Filomat 32:6 (2018), 2295–2305
https://doi.org/10.2298/FIL1806295W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Certain Class of q-Close-to-Convex Functions of Order α

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Abstract. For every $0 < q < 1$ and $0 \leq \alpha < 1$, we introduce a class of analytic functions f on the open unit disc $\mathbb{D}$ with the standard normalization $f(0) = 0 = f'(0) - 1$ and satisfying

$$\left| \frac{1}{1-\alpha} \left(\frac{z(D_q f)(z)}{h(z)} - \alpha \right) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (z \in \mathbb{D}),$$

where $h \in \mathcal{S}_q^*$. This class is denoted by $\mathcal{K}_q(\alpha)$, so called the class of q-close-to-convex- functions of order α. In this paper, we study some geometric properties of this class. In addition, we consider the famous Bieberbach conjecture problem on coefficients for the class $\mathcal{K}_q(\alpha)$. We also find some sufficient conditions for the function to be in $\mathcal{K}_q(\alpha)$ for some particular choices of the functions h. Finally, we provide some applications on q-analogue of Gaussian hypergeometric function.

1. Introduction and Preliminaries

Let $\mathcal{H}(\mathbb{D})$ be the class of analytic functions in $\mathbb{D}$ and $\mathcal{A}$ be the class of analytic functions normalized by the conditions that $f(0) = 0$ and $f'(0) = 1$, that is $f \in \mathcal{A}$ can be written as of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions, which are univalent on $\mathbb{D}$.

In the field of geometric functions theory, the class, as well as subclasses, of univalent functions has been widly studied by several researchers. There are many distinguished geometric properties that played an important role in the theory of univalent functions, such as starlikeness, convexity, close-to-convexity. A function $f \in \mathcal{A}$ is said to be starlike with respect to w_0 if f maps $\mathbb{D}$ onto starlike domain with respect to w_0, In the special case that $w_0 = 0$, we say that f is a starlike function. Also, a function $f \in \mathcal{A}$ is said to be convex function if f maps $\mathbb{D}$ onto a convex domain. The classes of all starlike and convex functions are respectively denoted by $\mathcal{S}^*$ and $\mathcal{C}$. More generally, for $0 \leq \alpha \leq 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of $\mathcal{S}$ consisting of respectively starlike functions of order α, and convex functions of order α. Analytically,

2010 *Mathematics Subject Classification*. Primary 30C45
Keywords. Analytic function, Univalent function, q-Calculus, Close-to-convex function of order α
Received: 17 May 2017; Accepted: 27 October 2017
Communicated by Hari M. Srivastava
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these classes are defined by the following characterizations. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \le \alpha < 1)$ in $\mathbb{D}$ if f satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{D}). \tag{2}$$

We denote this class by $\mathcal{S}^*(\alpha)$. On the other hand, a function $f \in \mathcal{A}$ is said to be convex of order α in $\mathbb{D}$ if f satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

We denote this class by $C(\alpha)$. In particular, we set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ for a class of starlike functions and $C(0) \equiv C$ for a class of convex functions.

We say that an analytic function $f \in \mathcal{K}(\alpha)$, say close-to-convex function of order α, if there exists a function $h \in \mathcal{S}^*$ such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{h(z)}\right\} > \alpha \quad (z \in \mathbb{D}). \tag{3}$$

In particular, when $h(z) = z$, the class $\mathcal{K}(\alpha)$ is exactly the class of bounded turning function of order α.

There are many ways to study subclasses of analytic functions, especially subclasses of univalent functions. In [1], Deng studied some sharp properties on univalent functions with negative coefficients. In [2], M.N. Pascu and N.R. Pascu studied the best starlike univalent approximations of analytic functions problem, and they also solved this kind of problem on the subclass of the convex function in [3]. Many subclasses of starlike and convex functions were intensively introduced and studied by many authors. In [4], Aouf and Srivastava introduced and investigated some families of starlike functions with negative coefficient by using Salagean operator. Kanas and Wisniowska [5] introduced the class of k-uniformly convex functions and obtainded some necessary and sufficient for functions in this class. In [6], Kanas and Srivastava provided some sufficient conditions for the operators in term of the Hadamard product in order to map the class of starlike and univalent functions onto the class of k-uniformly convex and k-starlike functions. Some properties of the classes of k-uniformly close-to-convex functions and k-uniformly quasi-convex functions defined by the Dziok-Srivastava operator were studied by Srivastava et.al. [7]. For some interesting properties and related topics on subclass of univalent functions, we refer to [8–15].

