < \rho(W_2(\epsilon)) < 1$. Then by using equations (11) and (12), as in the proof of the Theorem 3.1, we can show that $\rho(W_2(\epsilon))A_\epsilon x_1(\epsilon) \geq 0$.

This implies that $A_\epsilon x_1(\epsilon) \geq 0$. Also, from equations (11) and (12), we get

$$\begin{aligned}
&W_1(\epsilon)x(\epsilon) - \rho(W_2(\epsilon))x(\epsilon)\\
&= \begin{pmatrix} P_1^\dagger R_1(\epsilon)x_1(\epsilon) - P_1^\dagger S_1(\epsilon)x_2(\epsilon) - \rho(W_2(\epsilon))x_1(\epsilon) \\ x_1(\epsilon) - \rho(W_2(\epsilon))x_2(\epsilon) \end{pmatrix}\\
&= \begin{pmatrix} (P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon) \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} \nabla \\ 0 \end{pmatrix},
\end{aligned}$$

where $\nabla = (P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon)$.

Case(*i*) Assume that $P_1^\dagger R_1 \geq P_2^\dagger R_2$. Since $0 < \rho(W_2(\epsilon)) < 1$, we get that $(P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) \leq \frac{1}{\rho(W_2(\epsilon))}(P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon)$. Therefore,

$$\begin{aligned}
\nabla &\leq \frac{1}{\rho(W_2(\epsilon))}(P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon)\\
&= \frac{1}{\rho(W_2(\epsilon))}[(P_1^\dagger(R_1(\epsilon) - S_1(\epsilon))x_1(\epsilon) - P_2^\dagger(R_2(\epsilon) - S_2(\epsilon))x_1(\epsilon)]\\
&= \frac{1}{\rho(W_2(\epsilon))}[P_1^\dagger P_1 - P_1^\dagger A_\epsilon - P_2^\dagger P_2 + P_2^\dagger A_\epsilon]x_1(\epsilon)\\
&= \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger - P_1^\dagger)A_\epsilon x_1(\epsilon),
\end{aligned}$$

where we have used the fact that $P_1^\dagger P_1 = P_2^\dagger P_2$. Since $A_\epsilon x_1(\epsilon) \geq 0$ and $P_1^\dagger \geq P_2^\dagger$, we get that $\nabla \leq 0$. Thus, $W_1(\epsilon)x(\epsilon) - \rho(W_2(\epsilon))x(\epsilon) = \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \leq 0$. This implies, $W_1(\epsilon)x(\epsilon) \leq \rho(W_2(\epsilon))x(\epsilon)$. So, by Lemma 2.1, $\rho(W_1(\epsilon)) \leq \rho(W_2(\epsilon))$.

Now, from the continuity of eigenvalues, we have

$$\rho(W_1) = \lim_{\epsilon \to 0} \rho(W_1(\epsilon)) \leq \lim_{\epsilon \to 0} \rho(W_2(\epsilon)) = \rho(W_2).$$

Case(*ii*) Assume that $P_1^\dagger S_1 \geq P_2^\dagger S_2$. We have $\rho(\epsilon A^\dagger J) < 1$. Choose the above ϵ small enough such that

$$P_1^\dagger S_1 - P_2^\dagger S_2 \geq \frac{\epsilon}{2}(P_1^\dagger - P_2^\dagger)J.$$

Since, $P_1^\dagger S_1(\epsilon) \geq P_2^\dagger S_2(\epsilon)$, $A_\epsilon^\dagger \geq 0$ and $0 < \rho(W_2) < 1$, we get

$$\begin{aligned}
\nabla &\leq (P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + (P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon)\\
&= (P_2^\dagger - P_1^\dagger)A_\epsilon x_1(\epsilon) \leq 0.
\end{aligned}$$

This implies that $W_1(\epsilon)x(\epsilon) - \rho(W_2(\epsilon))x(\epsilon) = \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \leq 0$.

So, $W_1(\epsilon)x(\epsilon) \leq \rho(W_2(\epsilon))x(\epsilon)$. Then, by Lemma 2.1, $\rho(W_1(\epsilon)) \leq \rho(W_2(\epsilon))$. Similar to the proof of case(*i*), this implies that $\rho(W_1) \leq \rho(W_2)$. □

The following example illustrates Theorem 3.5.

Example 3.6. *Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *then* $A^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \geq 0$. *Set* $P_1 = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. $P_2 = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $S_2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -3 & 0 \end{pmatrix}$. *Then* $P_1^\dagger = \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$,

$P_1^{\dagger}R_1 = \frac{1}{6}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ *and* $P_1^{\dagger}S_1 = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $P_2^{\dagger} = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$, $P_2^{\dagger}R_2 = \frac{1}{8}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ *and* $P_2^{\dagger}S_2 = \frac{1}{8}\begin{pmatrix} -1 & 0 & -1 \\ 0 & -6 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.
Note that $A = P_1 - R_1 + S_1$ *is a weak regular proper double splitting and* $A = P_2 - R_2 + S_2$ *is a regular proper double splitting. Also,* $e \in \mathcal{R}(A)$, $P_2^{\dagger}$ *has no zero row and* $P_2P_2^{\dagger} \geq 0$. *We can verify that* $P_1^{\dagger} \geq P_2^{\dagger}$, *and* $P_1^{\dagger}R_1 \geq P_2^{\dagger}R_2$. *Hence* $0.7676 = \rho(W_1) \leq \rho(W_2) = 0.8660 < 1$.

The following result is an obvious consequence of Theorem 3.5

Corollary 3.7. *(Theorem 3.2, [15]) Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a weak regular double splitting and* $A = P_2 - R_2 + S_2$ *be a regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and any one of the following conditions,*
(i) $P_1^{-1}R_1 \geq P_2^{-1}R_2$
(ii) $P_1^{-1}S_1 \geq P_2^{-1}S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

The next result proof is similar to the proof of Theorem 3.1. Thus we skip the proof.

Theorem 3.8. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $A^{\dagger} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a weak regular proper double splitting and* $A = P_2 - R_2 + S_2$ *be a weak regular proper double splitting. If* $P_1^{\dagger}A \geq P_2^{\dagger}A$ *and any of the following conditions,*
(i) $P_1^{\dagger}R_1 \geq P_2^{\dagger}R_2$
(ii) $P_1^{\dagger}S_1 \geq P_2^{\dagger}S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

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