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Unique Common Fixed Points for Mixed Contractive Mappings on Non-Normal Cone Metric Spaces over Banach Algebras

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Abstract. In this paper, we discuss and obtain some new unique common fixed point theorems for two mappings satisfying Kannan type mixed contractive conditions and Chatterjea type mixed contractive conditions respectively on cone metric spaces over Banach algebras without the assumption of normality and give some generalizations of Kannan type and Chatterjea type fixed point theorems.

1. Introduction

In 2007, cone metric spaces were reviewed by Huang and Zhang, as a generalization of metric spaces (see [1]). The distance $d(x, y)$ of two elements x and y in a cone metric space X is defined to be a vector in an ordered Banach space E, quite different from that which is defined a non-negative real numbers in general metric space. In 2011, Beg A, Azam A and Arshad M([2]) introduced the concept of topological vector space-valued cone metric spaces, where the ordered Banach space in the definition of cone metric spaces is replaced by a topological vector space.

Recently, some authors investigated the problems of whether cone metric spaces are equivalent to metric spaces in terms of the existence of fixed points of the mappings and successfully established the equivalence between some fixed point results in metric spaces and in (topological vector space-valued) cone metric spaces, see [3-6]. Actually, they showed that any cone metric space (X, d) is equivalent to a usual metric space (X, d^*), where the real-metric function d^* is defined by a nonlinear scalarization function ξ_e(see [4]) or by a Minkowski function q_e(see[5]). After that, some other interesting generalizations were developed, see. for instance, [7].

In 2013, Liu and Xu [8] introduced the concept of cone metric spaces over Banach algebras, replacing a Banach space E by a Banach algebra $\mathcal{A}$ as the underlying spaces of cone metric spaces. And the authors in [8-10] discussed and obtained Banach fixed point theorem, Kannan type fixed point theorem, Chatterjea type fixed point theorem and Ćirić type fixed point theorem in cone metric spaces over Banach algebras. Especially, the authors in [10] gave an example to show that fixed point results of mappings in this new space are indeed more different than the standard results of cone metric spaces presented in literature.

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In 1968, Kannan[11] obtained the generalization of Banach contractive principle, that is, Kannan fixed point theorem:

Theorem 1.1 Let X be a complete metric space and $f : X \to X$ a mapping. If there is a $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha\,[d(x, fx) + d(y, fy)].$$

Then f has a unique fixed point.

In 2011, Shukla and Tiwari[12] obtained the variant result of Kannan fixed point theorem:

Theorem 1.2 Let X be a complete metric space and $f : X \to X$ a mapping. If there is a $\alpha \in [0, \frac{1}{3})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha\,[d(x, fx) + d(y, fy) + d(x, y)].$$

Then f has a unique fixed point.

In 2010 and 2014, new generalizations of Kannan fixed point theorem are given in [13] and [14] respectively:

Theorem 1.3 Let X be a complete metric space, $T, S : X \to X$ two mappings such that T is one to one, continuous and subsequentially convergent(see [13-14]). If there is a $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(TSx, TSy) \leq \alpha\,[d(Tx, TSx) + d(Ty, TSy)].$$

Then S has a unique fixed point.

Theorem 1.4 Let X be a complete metric space, $T, f : X \to X$ two mappings such that T is one to one, continuous and subsequentially convergent. If there is a $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$F(d(Tfx, Tfy)) \leq \alpha\,[F(d(Tx, Tfx)) + F(d(Ty, Tfy))],$$

where $F : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing mapping satisfying $F^{-}(0) = \{0\}$. Then f has a unique fixed point.

In 1972, Chatterjea[15] obtained the another generalization of Banach contractive principle, that is, Chatterjea fixed point theorem:

Theorem 1.5 Let X be a complete metric space, $f : X \to X$ a mapping. If there is a $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha\,[d(x, fy) + d(y, fx)].$$

Then f has a unique fixed point.

In this paper, we obtain new common fixed point theorems for two mappings satisfying mixed contractive conditions on cone metric spaces over Banach algebras and give fixed point theorems. These results generalize and improve Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem and others on cone metric spaces over Banach algebras.

