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Approximation with Certain Genuine Hybrid Operators

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Abstract. In the present article, we introduce a general sequence of summation-integral type operators. We establish some direct results which include Voronovskaja type asymptotic formula, point-wise convergence for derivatives, error estimations in terms of modulus of continuity and weighted approximation for these operators. Furthermore, the convergence of these operators and their first order derivatives to certain functions and their corresponding derivatives respectively is illustrated by graphics using Matlab algorithms for some particular values of the parameters c and ρ.

1. Introduction

In the last five decades, several new operators of integral type have been introduced and their approximation properties were discussed by several researchers. Milovanovic et al. [12] (see also [13], [14]) in their book covered some topics on the behaviour of some polynomials in real and complex domains. In the last decade, q calculus was also applied extensively in the theory of approximation of functions by linear positive operators. Aral, Gupta and Agarwal compiled the results on convergence of various q-operators in their book [4].

Very recently, Gupta and Agarwal [9] and recently Gupta and Tachev [11] also presented convergence estimates of many operators in real and complex domains (see also some of the papers [1], [2], [6], [7], [8]). In continuation of the above work, Păltănea [16] (see also [18]) considered a Durrmeyer type modification of the genuine Szász-Mirakjan operators based on two parameters. Motivated by his modification, we now propose for $f : [0,\infty) \to \mathbb{R}$, a general hybrid family of summation-integral type operators based on the parameters $\rho > 0$ and $c \in \{0,1\}$ in the following way:

$$B_\alpha^\rho(f;x,c) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t) f(t)dt + p_{\alpha,0}(x,c) f(0), \tag{1}$$

where

$$p_{\alpha,k}(x,c) = \frac{(-x)^k}{k!}\phi_{\alpha,c}^{(k)}(x), \theta_{\alpha,k}^\rho(t) = \frac{\alpha\rho}{\Gamma(k\rho)} e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1}.$$

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For the space defined as:

$$C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Ce^{\gamma t}, \text{for some } \gamma > 0,\ t \in [0,\infty)\},$$

it is observed that the operators $B_\alpha^\rho(f;x,c)$ are well defined for $\alpha\rho > \gamma$. Further, we note that the operators (1) preserve the linear functions.

Special cases:

1. If $\phi_{\alpha,0}(x) = e^{-\alpha x}$, then $p_{\alpha,k}(x,0) = e^{-\alpha x}\frac{(\alpha x)^k}{k!}$, we get the operators due to Păltănea [16]. Also, for this case if $\rho = 1$, we get the Phillips operators [17].

2. If $\phi_{\alpha,1}(x) = (1+x)^{-\alpha}$ and $\alpha = n$, then $p_{\alpha,k}(x,1) = \dfrac{(\alpha)_k}{k!}\dfrac{x^k}{(1+x)^{\alpha+k}}$, with the rising factorial given by $(n)_i = n(n+1)\cdots(n+i-1)$, $(n)_0 = 1$. For $\rho = 1$, we get the operators studied in [3].

3. If $c = 0, \alpha = n$ and $\rho \to \infty$, then in view of ([18], Theorem 2.2), we get the Szász-Mirakjan operators.

4. Similarly, if $c = 1, \alpha = n,\ f \in \overline{\Pi}$, the closure of the space of algebraic polynomials in space $C[0,\infty)$) and $\rho \to \infty$, we obtain at once Baskakov operators.

The aim of the present paper is to discuss some direct results for the generalized operators (1). We obtain asymptotic formula, point-wise convergence for derivatives, error estimations in terms of modulus of continuity and weighted approximation. The convergence of these operators and their first order derivatives is also illustrated by using Matlab algorithms.

2. Basic Results

In the sequel, we need the following lemmas.

Lemma 2.1. *For $c = 0, 1$ if the $m-$th order central moment $\mu_{\alpha,m}(x)$ is defined as*

$$\mu_{\alpha,m}(x) := B_\alpha^\rho((t-x)^m;x,c) = \sum_{k=1}^{\infty} p_{\alpha,k}(x)\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^m dt + p_{\alpha,0}(x,c)(-x)^m,$$

then, $\mu_{\alpha,0}(x) = 1$, $\mu_{\alpha,1}(x) = 0$ and there holds the following recurrence relation:

$$\alpha\mu_{\alpha,m+1}(x) = x(1+cx)\mu'_{\alpha,m}(x) + mx\left[\frac{1}{\rho} + (1+cx)\right]\mu_{\alpha,m-1}(x) + \frac{m}{\rho}\mu_{\alpha,m}(x).$$

Consequently,

(i) $\mu_{\alpha,m}(x)$ is a polynomial in x of degree atmost m depending on the parameters c and α;

(ii) for every $x \in (0,\infty)$, $\mu_{\alpha,m}(x) = O\Big(\alpha^{-[(m+1)/2]}\Big)$, where $[s]$ denotes the integer part of s.

Proof. We shall prove the result for different values of c separately. First for $c \in \{0,1\}$, using the identity $x(1+cx)p'_{\alpha,k}(x,c) = (k-\alpha x)p_{\alpha,k}(x,c)$, we may write

$$\begin{aligned}
x(1+cx)\mu'_{\alpha,m}(x) &= \sum_{k=1}^{\infty}(k-\alpha x)p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\varrho(t)(t-x)^m dt \\
&\quad -mx(1+cx)\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\varrho(t)(t-x)^{m-1}dt \\
&\quad +(-\alpha x)p_{\alpha,0}(x,c)(-x)^m - mx(1+cx)p_{\alpha,0}(x,c)(-x)^{m-1} \\
&= \sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_0^\infty [(k-\alpha t)+\alpha(t-x)]\theta_{\alpha,k}^\varrho(t)(t-x)^m dt \\
&\quad -mx(1+cx)\mu_{\alpha,m-1}(x) + \alpha p_{\alpha,0}(x,c)(-x)^{m+1}.
\end{aligned}$$