Let f and g be an analytic function in $\mathbb{D}$, we say that f is subordinate to g if there exists a Schwarz function $\omega : \mathbb{D} \to \mathbb{D}$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$, written as $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. For any fixed complex number μ, a set $A \subset \mathbb{C}$ is called a μ–geometric set if for $z \in A$, $\mu z \in A$. Let f be a function defined on a q–geometric set. The Jackson's q-derivative and q-integral of a function on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [16], pp.19–22)

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0,\ q \neq 0), \tag{4}$$

and

$$\int_0^z f(t)d_qt = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k). \tag{5}$$

In case $f(z) = z^n$, the q-derivative and q-integral of $f(z)$, where n is a positive integer, is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q z^{n-1}$$

and

$$\int_0^z t^n d_q t = z(1-q)\sum_{k=0}^{\infty} q^k (zq^k)^n = \frac{z^{n+1}}{[n+1]_q}.$$

As $q \to 1^-$ and $n \in \mathbb{N}$, we have $[n]_q = \dfrac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1} \to n$. In the theory of q-calculus, the q–shifted factorial is defined for $\alpha, q \in \mathbb{C}$, $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ as a product of n factors by

$$(\alpha; q)_n = \begin{cases} 1 & , \ n = 0; \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & , \ n \in \mathbb{N}, \end{cases} \tag{6}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n > 0),$$

where the q–gamma function [16, 17] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0 < q < 1).$$

We note that, if $|q| < 1$, the q–shifted factorial (6) remains meaningful for $n = \infty$ as a convergent infinite product:

$$(\alpha; q)_\infty = \prod_{k=0}^{\infty} (1-\alpha q^k).$$

Here, we recall the following q-analogue definitions given by Gasper and Rahman [16]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x),$$

where $[x]_q = (1-q^x)/(1-q)$ and is called q–analogue of x. It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^-$, where $\Gamma(x)$ is the ordinary Euler gamma function.

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n,$$

we observe that the q–shifted factorial (6) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$. Here, we recall the q-analogue of Gaussian hypergeometric function is as the following form (see [18])

$${}_2\Phi_1(a, b; c; q; z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n, \quad (z \in \mathbb{D}).$$

Up to date, there are many applications of q-calculus on subclasses of analytic functions, especially generalization of subclasses of univalent functions (see [19–30]). In the context of geometric function theory, the usage of q-calculus was firstly applied in a book chapter by Srivastava [19], in which the basis q-hypergeometric functions was also provided. In [20], Ismail et al. introduced the generalized starlike function by replacing the usual derivative with q-difference operator D_q and the right-half plane $\{w : \mathrm{Re}\, w > \alpha\}$ was substituted by an appropriate domain. By extending this idea, Agrawal and Sahoo in [21], introduced $\mathcal{S}_q^*(\alpha)$ the class of q-starlike functions of order α. The definition turned out to be the following:

Definition 1.1. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*(\alpha)$, $0 \le \alpha < 1$, if*

$$\left|\frac{1}{1-\alpha}\left(\frac{z(D_q f)(z)}{f(z)} - \alpha\right) - \frac{1}{1-q}\right| < \frac{1}{1-q}, \quad (z \in \mathbb{D}). \tag{7}$$

From this definition, this equivalent to the following form

$$f \in \mathcal{S}_q^*(\alpha) \iff \left|\frac{z(D_q f)(z)}{f(z)} - \frac{1-\alpha q}{1-q}\right| < \frac{1-\alpha}{1-q}.$$

Moreover, by using the concept of the well-known Alexander duality between starlike and convex functions, this lead us to consider the class of q-convex function of order α by

$$f \in C_q(\alpha) \iff z(D_q f)(z) \in \mathcal{S}_q^*(\alpha).$$

We note that if $q \to 1^-$, the class $C_q(\alpha)$ reduces to the class of usual convex functions of order α. For several interesting geometric properties related to both classes, we refer to [22–24].