2. Preliminaries

Let $\mathcal{A}$ always be a Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties(for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\| xy \| \leq \| x \| \| y \|$.

In this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. an element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x denoted by x^{-1}. For more detail, we refer to [16].

We say that $\{x_1, x_2, \cdots, x_n\} \subset \mathcal{A}$ commute if $x_i x_j = x_j x_i$ for all $i, j \in \{1, 2, \cdots, n\}$.

Proposition 2.1([16]) Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}} = \inf_{n\to\infty} \| x^n \|^{\frac{1}{n}} < 1.$$

Then $(e - x)$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{+\infty} x^i.$$

Remark 2.1 1) $r(x) \leq \| x \|$ for any $x \in \mathcal{A}$(see [16]).

2) In Proposition 2.1, if the condition $r(x) < 1$ is replaced by the condition $\| x \| < 1$, then the conclusion remains true.

A subset P of a Banach algebra $\mathcal{A}$ is called a cone if

1. P is nonempty closed and $\{0, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{0\}$.

Where 0 denotes the null of the Banach algebra $\mathcal{A}$.

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ stand for $x \leq y$ and $x \neq y$. While $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called solid if $\operatorname{int} P \neq \emptyset$.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$.

$$0 \leq x \leq y \implies \| x \| \leq M \| y \| .$$

The least positive number satisfying the above is called the normal constant of P.

Here, we always assume that P is a solid and $\leq$ is the partial ordering with respect to P.

Definition 2.1([1, 9-10]) Let X be a non-empty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space(over a Banach algebra $\mathcal{A}$).

Remark 2.2 The examples of cone metric spaces(over a Banach algebra $\mathcal{A}$) can be found in [8-10].

Definition 2.2([1, 8]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then:

1. $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

2. $\{x_n\}$ is Cauchy sequence whenever for each $c \in \mathcal{A}$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

3. (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent.

Definition 2.3([17-18]) Let P be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

Proposition 2.2([17]) Let P be a solid cone in a Banach space $\mathcal{A}$, $\{x_n\}$ and $\{y_n\}$ two sequences in P. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.

Proposition 2.3([17]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and $\{x_n\}$ a sequence in P. Then the following conditions are equivalent:

(1) $\{x_n\}$ is a c-sequence;

(2) for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n < c$ for all $n \geq n_0$;

(3) for each $c \gg 0$ there exists $n_1 \in \mathbb{N}$ such that $x_n \leq c$ for all $n \geq n_1$.

Proposition 2.4([10]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and $\{u_n\}$ a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Proposition 2.5([10]) Let $\mathcal{A}$ be a Banach algebra with a unit e, P a cone in $\mathcal{A}$ and $\leq$ be the semi-order generated by the cone P. The following assertions hold true:

(i) For any $x, y \in \mathcal{A}$, $a \in P$ with $x \leq y$, $ax \leq ay$;

(ii) For any sequences $\{x_n\}, \{y_n\} \subset \mathcal{A}$ with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, where $x, y \in \mathcal{A}$, we have $x_n y_n \to xy$ as $n \to \infty$.

Proposition 2.6([10]) Let $\mathcal{A}$ be a Banach algebra with a unit e, P a cone in $\mathcal{A}$ and $\leq$ be the semi-order generated by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true:

(i) Suppose that x is invertible and that $x^{-1} > 0$ implies $x > 0$, then for any integer $n \geq 1$, we have $\lambda^n \leq \lambda \leq e$.

(ii) For any $u > 0$, we have $u \nleq \lambda\, u$, i.e., $\lambda\, u - u \notin P$.

(iii) If $\lambda \geq 0$, then $(e - \lambda)^{-1} \geq 0$.

Proposition 2.7([10]) Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P a solid cone in A and $\{x_n\}$ a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have

(i) $\{d(x_n, x)\}$ is a c-sequence.

(ii) For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.

Lemma 2.1([19]) If E is a real Banach space with a cone P and if $a \leq \lambda\, a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = 0$.

Lemma 2.2([20]) If E is a real Banach space with a cone P and if $0 \leq u \ll c$ for all $0 \ll c$, then $u = 0$.

Lemma 2.3([20]) If E is a real Banach space with a solid cone P and if $\| x_n \| \to 0$ as $n \to \infty$, then for any $0 \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.

