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On an Inversion Formula for the Fourier Transform on Distributions by Means of Gaussian Functions

B. J. González[a], E. R. Negrín[a]

[a] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de La Laguna (ULL). Campus de Anchieta. ES-38271 La Laguna (Tenerife), España*

Abstract. Gaussian functions are useful in order to establish inversion formulae for the classical Fourier transform. In this paper we show that they also are helpful in order to obtain a Fourier inversion formula for the distributional case.

1. Introduction

In a series of papers published by the authors, different aspects of the Fourier transform on the spaces of distributions denoted by S'_k (duals of the spaces $\mathcal{S}_k$ introduced by J. Horváth in [9]) were studied (see [3], [4], [5] and [6]).

These spaces can be identified with subspaces of the Schwartz space $\mathcal{S}'$ and its members can be considered as tempered distributions. Moreover, the usual distributional Fourier transform of $f \in \mathcal{S}'_k$ [12, Chap. VII, §6, p. 248] is the regular distribution generated by the function in $\mathbb{R}^n$ given by $(\mathcal{F}f)(y) = \left\langle f, e^{ixy} \right\rangle$.

In [4, Theorem 2.1] it was established that if $f \in \mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, then for all $\phi \in \mathcal{S}$ the Parseval equality

$$\left\langle f, \mathcal{F}\phi \right\rangle = \left\langle T_{<f,e^{ixy}>}, \phi(y) \right\rangle$$

holds, where $T_{<f,e^{ixy}>}$ is the member of $\mathcal{S}'$ given by

$$\left\langle T_{<f,e^{ixy}>}, \phi(y) \right\rangle = \int_{\mathbb{R}^n} \left\langle f, e^{ixy} \right\rangle \phi(y) dy,$$

and $\mathcal{F}\phi$ denotes the classical Fourier transform of ϕ, namely

$$(\mathcal{F}\phi)(t) = \int_{\mathbb{R}^n} \phi(y) e^{ity} dy, \quad t \in \mathbb{R}^n.$$

Moreover, in [4, Theorem 3.1] it was proved the following inversion formula:

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Email addresses: bjglez@ull.es (B. J. González), enegrin@ull.es (E. R. Negrín)

Let $f \in \mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, and set $(\mathcal{F}f)(y) = \left\langle f, e^{ixy} \right\rangle$ for $y \in \mathbb{R}^n$. Then for any $\phi_1, \cdots, \phi_n \in \mathcal{D}(\mathbb{R})$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $\phi(t) = \phi_1(t_1) \cdots \phi_n(t_n)$, one has

$$\left\langle f, \phi \right\rangle = \lim_{Y \to +\infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{C(0;Y)} (\mathcal{F}f)(y) e^{-ity} dy \, \phi(t) dt,$$

where $C(0;Y)$ is the n−cube $[-Y, Y], \times \overset{n}{\cdots} \times [-Y, Y] \subset \mathbb{R}^n$, $Y > 0$.

Later, in [6, Theorem 1], this inversion formula was extended to functions $\phi \in \mathcal{S}$ such that $\phi(t) = \phi_1(t_1) \cdots \phi_n(t_n)$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, where $\phi_1, \cdots, \phi_n \in \mathcal{S}(\mathbb{R})$.

The purpose of the present paper is to obtain a distributional Fourier inversion formula which be valid for any $\phi \in \mathcal{S}$. For it we follow to Lang in [10, Theorem 4, p. 264] for obtaining an inversion formula for the classical Fourier transform by means of Gaussian functions.

As a consequence of this distributional inversion formula we get a representation over $\mathcal{S}$ of the solution in $\mathcal{S}'_k$ of convolution equations and, consequently, of linear partial differential equations with complex constant coefficients.

A representation of the Fourier transform on distributions was obtained in [1] (amongst others).

Gaussian functions have been useful in the context of integral transforms, as has been revealed in recent papers (see [7] and [13]). We also recall some interesting recent advances concerning to integral transforms [15].

Related differential equations have been solved in [16] by using the operational method.

We recall that the spaces $\mathcal{S}_k$, $k \in \mathbb{Z}$ [9, p. 90], are defined as the vector spaces of all functions ϕ on $\mathbb{R}^n$ which possess continuous partial derivatives of all orders and which satisfy the condition that if $p \in \mathbb{N}^n$ and $\varepsilon > 0$, then there exists $A(\phi, p, \varepsilon) > 0$ such that

$$\left| (1 + |x|^2)^k \partial^p \phi(x) \right| \leq \varepsilon, \quad \text{for} \quad |x| > A(\phi, p, \varepsilon).$$

For every $p \in \mathbb{N}^n$, Horváth defines on $\mathcal{S}_k$ the seminorms

$$q_{k,p}(\phi) = \max_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^k \partial^p \phi(x) \right|.$$

The spaces $\mathcal{S}_k$ equipped with the countable family of seminorms $q_{k,p}$ are Fréchet spaces. The well known space of test functions $\mathcal{D}$ is a dense subspace of $\mathcal{S}_k$ (see [9], p. 419). As it is usual, $\mathcal{S}'_k$ denotes the dual of the space $\mathcal{S}_k$.