Next, using the identity

$$\frac{d}{dt}(t\theta^{\rho}_{\alpha,k}(t)) = \rho(k-\alpha t)\theta^{\rho}_{\alpha,k}(t),$$

we have

$$\begin{aligned}
x(1+cx)\mu'_{\alpha,m}(x) &= \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^{\infty}\frac{1}{\rho}(t\theta^{\rho}_{\alpha,k}(t))'(t-x)^m dt\\
&\quad -mx(1+cx)\mu_{\alpha,m-1}(x)+\alpha\mu_{\alpha,m+1}(x)\\
&= -\frac{m}{\rho}\sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^{\infty} t\theta^{\rho}_{\alpha,k}(t)(t-x)^{m-1}dt\\
&\quad -mx(1+cx)\mu_{\alpha,m-1}(x)+\alpha\mu_{\alpha,m+1}(x)\\
&= -\frac{m}{\rho}\Big[(\mu_{\alpha,m}(x)-p_{\alpha,0}(x,c)(-x)^m)+x\left(\mu_{\alpha,m-1}(x)-p_{\alpha,0}(x,c)(-x)^{m-1}\right)\Big]\\
&\quad -mx(1+cx)\mu_{\alpha,m-1}(x)+\alpha\mu_{\alpha,m+1}(x)\\
&= -\frac{m}{\rho}\Big[\mu_{\alpha,m}(x)+x\mu_{\alpha,m-1}(x)\Big]-mx(1+cx)\mu_{\alpha,m-1}(x)+\alpha\mu_{\alpha,m+1}(x),
\end{aligned}$$

which is the required recurrence relation. The consequences (i) and (ii) easily follow from the recurrence relation on using mathematical induction on m. □

Remark 2.2. *From Lemma 2.1, for each $x \in (0,\infty)$ and $c \in \{0,1\}$ we have*

$$\mu_{\alpha,2}(x) = \frac{x\{1+\rho(1+cx)\}}{\alpha\rho}; \tag{2}$$

$$\begin{aligned}
\mu_{\alpha,4}(x) &= \frac{x(1+cx)}{\alpha^3\rho^2}\left[3\rho(1+2cx)+\rho^2\left\{(1+2cx)^2+2cx(1+cx)\right\}+2\right]+\frac{3x^2\{1+\rho(1+cx)\}^2}{(\alpha\rho)^2}\\
&\quad +\frac{1}{(\alpha\rho)^3}\left[3\rho x(1+cx)\{3+\rho(1+2cx)\}+6x\right].
\end{aligned} \tag{3}$$

Corollary 2.3. *Let γ and δ be any two positive real numbers and $[a,b] \subset (0,\infty)$ be any bounded interval. Then, for any $m > 0$ there exists a constant M' depending on m only such that*

$$\left\|\sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{\gamma t}dt\right\| \leq M'\alpha^{-m},$$

where $\|.\|$ is the sup-norm over $[a,b]$.

3. Convergence Estimates

Our first main result is the Voronovskaja type theorem for the operators defined in (1).

Theorem 3.1. *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If f'' exists at a point $x \in [0,\infty)$ then, we have*

$$\lim_{\alpha\to\infty}\alpha(B^{\rho}_{\alpha}(f;x,c)-f(x)) = \frac{x\{1+\rho(1+cx)\}}{2\rho}f''(x).$$

Proof. From the Taylor's theorem, we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + \psi(t,x)(t-x)^2, \quad t \in [0,\infty) \tag{4}$$

where the function $\psi(t,x) \to 0$ as $t \to x$.
Applying $B_\alpha^\rho(.;x,c)$ and taking the limit as $\alpha \to \infty$ on both sides of (4), we have

$$\begin{aligned}\lim_{\alpha\to\infty} \alpha(B_\alpha^\rho(f;x,c) - f(x)) &= \lim_{\alpha\to\infty} \alpha B_\alpha^\rho((t-x);x,c)f'(x) + \frac{f''(x)}{2}\lim_{\alpha\to\infty} \alpha B_\alpha^\rho((t-x)^2;x,c)\\ &\quad + \lim_{\alpha\to\infty} \alpha B_\alpha^\rho(\psi(t,x)(t-x)^2;x,c).\end{aligned}$$

In view of Remark 2.2, we get

$$\lim_{\alpha\to\infty} \alpha B_\alpha^\rho((t-x);x,c) = 0 \tag{5}$$

and

$$\lim_{\alpha\to\infty} \alpha B_\alpha^\rho((t-x)^2;x,c) = \frac{x\{1+\rho(1+cx)\}}{\rho}. \tag{6}$$

Now, we prove that $\alpha B_\alpha^\rho(\psi(t,x)(t-x)^2;x,c) \to 0$, as $\alpha \to \infty$. From the Cauchy-Schwarz inequality, we have

$$B_\alpha^\rho(\psi(t,x)(t-x)^2;x,c) \leq \sqrt{B_\alpha^\rho(\psi^2(t,x);x,c)}\sqrt{B_\alpha^\rho((t-x)^4;x,c)}. \tag{7}$$

Since $\psi(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$ there exists $\delta > 0$ such that $|\psi(t,x)| < \epsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, there exists M_1 such that $|\psi(t,x)| \leq M_1 e^{\gamma t}$.