Later, Raghavendar and Swaminathan [25] defined and investigated the class of q-analog of close-to-convex functions by using the similar idea as above. The definition turned out to be as following:

Definition 1.2. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{PK}_q$, if there exists $h \in \mathcal{S}^*$ such that*

$$\left|\frac{z(D_q f)(z)}{h(z)} - \frac{1}{1-q}\right| < \frac{1}{1-q}, \quad (z \in \mathbb{D}). \tag{8}$$

Motivated by the study along this line, we introduce the class of q-close-to-convex functions of order α. Moreover, we replace the function $h \in \mathcal{S}^*$ in Definition 2 by a weaker condition $h \in \mathcal{S}_q^*$. The definition turns out to be as following:

Definition 1.3. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{K}_q(\alpha)$, $0 \le \alpha < 1$, if there exists $h \in \mathcal{S}_q^*$ such that*

$$\left|\frac{1}{1-\alpha}\left(\frac{z(D_q f)(z)}{h(z)} - \alpha\right) - \frac{1}{1-q}\right| < \frac{1}{1-q}, \quad (z \in \mathbb{D}). \tag{9}$$

Here, we note that the definition equivalent to the following form

$$f \in \mathcal{K}_q^*(\alpha) \iff \left|\frac{z(D_q f)(z)}{h(z)} - \frac{1-\alpha q}{1-q}\right| < \frac{1-\alpha}{1-q},$$

where $h \in \mathcal{S}_q^*$. In particular, we set $\mathcal{K}_q(0) \equiv \mathcal{K}_q$. In fact, the class $\mathcal{K}_q$ generalize the class $\mathcal{PK}_q$ in [25] due to the following relation (see [20])

$$\mathcal{S}^* = \bigcap_{0<q<1} \mathcal{S}_q^*.$$

As $(D_q f)(z) \to f'(z)$, as $q \to 1^-$, we observe that the class $\mathcal{K}_q(\alpha)$ satisfies the following relation

$$\bigcap_{0<q<1} \mathcal{K}_q(\alpha) \subset \mathcal{K}(\alpha) \subset \mathcal{K} \subset \mathcal{S}.$$

The main purpose of this paper is to introduce and investigate some geometric properties on the class $\mathcal{K}_q(\alpha)$. In the main results, we obtain a characterization for the function belonging to $\mathcal{K}_q(\alpha)$ by using the concept of subordinate property. We also study on radius of q-convexity and coefficient bounds for $\mathcal{K}_q(\alpha)$. Moreover, we derive some sufficient conditions for functions to be in $\mathcal{K}_q(\alpha)$ for some particular choices of the function h. Some applications on q-analogue of Gaussian hypergeometric function are also obtained.

Main results

We begin this section with a characterization for the function in $f \in \mathcal{K}_q(\alpha)$ via subordination symbol.

Theorem 1.4. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{K}_q(\alpha)$ if and only if*

$$\frac{zD_q f(z)}{h(z)} \prec \frac{1 + (1 - \alpha(1 + q))}{1 - qz},$$

where $h \in \mathcal{S}^*_q$.

Proof. Let $f \in \mathcal{K}_q(\alpha)$, then there exists a analytic function $h \in \mathcal{S}^*_q$ such that

$$\left| z\frac{D_q f(z)}{h(z)} - \frac{1 - \alpha q}{1 - q} \right| \le \frac{1 - \alpha}{1 - q}. \tag{10}$$

This leads us to introduce the function

$$\Psi(z) = \frac{1 - q}{1 - \alpha} z \frac{D_q f(z)}{h(z)} - \frac{1 - \alpha q}{1 - \alpha},$$

which is $|\Psi(z)| < 1$, for $z \in \mathbb{D}$. Let

$$\omega(z) = \frac{\Psi(z) - \Psi(0)}{1 - \overline{\Psi(0)}\Psi(z)} = \frac{\frac{zD_q f(z)}{h(z)} - 1}{(1 - \alpha) + q\frac{zD_q f(z)}{h(z)}}. \tag{11}$$