In this paper we make use of the well known fact that

$$(2\pi c)^{(-1/2)} \cdot \int_{-\infty}^{+\infty} \exp\left[\nu x - (x^2/2c) \right] dx = \exp(c\nu^2/2), \quad \nu \in \mathbb{C}, \quad c > 0. \tag{1}$$

Throughout this paper we shall use the terminology and notation of [9].

2. The inversion formula

Firstly, we will establish the next assertion

Lemma 2.1. *Let $\phi \in \mathcal{S}$, $k \in \mathbb{Z}$ and $k < 0$, then*

$$\frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw \longrightarrow \phi(x),$$

in $\mathcal{S}$ for $a \to 0^+$.

Proof. First, we claim that for all $\phi \in \mathcal{S}$ and all $a > 0$ one has

$$\frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw \in \mathcal{S}.$$

In fact, for any $p \in \mathbb{N}^n$ there exists a $M_{p,\phi} > 0$ such that $\left|\partial^p \phi(x)\right| \leq M_{p,\phi}$, for all $x \in \mathbb{R}^n$. Thus, for $\mathbf{0} = (0, \ldots, 0)$, it is clear that

$$\left|\phi(x + 2aw) e^{-\|w\|^2}\right| \leq M_{\mathbf{0},\phi} e^{-\|w\|^2}.$$

Also, for $r(j) = (r_1(j), \ldots, r_n(j))$, where $r_m(j) = 0$ for $m \neq j$ and $r_j(j) = 1, \quad j = 1, \ldots, n$, it follows that

$$\left|\frac{\partial}{\partial x_j} \phi(x + 2aw) e^{-\|w\|^2}\right| \leq M_{r(j),\phi} e^{-\|w\|^2}, \quad j = 1, \ldots, n, \text{ and all } x \in \mathbb{R}^n.$$

Since $M_{\mathbf{0},\phi} e^{-\|w\|^2}$ and $M_{r(j),\phi} e^{-\|w\|^2}, \quad j = 1, \ldots n$, are integrable functions over $\mathbb{R}^n$, the use of [2, Theorem 5.9, p. 238] yields to

$$\frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \phi(x + 2aw) e^{-\|w\|^2} dw.$$

A similar argument allows us to prove that for all $p_j \in \mathbb{N}$,

$$\frac{\partial^{p_j}}{\partial x_j^{p_j}} \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw$$

$$= \int_{\mathbb{R}^n} \frac{\partial^{p_j}}{\partial x_j^{p_j}} \phi(x + 2aw) e^{-\|w\|^2} dw,$$

for all $j = 1, \ldots, n$. Now, since for $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$, is $\partial^p = \dfrac{\partial^{p_1 + \cdots + p_n}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}}$, it follows that

$$\partial^p \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw = \int_{\mathbb{R}^n} \partial^p \phi(x + 2aw) e^{-\|w\|^2} dw.$$

On the other hand, being

$$\frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\|w\|^2} dw = 1,$$

we find that

$$\left|\left(1 + |x|^2\right)^k \frac{1}{\pi^{\frac{n}{2}}} \partial^p \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw\right|$$

$$\leq \left(1 + |x|^2\right)^k M_{p,\phi} \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\|w\|^2} dw = \left(1 + |x|^2\right)^k \cdot M_{p,\phi}\,, \tag{1}$$

from which, being $k < 0$, it follows that (1) tends to zero as $|x|$ tends to infinity.