Let $\chi_\delta(t)$ denote the characteristic function of $(x-\delta, x+\delta)$. Then

$$\begin{aligned}B_\alpha^\rho(\psi^2(t,x);x,c) &\leq B_\alpha^\rho(\psi^2(t,x)\chi_\delta(t);x,c) + B_\alpha^\rho(\psi^2(t,x)(1-\chi_\delta(t));x,c)\\ &\leq \epsilon^2 B_\alpha^\rho(1;x,c) + M_1^2 B_\alpha^\rho(e^{2\gamma t}(1-\chi_\delta(t));x,c)\\ &\leq \epsilon^2 + M_2\alpha^{-m}, \quad \text{in view of Corollary 2.3.}\end{aligned}$$

Hence, we have

$$\lim_{\alpha\to\infty} B_\alpha^\rho(\psi^2(t,x);x,c) = 0. \tag{8}$$

Further from Remark 2.2,

$$\lim_{\alpha\to\infty} \alpha^2 B_\alpha^\rho((t-x)^4;x,c) = 3x^2\left(\frac{1}{\rho} + (1+cx)\right)^2, \tag{9}$$

which is a finite quantity for each fixed $x \in [0,\infty)$ thus from (7)-(9), we get

$$\lim_{\alpha\to\infty} \alpha B_\alpha^\rho\Big(\psi(t,x)(t-x)^2;x,c\Big) = 0. \tag{10}$$

Combining (5), (6) and (10), we obtain the desired result. □

Example 3.2. *For $\alpha = 20, 50, 100$, the convergence of the operators $B_\alpha^\rho(f;x,c)$ to the function $f(x) = x^8 - 6x^7 + 5x^4 - 4x^3 + 2x^2 + 3$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures $1-3$, respectively.*

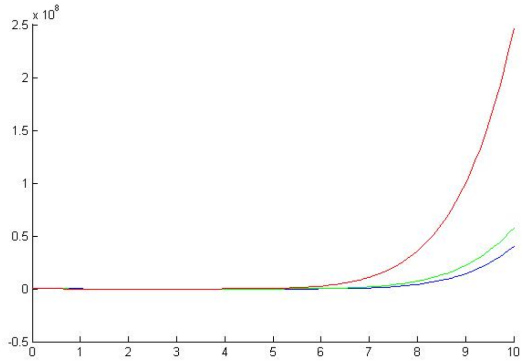

Figure 1

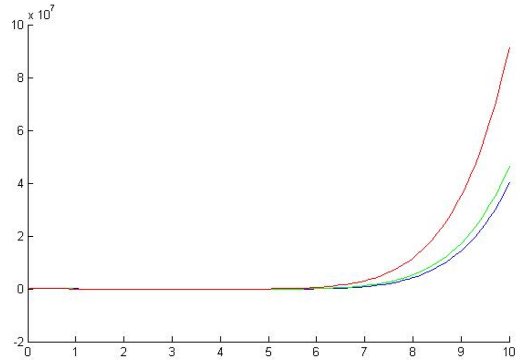

Figure 2

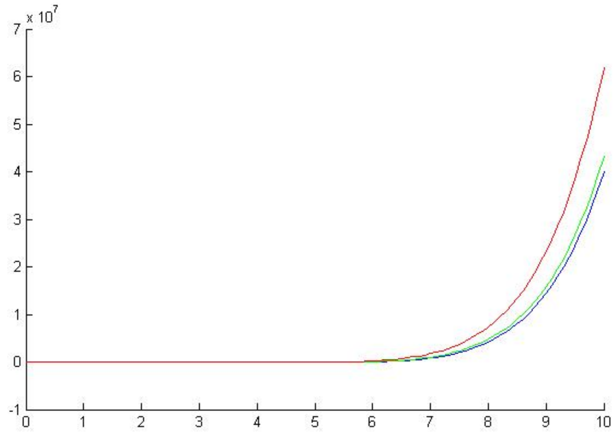

Figure 3

Example 3.3. *For $\alpha = 20, 50, 100$, the convergence of the operators $B_\alpha^\rho(f; x, c)$ to the function $f(x) = x^4 e^{(-2\pi x)}$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures $4-6$, respectively.*

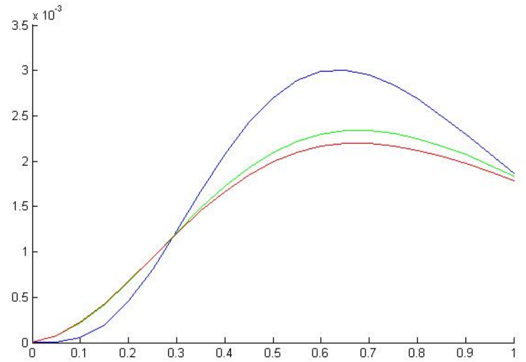

Figure 4

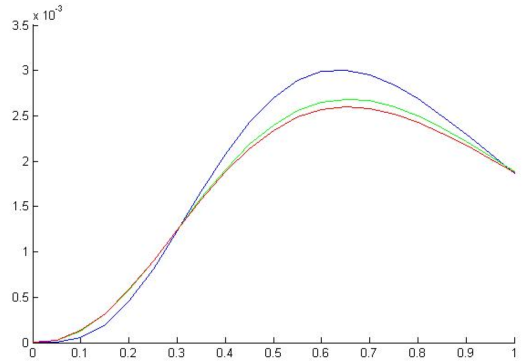

Figure 5

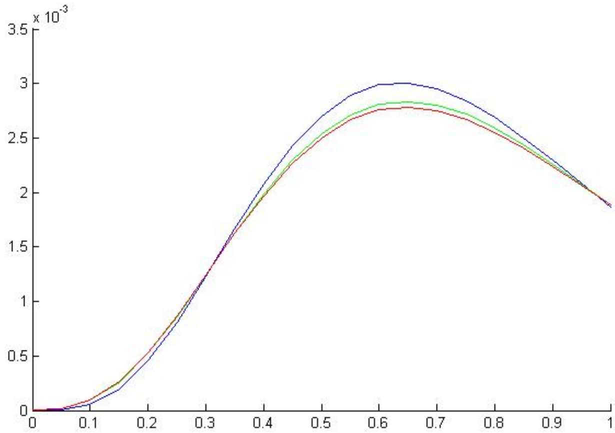

Figure 6

In the following theorem, we show that the derivative $\left(\frac{d}{d\omega}B_\alpha^\rho(f;\omega,c)\right)_{\omega=x}$ is also an approximation process for $f'(x)$.