We note that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Moreover, from Eq. (11), we have

$$\frac{zD_q(z) f(z)}{h(z)} = \frac{1 + (1 - \alpha(1 + q))\omega(z)}{1 - q\omega(z)}. \tag{12}$$

This implies that

$$\frac{zD_q(z) f(z)}{h(z)} \prec \frac{1 + (1 - \alpha(1 + q))z}{1 - qz}. \tag{13}$$

Conversely, we assume that Eq. (13) holds. That is there exists a function $\omega : \mathbb{D} \to \mathbb{D}$ such that

$$\frac{zD_q(z) f(z)}{h(z)} = \frac{1 + (1 - \alpha(1 + q))\omega(z)}{1 - q\omega(z)}.$$

We observe that

$$\begin{aligned} \frac{zD_q(z) f(z)}{h(z)} - \frac{1 - \alpha q}{1 - q} &= \frac{1 + (1 - \alpha(1 + q))\omega(z)}{1 - q\omega(z)} - \frac{1 - \alpha q}{1 - q} \\ &= \frac{1 - \alpha}{1 - q}\left(\frac{-q + \omega(z)}{1 - q\omega(z)}\right). \end{aligned}$$

Since the pseudo-hyperbolic distance between $-q$ and $\omega(z)$ is less than 1, then we have

$$\left| \frac{zD_q(z) f(z)}{h(z)} - \frac{1 - \alpha q}{1 - q} \right| = \frac{1 - \alpha}{1 - q} \left| \frac{-q + \omega(z)}{1 - q\omega(z)} \right| \le \frac{1 - \alpha}{1 - q}.$$

The proof is completed. $\square$

Using the same argument as Theorem 1.4, we obtain the following corollary.

Corollary 1.5. *Let* $f \in \mathcal{A}$. *Then* $f \in \mathcal{S}_q^*(\alpha)$ *if and only if*

$$\frac{zD_qf(z)}{f(z)} \prec \frac{1+(1-\alpha(1+q))z}{1-qz}.$$

Next, we consider the radius of q-convexity for the class $\mathcal{K}_q(\alpha)$.

Theorem 1.6. *If* $\alpha < 1/(1+q)$, *then the radius of q-convexity of the class* $\mathcal{K}_q(\alpha)$ *is the unique root of the polynomial*

$$p(r) = (1-\alpha)(1-q^2)(1-qr)r + ((1+q)r-(1-q))(1-Ar-Br^2)$$

defined on the interval $(0,1)$, *where* $A = |1-q-\alpha(1+q)|$ *and* $B = |1-\alpha(1+q)|$.

Proof. Using Theorem 1.4, there exits an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ such that

$$\frac{zD_qf(z)}{h(z)} = \frac{1+(1-\alpha(1+q))\omega(z)}{1-q\omega(z)}. \tag{14}$$

By logarithmic q-differentiation of Eq. (14), we have

$$\frac{\ln q}{q-1}\left[\frac{D_q(zD_qzf(z))}{zD_qzf(z)} - \frac{D_qh(z)}{h(z)}\right] = \frac{\ln q}{q-1}\left[\frac{(1-\alpha(1+q)D_q\omega(z))D_q\omega(z)}{1+(1-\alpha(1+q))\omega(z)} + \frac{qD_q\omega(z)}{1-q\omega(z)}\right]. \tag{15}$$

Hence

$$\frac{D_q(zD_qzf(z))}{zD_qzf(z)} - \frac{D_qh(z)}{h(z)} = \frac{(1-\alpha)(1+q)D_q\omega(z)}{1+(1-q-\alpha(1+q))\omega(z)-q(1-\alpha(1+q)))\omega^2(z)}.$$