Now, for all $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$,

$$\max_{x \in \mathbb{R}^n} \left|\left(1 + |x|^2\right)^k \frac{1}{\pi^{\frac{n}{2}}} \partial^p \left\{\int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw - \phi(x)\right\}\right|$$

$$= \max_{x \in \mathbb{R}^n} \left|\left(1 + |x|^2\right)^k \frac{1}{\pi^{\frac{n}{2}}} \partial^p \left\{\int_{\mathbb{R}^n} \left[\phi(x + 2aw) - \phi(x)\right] e^{-\|w\|^2} dw\right\}\right|, \tag{2}$$

which, applying again [2, Theorem 5.9, p. 238], we have that the last expression is equal to

$$\frac{1}{\pi^{\frac{n}{2}}} \max_{x \in \mathbb{R}^n} \left|\int_{\mathbb{R}^n} \left\{\partial^p \phi(x + 2aw) - \partial^p \phi(x)\right\} e^{-\|w\|^2} dw\right|$$

$$\leq \frac{1}{\pi^{\frac{n}{2}}} \max_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \left|\partial^p\phi(x+2aw) - \partial^p\phi(x)\right| e^{-\|w\|^2} dw,$$

and by the Mean-Value theorem it is less than or equal to

$$\frac{2a}{\pi^{\frac{n}{2}}} \cdot \left\{ \sum_{j=1}^{n} M_{p(j),\phi} \right\} \cdot \int_{\mathbb{R}^n} \| w \| \, e^{-\|w\|^2} dw,$$

where $p(j) = (p_1, \ldots, p_j + 1, \ldots, p_n)$.

Also, using spherical coordinates in $\mathbb{R}^n$ it is easily obtained that

$$\int_{\mathbb{R}^n} \| w \| \, e^{-\|w\|^2} dw = \pi^{n-1}\Gamma\left(\frac{n+1}{2}\right),$$

from which (2) is less than or equal to

$$\frac{2a}{\pi^{\frac{n}{2}}} \cdot \left\{ \sum_{j=1}^{n} M_{p(j),\phi} \right\} \cdot \pi^{n-1}\Gamma\left(\frac{n+1}{2}\right),$$

and, thus, the Lemma holds. □

We are now ready to prove the main result

Theorem 2.2. *Let* $f \in \mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, *and* $(\mathcal{F}f)(y) = \left\langle f, e^{ixy} \right\rangle$, $\;y \in \mathbb{R}^n$, *then, for all* $\phi \in \mathcal{S}$ *it follows*

$$\left\langle f, \phi \right\rangle = \lim_{a\to 0+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F}f)(y) e^{-ity} e^{-a^2\|y\|^2} dy \, \phi(t) dt. \tag{3}$$

Proof.

First, from [9, Proposition 2, p. 97], there exist a $C > 0$ and a nonnegative integer r, both depending on f, such that

$$|(\mathcal{F}f)(y)| = \left|\left\langle f, e^{ixy} \right\rangle\right| \leq C \max_{|p|\leq r} \max_{x\in\mathbb{R}^n} \left| \left(1+|x|^2\right)^k \partial_x^p e^{ixy} \right| = C \max_{|p|\leq r} |y^p|.$$

Thus, for any $\phi \in \mathcal{S}$, one has

$$\begin{aligned} &\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F}f)(y) e^{-ity} e^{-a^2\|y\|^2} dy \, \phi(t) dt \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\langle f, e^{ixy} \right\rangle e^{-ity} e^{-a^2\|y\|^2} dy \, \phi(t) dt, \end{aligned}$$

and by Fubini theorem it is equal to

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left\langle f, e^{ixy} \right\rangle e^{-a^2\|y\|^2} \int_{\mathbb{R}^n} e^{-ity} \phi(t) dt \, dy. \tag{4}$$

Note that, since $\phi \in \mathcal{S}$ it follows that

$$e^{-a^2\|y\|^2} \int_{\mathbb{R}^n} e^{-ity} \phi(t) dt \in \mathcal{S}.$$

Thus, as a consequence of [4, Theorem 2.1], we have that (4) is equal to

$$\left\langle f, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ixy} e^{-a^2\|y\|^2} \int_{\mathbb{R}^n} e^{-ity} \phi(t) dt \, dy \right\rangle,$$

which, making use again of Fubini theorem, is equal to

$$\left\langle f, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ixy} e^{-ity} e^{-a^2\|y\|^2} dy\, \phi(t) dt \right\rangle. \tag{5}$$

Now, observe that by (1) we have

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x-t)y} e^{-a^2y^2} dy = \frac{1}{2\sqrt{\pi}a\sqrt{2\pi\frac{1}{2a^2}}} \int_{-\infty}^{+\infty} e^{i(x-t)y} e^{-\frac{y^2}{2\frac{1}{2a^2}}} dy$$

$$= \frac{1}{2\sqrt{\pi}a} e^{-\frac{1}{2a^2}\frac{(x-t)^2}{2}} = \frac{1}{2\sqrt{\pi}a} e^{-\frac{(x-t)^2}{4a^2}},$$

and thus we get that

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-t)y} e^{-a^2\|y\|^2} dy = \frac{1}{2^n \pi^{n/2} a^n} e^{-\frac{\|x-t\|^2}{4a^2}}. \tag{6}$$