Theorem 3.4. *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If f' exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{\alpha\to\infty}\left(\frac{d}{dw}B_\alpha^\rho(f;w,c)\right)_{w=x} = f'(x).$$

Proof. By our hypothesis, we have

$$f(t) = f(x) + (t-x)f'(x) + \psi(t,x)(t-x), \quad t \in [0,\infty),$$

where the function $\psi(t,x) \to 0$ as $t \to x$.
From the above equation, we can write

$$\begin{aligned}\left(\frac{d}{d\omega}B_\alpha^\rho(f(t);\omega,c)\right)_{\omega=x} &= f(x)\left(\frac{d}{d\omega}B_\alpha^\rho(1;\omega,c)\right)_{\omega=x} + f'(x)\left(\frac{d}{d\omega}B_\alpha^\rho((t-x);\omega,c)\right)_{\omega=x}\\ &\quad+\left(\frac{d}{d\omega}B_\alpha^\rho(\psi(t,x)(t-x);\omega,c)\right)_{\omega=x}\\ &= f'(x) + \left(\frac{d}{d\omega}B_\alpha^\rho(\psi(t,x)(t-x);\omega,c)\right)_{\omega=x}.\end{aligned}$$

Taking limit as $\alpha \to \infty$, the result follows immediately, if we show that

$$\lim_{\alpha\to\infty}\left(\frac{d}{d\omega}B_\alpha^\rho(\psi(t,x)(t-x);\omega,c)\right)_{\omega=x} = 0.$$

By using the identity $x(1+cx)\left(\frac{d}{d\omega}(p_{\alpha,k}(\omega,c))\right)_{\omega=x} = (k-\alpha x)\ p_{\alpha,k}(x,c)$, we have

$$\begin{aligned}&\left|\left(\frac{d}{d\omega}B_\alpha^\rho(\psi(t,x)(t-x);\omega,c)\right)_{\omega=x}\right|\\ &= \left|\left(\sum_{k=1}^\infty \frac{d}{d\omega}(p_{\alpha,k}(\omega,c))\int_0^\infty \theta_{\alpha,k}^\rho(t)\psi(t,x)(t-x)dt + \frac{d}{d\omega}\Big(p_{\alpha,0}(\omega,c)\Big)\psi(0,x)(-x)\right)_{\omega=x}\right|\\ &\leq \sum_{k=1}^\infty \frac{|k-\alpha x|}{x(1+cx)}p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)|\psi(t,x)||t-x|dt + \left|\left(\frac{d}{d\omega}\phi_{\alpha,c}(\omega)\right)_{\omega=x}\right||\psi(0,x)|x \qquad (11)\\ &=: I_1 + I_2.\end{aligned}$$

Since $\psi(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$ there exists $\delta > 0$ such that $|\psi(t,x)| < \epsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, we have $|\psi(t,x)(t-x)| \leq M_1 e^{\gamma t}$, for some $M_1 > 0$. Thus, from equation (11) we may write

$$\begin{aligned} I_1 &\leq \sum_{k=1}^{\infty}\Big(\frac{|(k-\alpha x)|}{x(1+cx)}\Big)p_{\alpha,k}(x,c)\Big(\epsilon\int_{|t-x|<\delta}\theta^{\rho}_{\alpha,k}(t)|t-x|dt \\ &\quad +M_1\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{\gamma t}dt\Big) \\ &:= J_1 + J_2. \end{aligned} \tag{12}$$

Using Schwarz inequality for integration and then for summation, we can write

$$\begin{aligned} J_1 &\leq \epsilon\sum_{k=1}^{\infty}\Big(\frac{|(k-\alpha x)|}{x(1+cx)}\Big)p_{\alpha,k}(x,c)\int_{|t-x|<\delta}\theta^{\rho}_{\alpha,k}(t)|t-x|dt \\ &\leq \frac{\epsilon}{x(1+cx)}\sum_{k=1}^{\infty}|(k-\alpha x)|p_{\alpha,k}(x,c)\Big(\int_0^{\infty}\theta^{\rho}_{\alpha,k}(t)(t-x)^2dt\Big)^{1/2} \\ &\leq \frac{\epsilon}{x(1+cx)}\sum_{k=1}^{\infty}|(k-\alpha x)|p_{\alpha,k}(x,c)\Big(\int_0^{\infty}\theta^{\rho}_{\alpha,k}(t)(t-x)^2dt\Big)^{1/2} \\ &\leq \frac{\epsilon}{x(1+cx)}\Big(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)(k-\alpha x)^2\Big)^{1/2}\Big(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_0^{\infty}\theta^{\rho}_{\alpha,k}(t)(t-x)^2dt\Big)^{1/2} \\ &= \frac{\epsilon}{x(1+cx)}\left(\alpha^2\left(\sum_{k=0}^{\infty}p_{\alpha,k}(x,c)\Big(\frac{k}{\alpha}-x\Big)^2 - x^2p_{\alpha,0}(x,c)\right)\right)^{1/2}(\mu_{\alpha,2}(x)-x^2p_{\alpha,0}(x,c))^{1/2} \\ &= \epsilon\Big\{\alpha.O\Big(\frac{1}{\alpha^{1/2}}\Big)\Big\}O\Big(\frac{1}{\alpha^{1/2}}\Big), \text{ in view of Lemma 2.1} \\ &= \epsilon.O(1). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $J_1 \to 0$ as $\alpha \to \infty$. Next