Therefore, for $|z| < r$, we have

$$\begin{aligned}\left|\frac{zD_q(zD_qzf(z))}{zD_qzf(z)} - \frac{1}{1-q}\right| &\le \frac{(1-\alpha)(1+q)|zD_q\omega(z)|}{1-|(1-q-\alpha(1+q))||\omega(z)|-|q(1-\alpha(1+q)))||\omega(z)|^2} + \left|\frac{zD_qh(z)}{h(z)} - \frac{1}{1-q}\right| \\ &\le \frac{(1-\alpha)(1+q)|zD_q\omega(z)|}{1-Ar-Br^2} + \left|\frac{zD_qh(z)}{h(z)} - \frac{1}{1-q}\right|,\end{aligned} \tag{16}$$

where $A = |1-q-\alpha(1+q)|$ and $B = |1-\alpha(1+q)|$. Applying Corollary 1.5 with $\alpha = 0$, there exits an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ such that

$$\frac{zD_qf(z)}{h(z)} = \frac{1+\varphi(z)}{1-q\varphi(z)}.$$

So, for $|z| < r$, we have

$$\left|\frac{zD_qh(z)}{h(z)} - \frac{1}{1-q}\right| = \left|\frac{1+\varphi(z)}{1-q\varphi(z)} - \frac{1}{1-q}\right| \le \frac{q+r}{(1-q)(1-qr)}. \tag{17}$$

On the other hand, using the Lemma 6 in [27], we obtain

$$|D_q\omega(z)| \le \frac{1-|\omega(z)|^2}{1-|z|^2}. \tag{18}$$

From Eqs. (16)-(18), we get

$$\left|\frac{D_q(zD_qzf(z))}{zD_qzf(z)} - \frac{1}{1-q}\right| \le \frac{(1-\alpha)(1+q)r}{1-Ar-Br^2} + \frac{q+r}{(1-q)(1-qr)}.$$

To complete the proof, we need to find the smallest $r_0 \in (0,1)$ satisfying the equation

$$\frac{(1-\alpha)(1+q)r}{1-Ar-Br^2}+\frac{q+r}{(1-q)(1-qr)}=\frac{1}{1-q},$$

or

$$\frac{(1-\alpha)(1-q^2)(1-qr)r-((1+q)r+(1-q))(1-Ar-Br^2)}{(1-qr)(1-Ar-Br^2)}=0.$$

Now let us consider the polynomial

$$p(r)=(1-\alpha)(1-q^2)(1-qr)r+((1+q)r-(1-q))(1-Ar-Br^2).$$

We observe that $p(0)=-(1-q)<0$. By the assumption $\alpha<\frac{1}{1+q}$, we then distinguish following cases ($\alpha<\frac{1-q}{1+q}$ and $\frac{1-q}{1+q}<\alpha<\frac{1}{1+q}$).

If $\alpha<\frac{1-q}{1+q}$, we have $A=1-q-\alpha(1+q)$ and $B=1-\alpha(1+q)$. Then

$$p(1)=(1-\alpha)(1-q^2)+2q(2\alpha(1+q)-(1-q))=(1-q)(1-q-\alpha(1+q))+4q\alpha(1+q)>0.$$

If $\frac{1-q}{1+q}<\alpha<\frac{1}{1+q}$, we have $A=\alpha(1+q)-(1-q)$ and $B=1-\alpha(1+q)$. Then

$$p(1)=(1-\alpha)(1-q^2)+2q(1-q)>0.$$

Both cases show us that $p(1)>0$ for all $\alpha<1/(1+q)$. This guarantees the existence of the smallest positive root r_0 of the equation $p(r)=0$ lies between 0 and 1. This completes the proof. □

In order to obtain the optimal coefficient estimates for the class $\mathcal{K}_q(\alpha)$, we need another characterization for functions in $\mathcal{K}_q(\alpha)$. The proof follows immediately by the definition of q-difference operator.

Lemma 1.7. *Let $f\in\mathcal{A}$. Then $f\in\mathcal{K}_q(\alpha)$ if and only if*

$$\frac{|(1-\alpha q)h+f(qz)-f(z)|}{h(z)}\le 1-\alpha.$$

In the next theorem, we obtain the coefficient estimates for the class $\mathcal{K}_q(\alpha)$.