Therefore, (5) is equal to

$$\left\langle f, \frac{1}{2^n \pi^{\frac{n}{2}} a^n} \int_{\mathbb{R}^n} \phi(t) e^{\frac{-\|x-t\|^2}{4a^2}} dt \right\rangle. \tag{7}$$

Now, performing the change of variables $t = x + 2aw$, (7) becomes

$$\left\langle f, \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\|w\|^2} dw \right\rangle, \tag{8}$$

from which, since $f \in \mathcal{S}_k'$ by Lemma 2.1, the equality (3) follows. □

As it is well known, the Dirac distribution δ_u at $u \in \mathbb{R}^n$ given by $\langle \delta_u, \phi \rangle = \phi(u)$, for all $\phi \in \mathcal{S}_k$, is a member in $\mathcal{S}_k'$. As it is usual we denote $\delta = \delta_0$. Also, for all $m \in \mathbb{N}^n$, $\partial^m \delta_u$ at $u \in \mathbb{R}^n$ given by $\langle \partial^m \delta_u, \phi \rangle = \langle \delta_u, (-1)^{|m|} \partial^m \phi \rangle = (-1)^{|m|} \partial^m \phi(u)$, for all $\phi \in \mathcal{S}_k$, is a member in $\mathcal{S}_k'$.

Now, one obtains the next result

Corollary 2.3. *For all $\phi \in \mathcal{S}$, $u \in \mathbb{R}^n$ and all $m \in \mathbb{N}^n$, one has*

$$\langle \partial^m \delta_u, \phi \rangle = \lim_{a \to 0^+} \frac{(-1)^{|m|}}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} e^{-\frac{\|u-t\|^2}{4a^2}} \partial^m \phi(t) dt,$$

and

$$\partial^m \phi(u) = \lim_{a \to 0^+} \frac{1}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} e^{-\frac{\|u-t\|^2}{4a^2}} \partial^m \phi(t) dt.$$

Proof.

Since $< \delta_u, e^{ixy} >= e^{iuy}$, $y \in \mathbb{R}^n$, and according to the above inversion formula, for any $\phi \in \mathcal{S}$, one has

$$< \partial^m \delta_u, \phi >= \lim_{a \to 0^+} \frac{(-1)^{|m|}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-t)y} e^{-a^2\|y\|^2} dy\, \partial^m \phi(t) dt. \tag{9}$$

Now, using (6), formula (9) becomes

$$\langle \partial^m \delta_u, \phi \rangle = (-1)^{|m|} \partial^m \phi(u) = \lim_{a \to 0^+} \frac{(-1)^{|m|}}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} e^{-\frac{\|u-t\|^2}{4a^2}} \partial^m \phi(t) dt.$$

□

Also, using Theorem 2.2 above and [6, Theorem 2.1] one has

Corollary 2.4. *Set $f \in \mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$. Then*

$$\lim_{Y\to+\infty} \int_{\mathbb{R}^n} \int_{C(0;Y)} (\mathcal{F} f)(y) e^{-ity} dy\, \phi(t) dt$$

$$= \lim_{a\to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F} f)(y) e^{-ity} e^{-a^2\|y\|^2} dy\, \phi(t) dt,$$

for all $\phi \in \mathcal{S}$ such that $\phi(t) = \phi_1(t_1)\cdots\phi_n(t_n)$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, where $\phi_1, \dots, \phi_n \in \mathcal{S}(\mathbb{R})$.

The next result is a variant of [5, Corollary 2.1] concerning the solution of convolution equations.

Corollary 2.5. *Set $h, g \in \mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$. Assume that $\mathcal{F}h$ has no zeros in $\mathbb{R}^n$, suppose that $\mathcal{F}h \in C^{-2k+2n}(\mathbb{R}^n)$ and there exists a polynomial P such that*

$$\left|\partial^m \left(\frac{1}{(\mathcal{F}h)(y)}\right)\right| \le P(|y|), \ \ \forall y \in \mathbb{R}^n, \ \ \forall m \in \mathbb{N}^n, \ \ |m| \le -2k + 2n.$$

Then, the convolution equation

$$h * f = g, \tag{10}$$

has a unique solution $f \in \mathcal{S}'_k$ and this solution has the next representation over members in $\mathcal{S}$

$$< f, \phi >= \lim_{a\to 0^+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\mathcal{F}g)(y)}{(\mathcal{F}h)(y)} e^{-ity} e^{-a^2\|y\|^2} dy\, \phi(t) dt, \quad \phi \in \mathcal{S}. \tag{11}$$

Proof.