$$\begin{aligned} J_2 &\leq \frac{M_1}{x(1+cx)}\Big(\sum_{k=1}^{\infty}(k-\alpha x)^2p_{\alpha,k}(x,c)\Big)^{1/2}\Big(\int_0^{\infty}\theta^{\rho}_{\alpha,k}(t)dt\Big)^{1/2} \\ &\quad\times\Big(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{2\gamma t}dt\Big)^{1/2} \\ &\leq \frac{M_1}{x(1+cx)}\Big(\alpha^2\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\Big(\frac{k}{\alpha}-x\Big)^2\Big)^{1/2}\Big(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{2\gamma t}dt\Big)^{1/2} \\ &= \Big\{\alpha.O\Big(\frac{1}{\alpha^{1/2}}\Big)\Big\}M_1'\alpha^{-m} = O\Big(\frac{1}{\alpha^{\frac{2m-1}{2}}}\Big), \text{ for any } m > 0. \end{aligned}$$

By taking $m > 1/2, J_2 \to 0$ as $\alpha \to \infty$. Combining the estimates of J_1 and J_2, we get $I_1 \to 0$ as $\alpha \to \infty$. Finally, we show that $I_2 \to 0$ as $\alpha \to \infty$. Since $|\psi(0,x)x| \leq M_2$ and for $c = 0, 1$, we get $\phi'_{\alpha,c}(x) \to 0$ as $\alpha \to \infty$, it follows that $I_2 \to 0$ as $\alpha \to \infty$. By combining the estimates of I_1 and I_2, we get the required result. □

Example 3.5. *For $\alpha = 50, 100, 140$, the convergence of the operators $\left(\frac{d}{d\omega}B^{\rho}_{\alpha}(f;\omega,c)\right)_{\omega=x}$ to the function $\frac{d}{dx}f(x) = \frac{d}{dx}(x^8 - 6x^7 + 5x^4 - 4x^3 + 2x^2 + 3)$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures $7-9$, respectively.*

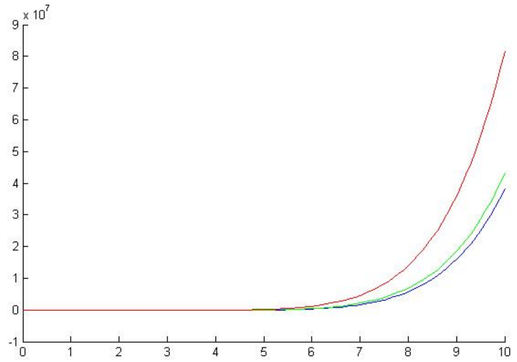

Figure 7

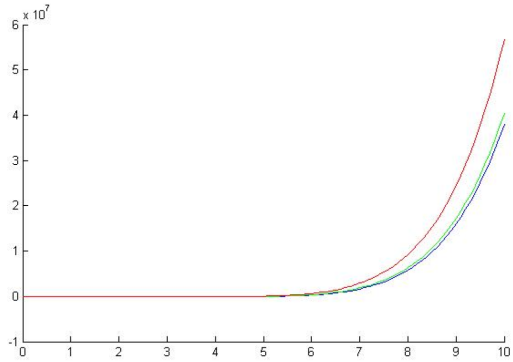

Figure 8

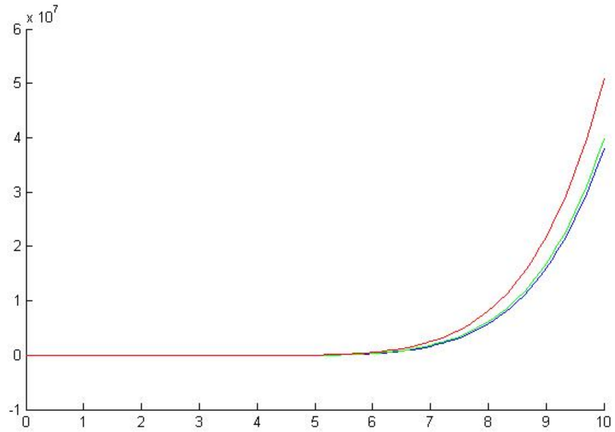

Figure 9

Example 3.6. *For* $\alpha = 50, 100, 140$, *the convergence of the operators* $\left(\frac{d}{d\omega}B_{\alpha}^{\rho}(f;\omega,c)\right)_{\omega=x}$ *to the function* $\frac{d}{dx}f(x) = \frac{d}{dx}(x^4 e^{(-2\pi x)})$ *(blue) is illustrated for* $c = 0, \rho = 1$ *(green) and* $c = 1, \rho = 1$ *(red) in figures* $10-12$, *respectively.*