Theorem 1.8. *Let $f\in\mathcal{K}_q(\alpha)$, then*

$$|a_n|\le\frac{1-q}{1-q^n}\left[\frac{1-q^2}{1-q^n}\prod_{k=2}^{n-1}\left(1+\frac{1-q^2}{q-q^k}\right)+(1-\alpha)(1+q)\sum_{k=1}^{n}\prod_{j=2}^{k-1}\left(1+\frac{1-q^2}{q-q^j}\right)\right],\quad n\ge 2.$$

Proof. Let $f\in\mathcal{K}_q(\alpha)$. By applying Lemma 1.7, there exists $w:\mathbb{D}\to\overline{\mathbb{D}}$ such that

$$(1-\alpha q)h(z)+f(qz)-f(z)=(1-\alpha)w(z)h(z).$$

Here we note that $w(0)=q$. Using Taylor series expansion of f,h and w, we have

$$\sum_{n=1}^{\infty}((1-\alpha q)b_n+a_nq^n-a_n)z^n=(1-\alpha)\sum_{n=1}^{\infty}qb_nz^n+(1-\alpha)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}w_{n-k}b_k\right),$$

where $a_1=b_1=1$ and $h(z)=\sum_{k=1}^{\infty}b_kz^k$ and $w(z)=q+\sum_{k=1}^{\infty}w_kz^k$. Let consider the coefficient of z^n, for $n\ge 2$, we have

$$a_n(q^n-1)=(q-1)b_n+(1-\alpha)\sum_{k=1}^{n-1}w_{n-k}b_k.$$

Since $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$, we get

$$|a_n| \leq \frac{1-q}{1-q^n}\left(|b_n| + (1-\alpha)(1+q)\sum_{k=1}^{n-1}|b_k|\right). \tag{19}$$

To complete the proof, we use the Bieberbach conjecture result for $\mathcal{S}_q^*$ in [26], we obtain the estimate of $|b_n|$ as

$$|b_n| \leq \frac{1-q^2}{q-q^n}\prod_{k=2}^{n-1}\left(1 + \frac{1-q^2}{q-q^k}\right), \tag{20}$$

for $n \geq 2$. From Eqs. (19)-(20), we complete the proof. □

When the function f is in the class $\mathcal{K}_q(\alpha)$ with the function $h \in \mathcal{S}^*$, we obtain the following result.

Corollary 1.9. *If $f \in \mathcal{K}_q(\alpha)$ with $h \in \mathcal{S}^*$ then*

$$|a_n| \leq \frac{1-q}{q-q^n}\left[n + (1-\alpha)(1+q)\frac{n(n-1)}{2}\right].$$

Proof. The proof is directly obtained by the Bieberbach-de Branges theorem for starlike functions [31] and Eq. (20). □

When $q \to 1^-$, Theorem 1.8 and Corollary 1.9 yield the Bieberbach conjecture problem for closed-to-convex functions of order α. Moreover, in corollary 1.9, we note that the series

$$z + \sum_{n=2}^{\infty}\frac{1-q}{1-q^n}\left[n + (1-\alpha)(1+q)\frac{n(n-1)}{2}\right]z^n \tag{21}$$

converges for $|z| < 1$ by using the ratio test. In fact, by using the definition of the Heine hypergeometric function one can easily see that the series given by Eq. (21) converges to the function

$$\frac{(1-\alpha)(1+q)}{2}z^2\frac{d^2(z_2\Phi_1(q,q;q;q;z))}{dz^2} + z\frac{d(z_2\Phi_1(q,q;q;q;z))}{dz}.$$

Next, modification the idea in [25], we concentrate on sufficient conditions when we take the particular choice of h. These are as indicated below:

$$\frac{1}{1 \pm z}, \quad \frac{1}{1 \pm z^2}, \quad \frac{1}{(1 \pm z)^2}, \quad \frac{1}{1 \pm z + z^2},$$

and identity function. Each of these maps the unit disk $\mathbb{D}$ onto starlike domains. So, every choice of h is in the class $\mathcal{S}^* \subset \mathcal{S}_q^*$.

Theorem 1.10. *Let $f \in \mathcal{A}$ and $B_0 = 0$, $B_1 = 1$, and $B_n = [n]_q a_n := \frac{1-q^n}{1-q}a_n$. Then we have the following:*

(1) If $\sum_{n=2}^{\infty}|B_n| \leq 1 - \alpha$, then $f \in \mathcal{K}_q(\alpha)$ with $h(z) = z$.