In fact, from the hypothesis of this Corollary and using [5, Theorem 2.1] it follows that there exists an element $w \in \mathcal{S}'_k$ such that $\mathcal{F}w = \frac{1}{\mathcal{F}h}$. Therefore, using [4, Proposition 4.1] one has

$$\mathcal{F}[h * w] = \mathcal{F}h \cdot \frac{1}{\mathcal{F}h} = 1 = \mathcal{F}\delta.$$

So, using [4, Corollary 3.1], it follows that $h * w = \delta$.

Now, the member of $\mathcal{S}'_k$ given by $f = w * g$ is a solution of equation (10).

In fact,

$$h * (w * g) = (h * w) * g = \delta * g = g.$$

Note that if $f_1, f_2 \in \mathcal{S}'_k$ satisfy $h * f_1 = g$ and $h * f_2 = g$ then $f_1 = f_2$. Indeed, taking Fourier transform it follows that

$$\mathcal{F} f_1 = \mathcal{F} f_2 = \frac{\mathcal{F} g}{\mathcal{F} h},$$

and, again by [5, Corollary 3.1], we have $f_1 = f_2$.

Also, since $\mathcal{F}[h * f] = \mathcal{F} g$ and using again [5, Proposition 4.1] one obtain that

$$\mathcal{F} f = \frac{\mathcal{F} g}{\mathcal{F} h},$$

which by Theorem 2.2 above allows us to the representation over $\mathcal{S}$ given by (11). □

Remark (invertible elements of $\mathcal{S}'_k$).

Observe that the distribution $w = h^{-1}$ in $\mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, which satisfies the equation $h * w = \delta$, is the inverse by convolution of the member $h \in \mathcal{S}'_k$. So, when the distributional Fourier transform of h has no zeros in $\mathbb{R}^n$, with $\mathcal{F}h \in C^{-2k+2n}(\mathbb{R}^n)$ and it satisfies the inequality

$$\left|\partial^m\left(\frac{1}{(\mathcal{F}h)(y)}\right)\right| \leq P(|y|),\ \ \forall y \in \mathbb{R}^n,\ \ m \in \mathbb{N}^n,\ \ |m| \leq -2k + 2n,$$

for some polynomial P, this distribution h^{-1} has the next representation over $\mathcal{S}$

$$< h^{-1}, \phi >= \lim_{a\to 0^+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(\mathcal{F}h)(y)} e^{-ity} e^{-a^2\|y\|^2} dy\, \phi(t) dt, \quad \phi \in \mathcal{S}.$$

Final observation

As in [8] and [11], we consider linear partial differential equations with constant coefficients of the form

$$P(\partial)\, u = v\,, \tag{1}$$

where as it is usual P is a polynomial in $\mathbb{R}^n$ (with complex coefficients) and $P(\partial)$ denotes the corresponding polynomial differential operator given by

$$\sum_{|\alpha|\leq m} a_\alpha \partial^\alpha, \quad \alpha \in \mathbb{N}^n, \quad a_\alpha \in \mathbb{C}, \quad m \in \mathbb{N},$$

and v is an element of $\mathcal{S}'_k$, $k \in \mathbb{Z}$, $k < 0$.

Note that, since

$$P(\partial)u = (P(\partial)\delta) * u,$$

equation (1) can be written as a convolution equation.

Having into account that

$$(\mathcal{F}[P(\partial)\delta])(y) = P(-iy),\ \ y \in \mathbb{R}^n,$$

and using Corollary 2.5 above, one has that when P has no zeros of type αi, where $\alpha \in \mathbb{R}^n$, then there exists a unique solution u in $\mathcal{S}'_k$ of (1).

Also, one obtains the next representation over $\mathcal{S}$ of the solution u of equation (1):

$$< u, \phi >= \lim_{a\to 0+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\mathcal{F}v)(y)}{P(-iy)} e^{-ity} e^{-a^2\|y\|^2} dy\, \phi(t) dt,$$

for all $\phi \in \mathcal{S}$.

Furthermore, observe that if in (1) we set $v = \delta$, then one obtains a representation over $\mathcal{S}$ of the fundamental solution E of equation (1). In fact, having into account that $\mathcal{F}\delta = 1$, then one has

$$< E, \phi >= \lim_{a\to 0+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{P(-iy)} e^{-ity} e^{-a^2\|y\|^2} dy\, \phi(t) dt,$$

for all $\phi \in \mathcal{S}$.

Observe that this fundamental solution E is the inverse by convolution of the member h of $\mathcal{S}'_k$ given by $h = P(\partial)\delta$.

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