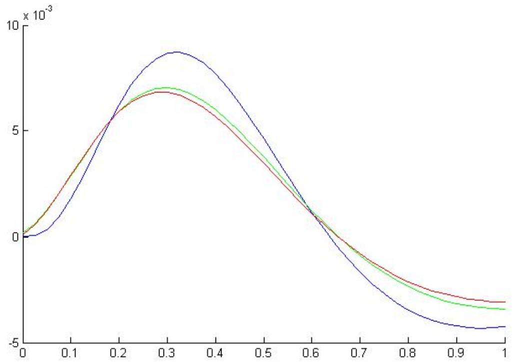

Figure 10

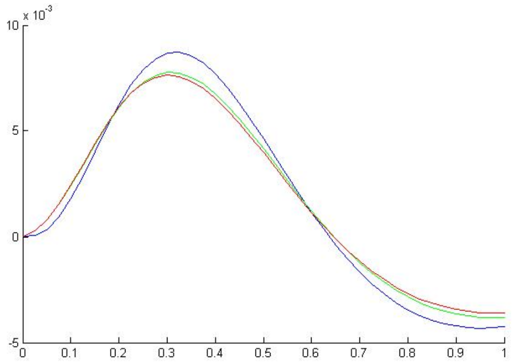

Figure 11

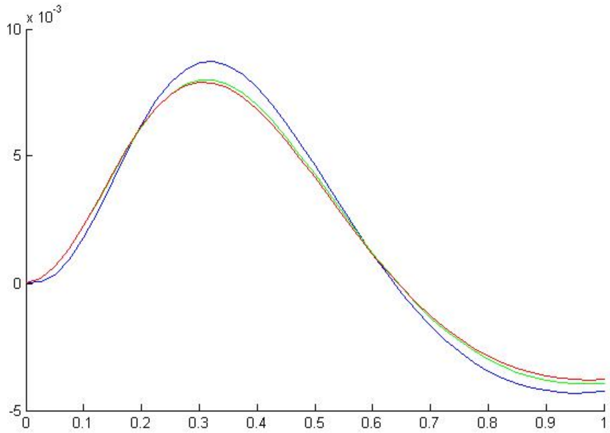

Figure 12

Next, we prove Voronovskaja type asymptotic formula for $\left(\frac{d}{d\omega}B_\alpha^\rho(f;\omega,c)\right)_{\omega=x}$.

Theorem 3.7. *Let $f \in C_\gamma[0,\infty)$ admitting the derivative of 3rd order at a fixed point $x \in (0,\infty)$, we have*

$$\lim_{\alpha\to\infty}\alpha\left(\left(\frac{d}{d\omega}B_\alpha^\rho(f;\omega,c)-f'(x)\right)_{\omega=x}\right)=f''(x)\left(cx+\frac{1}{2}\left(\frac{1}{\rho}+1\right)\right)+\frac{f'''(x)}{2}\left(cx^2+\left(\frac{1}{\rho}+1\right)x\right).$$

Proof. From the Taylor's theorem, we may write

$$f(t)=\sum_{k=0}^{3}\frac{(t-x)^k}{k!}f^{(k)}(x)+\psi(t,x)(t-x)^3,\ t\in[0,\infty), \tag{13}$$

where $\lim_{t\to x}\psi(t,x)=0$.
From equation (13), we obtain

$$\begin{aligned}\left(\frac{d}{d\omega}B_\alpha^\rho(f(t);\omega,c)\right)_{\omega=x} &= f'(x)\left(\frac{d}{d\omega}(B_\alpha^\rho(t;\omega,c)-x)\right)_{\omega=x}\\ &\quad+\frac{f''(x)}{2}\left(\frac{d}{d\omega}(B_\alpha^\rho(t^2;\omega,c)-2xB_\alpha^\rho(t;\omega,c)+x^2)\right)_{\omega=x}\\ &\quad+\frac{f'''(x)}{3!}\left(\frac{d}{d\omega}(B_\alpha^\rho(t^3;\omega,c)-3xB_\alpha^\rho(t^2;\omega,c)+3x^2B_\alpha^\rho(t;\omega,c)-x^3)\right)_{\omega=x}\\ &\quad+\left(\frac{d}{d\omega}(B_\alpha^\rho(\psi(t,x)(t-x)^3;\omega,c)\right)_{\omega=x}.\end{aligned}$$

Using Lemma 2.1, we get

$$\begin{aligned}\left(\frac{d}{d\omega}B_\alpha^\rho(f(t);\omega,c)\right)_{\omega=x} &= f'(x)+\frac{f''(x)}{2}\left\{2x\left(\frac{c}{\alpha}+1\right)+\frac{1}{\alpha}\left(\frac{1}{\rho}+1\right)-2x\right\}+\frac{f'''(x)}{3!}\left\{\frac{3}{\alpha}\left(1+\frac{2c}{\alpha}\right)x\right.\\ &\quad\left.\left(cx+1+\frac{1}{\rho}\right)+\frac{1}{\alpha^2}\left(\frac{2}{\rho^2}+\frac{3}{\rho}+1\right)\right\}+\left(\frac{d}{d\omega}(B_\alpha^\rho(\psi(t,x)(t-x)^3;\omega,c)\right)_{\omega=x}.\end{aligned}$$

Taking limit as $\alpha\to\infty$ on both sides of the above equation, we have

$$\begin{aligned}\lim_{\alpha\to\infty}\alpha\left(\left(\frac{d}{d\omega}B_\alpha^\rho(f;\omega,c)\right)_{\omega=x}-f'(x)\right) &= f''(x)\left(cx+\frac{1}{2}\left(\frac{1}{\rho}+1\right)\right)+\frac{f'''(x)}{2}\left(cx^2+\left(\frac{1}{\rho}+1\right)x\right)\\ &\quad+\lim_{\alpha\to\infty}\alpha\left(\frac{d}{d\omega}(B_\alpha^\rho(\psi(t,x)(t-x)^3;\omega,c)\right)_{\omega=x}.\end{aligned}$$

Proceeding in the same manner as in Theorem 3.4, we can easily show that

$$\lim_{\alpha\to\infty} \alpha\left(\frac{d}{d\omega}(B_\alpha^\rho(\psi(t,x)(t-x)^3;\omega,c)\right)_{\omega=x} = 0,$$

since $\lim_{\alpha\to\infty} \alpha^3(B_\alpha^\rho(t-x)^6;x,c)$ is finite for each $x \in [0,\infty)$ in view of the consequences (i) and (ii) of Lemma 2.1. Thus, the proof is completed. □

For $f \in C_B[0,\infty)$ (the space of all bounded and uniformly continuous functions on $[0,\infty)$), the Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf\{\| f-g \| + \delta \| g'' \|; g \in C_B^2[0,\infty)\}, \tag{14}$$

where $\delta > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Devore and Lorentz ([5] ,p.177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{15}$$

where the second order modulus of continuity is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) | .$$