(2) If $\sum_{n=1}^{\infty}|B_{n+1} \pm B_n| \leq 1 - \alpha$, then $f \in \mathcal{K}_q(\alpha)$ with $h(z) = z/(1 \pm z)$.

(3) If $\sum_{n=1}^{\infty}|B_{n+1} \pm B_n + B_{n+1}| \leq 1 - \alpha$, then $f \in \mathcal{K}_q(\alpha)$ with $h(z) = z/(1 \pm z + z^2)$.

(4) If $\sum_{n=1}^{\infty}|B_{n+1} \pm 2B_n + B_{n+1}| \leq 1 - \alpha$, then $f \in \mathcal{K}_q(\alpha)$ with $h(z) = z/(1 \pm z)^2$.

(5) If $\sum_{n=1}^{\infty}|B_{n+1} \pm B_{n+1}| \leq 1 - \alpha$, then $f \in \mathcal{K}_q(\alpha)$ with $h(z) = z/(1 \pm z^2)$.

Proof. (1) Suppose that $\sum_{n=2}^{\infty} |B_n| \le 1 - \alpha$, we see that

$$|a_n| \le \frac{1-\alpha}{1+q+q^2+\cdots+q^{n-1}}.$$

By applying the root test, we see that the radius of convergence of $f(z)$ is not less than unity. That is $f \in \mathcal{A}$. We then show that f is in $\mathcal{K}_q(\alpha)$ with $g(z) = z$. That is we need to show that

$$\left|(D_q f)(z) - \frac{1-\alpha q}{1-q}\right| \le \frac{1-\alpha}{1-q}.$$

By the definition of $q-$difference operator and applying the triangle inequality, we see that

$$\begin{aligned}
\frac{1-\alpha}{1-q} - \left|(D_q f)(z) - \frac{1-\alpha q}{1-q}\right| &= \frac{1-\alpha}{1-q} - \left|1 + \sum_{k=2}^{\infty} B_n z^{n-1} - \frac{1-\alpha q}{1-q}\right| \\
&\ge \frac{1-\alpha}{1-q} - \left|1 - \frac{1-\alpha q}{1-q}\right| - \left|\sum_{k=2}^{\infty} B_n z^{n-1}\right| \\
&\ge 1-\alpha - \left|\sum_{k=2}^{\infty} B_n z^{n-1}\right| \ge 0,
\end{aligned}$$

by hypothesis. Hence $f \in \mathcal{K}_q(\alpha)$ with $h(z) = z$.

(2) Suppose that $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \le 1 - \alpha$. Consider

$$|B_n| = \left|1 + \sum_{k=1}^{n} (B_k - B_{k-1})\right| \le 1 + \sum_{k=1}^{\infty} |B_k - B_{k-1}| \le 2 - \alpha,$$

hence, we have

$$|a_n| \le \frac{2-\alpha}{1+q+q^2+\cdots+q^{n-1}}.$$

By applying the root test, we see that the radius of convergence of $f(z)$ is not less than unity. That is $f \in \mathcal{A}$. We then show that f is in $\mathcal{K}_q(\alpha)$ with $h(z) = 1/(1-z)$. That is we need to show that

$$\left|(1-z)(D_q f)(z) - \frac{1-\alpha q}{1-q}\right| \le \frac{1-\alpha}{1-q}, \quad z \in \mathbb{D}. \tag{22}$$

By the definition of $q-$difference operator and B_n, we have

$$\begin{aligned}
(1-z)(D_q f)(z) &= (1-z)\left(1 + \sum_{k=2}^{\infty} \frac{1-q^n}{1-q} a_n z^{n-1}\right) \\
&= 1 + \sum_{k=1}^{\infty} (B_{n+1} - B_n) z^n.
\end{aligned} \tag{23}$$

From Eqs. (22)-(23), we have to show

$$\left|1 + \sum_{k=1}^{\infty} (B_{n+1} - B_n) z^n - \frac{1-\alpha q}{1-q}\right| \le \frac{1-\alpha}{1-q}, \quad z \in \mathbb{D}. \tag{24}$$