Theorem 3.8. *Let $f \in C_B[0,\infty)$ and $x \geq 0$. Then, there exists a constant $C > 0$ such that*

$$|B_\alpha^\rho(f;x,c) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{x\{1+\rho(1+cx)\}}{\alpha\rho}}\right).$$

Proof. Let $g \in C_B^2[0,\infty)$. From the Taylor's theorem, we may write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

which implies that

$$\left|B_\alpha^\rho(g;x,c) - g(x)\right| = \left|B_\alpha^\rho\left(\int_x^t (t-v)g''(v)dv;x,c\right)\right|.$$

Since

$$\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2\|g''\|,$$

by Remark 2.2, we have

$$|B_\alpha^\rho(g;x,c) - g(x)| \leq \frac{x\{1+\rho(1+cx)\}}{\alpha\rho}\|g''\|.$$

From (1) it follows that

$$|B_\alpha^\rho(f;x,c)| \leq \|f\|.$$

Hence

$$\begin{aligned} |B_\alpha^\rho(f;x,c)(f,x) - f(x)| &\leq |B_\alpha^\rho(f-g;x,c) - (f-g)(x)| + |B_\alpha^\rho(g;x,c) - g(x)| \\ &\leq 2\|f-g\| + \frac{x\{1+\rho(1+cx)\}}{\alpha\rho}\|g''\|. \end{aligned} \tag{16}$$

Taking infimum on the right hand side over all $g \in C_B^2[0,\infty)$ and using (15), we obtain the desired result. Hence, the proof is completed. □

Let us now consider the Lipschitz-type space [15]:

$$Lip_M^*(r) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \le M \frac{|t-x|^r}{(t+x)^{\frac{r}{2}}}; x, t \in (0,\infty) \right\},$$

where M is a positive constant and $r \in (0,1]$.

Theorem 3.9. *Let $f \in Lip_M^*(r)$. Then, for all $x > 0$, we have*

$$|B_\alpha^\rho(f;x,c) - f(x)| \le M\left(\frac{1+\rho(1+cx)}{\alpha\rho}\right)^{\frac{r}{2}}.$$

Proof. Initially for $r = 1$, we may write

$$\begin{aligned} |B_\alpha^\rho(f;x,c) - f(x)| &\le \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t) - f(x)|dt \\ &\le M \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t) \frac{|t-x|}{\sqrt{t+x}} dt. \end{aligned}$$

Using the fact that $\frac{1}{\sqrt{t+x}} < \frac{1}{\sqrt{x}}$ and the Cauchy-Schwarz inequality, the above inequality implies that

$$|B_\alpha^\rho(f;x,c) - f(x)| \le \frac{M}{\sqrt{x}} \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|dt = \frac{M}{\sqrt{x}} B_\alpha^\rho(|t-x|;x,c) \le M\left(\sqrt{\frac{1+\rho(1+cx)}{\alpha\rho}}\right),$$

which proves the required result for $r = 1$. Now for $r \in (0,1)$, applying the Hölder inequality with $p = \frac{1}{r}$ and $q = \frac{1}{1-r}$, we have

$$\begin{aligned} |B_\alpha^\rho(f;x,c) - f(x)| &\le \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t) - f(x)|dt \\ &\le \left\{ \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \left(\int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t) - f(x)|dt \right)^{\frac{1}{r}} \right\}^r \\ &\le \left\{ \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t) - f(x)|^{\frac{1}{r}} dt \right\}^r \\ &\le M\left\{ \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t) \frac{|t-x|}{\sqrt{t+x}} dt \right\}^r \\ &\le \frac{M}{x^{\frac{r}{2}}} \left\{ \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|dt \right\}^r \\ &\le \frac{M}{x^{\frac{r}{2}}} (B_\alpha^\rho(|t-x|;x,c))^r \le M\left(\frac{1+\rho(1+cx)}{\alpha\rho}\right)^{\frac{r}{2}}. \end{aligned}$$

Thus, the proof is completed. □

Let $H_{x^2}[0,\infty)$ be space of all functions f defined on $[0,\infty)$ with the property $|f(x)| \le M_f(1+x^2)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions from $H_{x^2}[0,\infty)$. If $f \in C_{x^2}[0,\infty)$ and $\lim_{x\to\infty} |f(x)|(1+x^2)^{-1}$ exists, we write $f \in C_{x^2}^*[0,\infty)$. The norm on $f \in C_{x^2}^*[0,\infty)$ is given by

$$\| f \|_{x^2} := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

On the closed interval $[0, a]$. for any $a > 0$, we define the usual modulus of continuity of f as

$$\omega_a(f, \delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,a]} |f(t) - f(x)|.$$

Theorem 3.10. *Let $f \in C_{x^2}[0, \infty)$ and $\omega_{a+1}(f, \delta)$ be its modulus of continuity on the finite interval $[0, a + 1] \subset [0, \infty)$. Then for any $\alpha > 0$, we have*

$$\| B_\alpha^\rho(f; x, c) - f(x) \|_{C[0,a]} \leq 4M_f(1 + a^2)\mu_{\alpha,2}(a) + 2\omega_{a+1}\Big(f, \sqrt{\mu_{\alpha,2}(a)}\Big),$$

where $\mu_{\alpha,2}(a) = \dfrac{a\{1 + \rho(1 + ca)\}}{\alpha\rho}$.

Proof. From ([10], p.378), for $x \in [0, a]$ and $t \in [0, \infty)$, we have

$$|f(t) - f(x)| \leq 4M_f(1 + a^2)(t - x)^2 + \Big(1 + \frac{|t - x|}{\delta}\Big)\omega_{a+1}(f, \delta), \delta > 0.$$

Applying $B_\alpha^\rho(.; x, c)$ and then Cauchy-Schwarz inequality to the above inequality, we get

$$\begin{aligned} |B_\alpha^\rho(f; x, c) - f(x)| &\leq 4M_f(1 + a^2)B_\alpha^\rho((t - x)^2; x, c) + \omega_{a+1}\Big(1 + \frac{1}{\delta}B_\alpha^\rho(|t - x|; x, c)\Big) \\ &\leq 4M_f(1 + a^2)\mu_{\alpha,2}(a) + \omega_{a+1}\Big(1 + \frac{1}{\delta}\sqrt{\mu_{\alpha,2}(a)}\Big). \end{aligned}$$

By choosing $\delta = \sqrt{\mu_{\alpha,2}(a)}$, we obtain the desired result. □

Theorem 3.11. *For each $f \in C^*_{x^2}[0, \infty)$, we have*

$$\lim_{\alpha\to\infty} \| B_\alpha^\rho(f) - f \|_{x^2} = 0.$$

Proof. From the Korovkin theorem, we see that it is sufficient to verify the following three conditions

$$\lim_{\alpha\to\infty} \| B_\alpha^\rho(t^k; x, c) - x^k) \|_{x^2} = 0, \; k = 0, 1, 2. \tag{17}$$

Since $B_\alpha^\rho(1; x, c) = 1$, the condition in (17) holds for $k = 0$.
By Lemma 2.1, we have for $\alpha > 0$
$\| B_\alpha^\rho(t; x, c) - x) \|_{x^2} = 0$, which implies that the condition in (17) holds for $k = 1$.
Similarly, we can write for $\alpha > 0$

$$\begin{aligned} \| B_\alpha^\rho(t^2; x, c) - x^2 \|_{x^2} &\leq \left\| \frac{\rho x(1 + cx) + x}{\rho\alpha} \right\|_{x^2} \leq \frac{\rho + 1}{\rho\alpha} \sup_{x\in[0,\infty)} \frac{x}{1 + x^2} + \frac{c}{\alpha} \sup_{x\in[0,\infty)} \frac{x^2}{1 + x^2} \\ &\leq \Big(1 + \frac{1}{\rho} + c\Big)\frac{1}{\alpha} \end{aligned}$$

which implies that $\lim_{\alpha\to\infty} \| B_\alpha^\rho(t^2; x, c) - x^2 \|_{x^2} = 0$, the equation (17) holds for $k = 2$. This completes the proof. □

Let $f \in C^*_{x^2}[0, \infty)$. The weighted modulus of continuity is defined as :

$$\Omega_2(f, \delta) = \sup_{x\geq0, 0<h\leq\delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}.$$

Lemma 3.12. *[10] Let $f \in C^*_{x^2}[0, \infty)$, then:*

(i) $\Omega(f,\delta)$ *is a monotone increasing function of* δ;

(ii) $\lim\limits_{\delta\to 0^+}\Omega(f,\delta)=0$;

(iii) *for each* $m\in\mathbb{N},\Omega(f,m\delta)\leq m\Omega(f,\delta)$;

(iv) *for each* $\lambda\in[0,\infty),\Omega(f,\lambda\delta)\leq(1+\lambda)\Omega(f,\delta)$.

Theorem 3.13. *Let* $f\in C^*_{x^2}[0,\infty)$, *then there exists a positive constant* K *such that*

$$\sup_{x\in(0,\infty)}\frac{|B^{\rho}_{\alpha}(f;x,c)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\leq K\Omega\Big(f,\frac{1}{\alpha}\Big).$$

Proof. For $t>0, x\in[0,\infty)$ and $\delta>0$, by definition of $\Omega(f,\delta)$ and Lemma 3.12, we get

$$\begin{aligned}|f(t)-f(x)| &\leq (1+(x+|x-t|^2))\Omega(f,|t-x|)\\ &\leq 2(1+x^2)(1+(t-x)^2)\Big(1+\frac{|t-x|}{\delta}\Big)\Omega(f,\delta).\end{aligned}$$

Since B^{ρ}_{α} is linear and positive, we have

$$|B^{\rho}_{\alpha}(f;x,c)-f(x)|\leq 2(1+x^2)\Omega(f,\delta)\Big\{1+B^{\rho}_{\alpha}((t-x)^2;x,c)+B^{\rho}_{\alpha}\Big((1+(t-x)^2)\frac{|t-x|}{\delta};x,c\Big)\Big\}. \tag{18}$$

Using (2), we have

$$B^{\rho}_{\alpha}((t-x)^2;x,c)\leq K_1\frac{(1+x^2)}{\alpha}, \text{for some positive number } K_1. \tag{19}$$

Applying Cauchy-Schwarz inequality to the second term of equation (18), we have

$$\begin{aligned}B^{\rho}_{\alpha}\Big((1+(t-x)^2)\frac{|t-x|}{\delta};x,c\Big) &\leq \frac{1}{\delta}\sqrt{B^{\rho}_{\alpha}((t-x)^2;x,c)}\\ &\quad+\frac{1}{\delta}\sqrt{B^{\rho}_{\alpha}((t-x)^4;x,c)}\sqrt{B^{\rho}_{\alpha}((t-x)^2;x,c)}.\end{aligned} \tag{20}$$

By using Lemma 2.1, there exists a positive constant K_2 such that

$$\sqrt{\Big(B^{\rho}_{\alpha}(t-x)^4;x,c)\Big)}\leq K_2\frac{(1+x^2)}{\alpha}. \tag{21}$$

Combining the estimates of (18)-(21) and taking $K=2(1+K_1+\sqrt{K_1}+K_2\sqrt{K_1})$, $\delta=\frac{1}{\sqrt{\alpha}}$, we obtain the required result. □

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