Applying the triangle inequality, we see that

$$\frac{1-\alpha}{1-q} - \left|1 + \sum_{k=1}^{\infty}(B_{n+1} - B_n)z^n - \frac{1-\alpha q}{1-q}\right| \geq \frac{1-\alpha}{1-q} - \left|1 - \frac{1-\alpha q}{1-q}\right| - \left|\sum_{k=1}^{\infty}(B_{n+1} - B_n)z^n\right|$$
$$= 1 - \alpha - \left|\sum_{k=1}^{\infty}(B_{n+1} - B_n)z^n\right| \geq 0,$$

by hypothesis. Hence $f \in \mathcal{K}_q(\alpha)$ with $h(z) = 1/(1-z)$.

Next, we suppose that $\sum_{n=1}^{\infty}|B_{n+1} + B_n| \leq 1 - \alpha$. In order to show that $f \in \mathcal{K}_q(\alpha)$ with $h(z) = 1/(1+z)$ we define $C_n = (-1)^n B_n$, for $n \in \mathbb{N}$. By using the same techniques as above with the equality

$$\sum_{n=1}^{\infty}|B_{n+1} + B_n| = \sum_{n=1}^{\infty}|C_{n+1} - C_n|,$$

we can easily obtain the result. The rest of theorem are immediate from the proof of (2). □

Next, we apply the Theorem 1.10 to find the conditions on the basic hypergeometric function $z{}_2\Phi_1(a, b; c; q, z)$ to be in the class $\mathcal{K}_q(\alpha)$ with particular function h. Here, we provide only the sufficient condition for the functions to be in $\mathcal{K}_q(\alpha)$ with $h(z) = z/(1-z)$. In order to find the sufficient condition, we modify the techniques of Theorem 1.1 in [25]. So, we omit the proof. For more applications and details, we refer to [25].

Corollary 1.11. *If a and b satisfy any one of the following conditions*

(i) $(1-q^a)(1-q^b) > 1-q$ *and* $\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)} \leq \frac{2-\alpha}{q}$.

(ii) $a + b > 2$, $(1-q^{a-1})(1-q^{b-1}) < -(1-q)$, *and* $\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)} \geq \frac{\alpha}{q}$.

Then the function $z{}_2\Phi_1(a, b; a+b; q, z) \in \mathcal{K}_q(\alpha)$ *with* $z/(1-z)$.

Example 1.12. *Let* $q = 0.4$ *and* $0 \leq \alpha < 1$. *The function* $z{}_2\Phi_1(-1.9, -0.2; -2.1; q; z)$ *satisfies the condition (i). That is* $z{}_2\Phi_1(-1.9, -0.2; -2.1; q; z) \in \mathcal{K}_q(\alpha)$ *for all* $0 \leq \alpha < 1$.

Let $a = -1.9$ and $b = -0.2$. Then

$$(1-q^{a-1})(1-q^{b-1}) = 1.3072... > 0.6 = 1-q.$$

Numerical computations give $\Gamma_q(a) = 0.3186...$, $\Gamma_q(b) = -1.7997...$, and $\Gamma_q(a+b) = 0.3966....$ Hence

$$\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)} = -0.6916... < 0 < \frac{2-\alpha}{q}.$$

That satisfies the condition (i).

Example 1.13. *Let* $q = 0.4$ *and* $\alpha = 0.1$. *The function* $z{}_2\Phi_1(2, 0.2; 2.2; q; z)$ *satisfies the condition (ii). That is* $z{}_2\Phi_1(2, 0.2; 2.2; q; z) \in \mathcal{K}_q(\alpha)$.

Let $a = 2$ and $b = 0.2$. Then $a + b = 2.2 > 2$ and

$$(1-q^{a-1})(1-q^{b-1}) = -0.6488... < -0.6 = -(1-q).$$

Numerical computations give $\Gamma_q(a) = 1$, $\Gamma_q(b) = 3.3958...$, and $\Gamma_q(a+b) = 1.0535....$ Hence

$$\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)} = 0.3102... \geq 0.25 = \frac{\alpha}{q}.$$

That satisfies the condition (ii).

Acknowledgments

The authors are greatly indebted to the anonymous referees for their valuable suggestions and careful reading of the paper.

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