Lemma 2.4([10]) If $\mathcal{A}$ is a Banach algebra and $k \in \mathcal{A}$ with $r(k) < 1$, then $\| k^n \| \to 0$ as $n \to \infty$.

Lemma 2.5([10]) Let $\mathcal{A}$ be a Banach algebra and $x, y \in \mathcal{A}$. If x and y commute, then the following hold:

(i) $r(xy) \leq r(x)r(y)$;

(ii) $r(x + y) \leq r(x) + r(y)$;

(iii) $| r(x) - r(y) | \leq r(x - y)$.

Lemma 2.6([10]) Let $\mathcal{A}$ be a Banach algebra and $\{x_n\}$ a sequence in $\mathcal{A}$. Suppose that $\{x_n\}$ converge to $x \in \mathcal{A}$ and that x_n and x commute for all n, then $r(x_n) \to r(x)$ as $n \to \infty$.

Lemma 2.7([21]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and $\{\alpha, \beta, \gamma\} \subset \mathcal{A}$ with $r(\gamma) < 1$. If $\{\alpha, \beta, \gamma\}$ commute, then

$$r\Big((e - \gamma)^{-1}(\alpha + \beta)\Big) \leq \frac{r(\alpha + \beta)}{1 - r(\gamma)} \leq \frac{r(\alpha) + r(\beta)}{1 - r(\gamma)}.$$

In particular,

$$r\Big((e - \gamma)^{-1}\Big) \leq \frac{1}{1 - r(\gamma)} \leq \frac{r(\alpha) + r(\beta)}{1 - r(\gamma)}$$

for all $\alpha, \beta, \gamma \in \mathcal{A}$ with $r(\gamma) < 1$ and $\alpha + \beta = e$ and $\{\alpha, \beta, \gamma\}$ commute.

Lemma 2.8 ([21])(Cauchy Principle) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, P a solid cone in $\mathcal{A}$ and $k \in P$ with $r(k) < 1$. If a sequence $\{x_n\} \subset X$ satisfies that

$$d(x_{n+1}, x_{n+2}) \leq k d(x_n, x_{n+1}), \forall\, n = 0, 1, 2, \cdots.$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.9([21]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, P a solid cone in $\mathcal{A}$ and $\{x_n\} \subset X$ a sequence. If $\{x_n\}$ is convergent, then the limits of $\{x_n\}$ is unique.

Definition 2.4 Let (X, d) and (Y, ρ) be two cone metric spaces over Banach algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, P and Q be the corresponding solid cones in $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that a mapping $f : X \to Y$ is continuous at $x^* \in X$ if for each $c \in \mathcal{B}$ with $0 \ll c$, there exists $b \in \mathcal{A}$ with $0 \ll b$ such that $d(x, x^*) \ll b$ for all $x \in X$ implies $\rho(fx, fx^*) \ll c$.

If f is continuous at all $x \in X$, then we say that f is continuous on X.

Lemma 2.10 Let (X, d) and (Y, ρ) be two cone metric spaces over Banach algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, P and Q be the corresponding solid cones in $\mathcal{A}$ and $\mathcal{B}$ respectively. If $f : X \to Y$ is a mapping, then f is continuous at $x^* \in X$ if and only if $fx_n \to fx^*$ as $n \to \infty$ whenever $\{x_n\} \subset X$ converges to $x^* \in X$.

Proof ($\Longrightarrow$) Suppose that $x_n \to x^*$. For any $c \in \mathcal{B}$ with $0 \ll c$, by given conditions there exists $b \in \mathcal{A}$ with $0 \ll b$ such that $d(x, x^*) \ll b$ implies $\rho(fx, fx^*) \ll c$. For b, there exists $N \in \mathbb{N}$ such that $d(x_n, x^*) \ll b$ for all $n > N$, so $d(fx_n, fx^*) \ll c$ for all $n > N$. Hence $fx_n \to fx^*$ as $n \to \infty$.

($\Longleftarrow$) Suppose that f is not continuous at x^*, then there exists $c \in \mathcal{B}$ with $0 \ll c$ such that for any $b \in \mathcal{A}$ with $0 \ll b$ there is $x \in X$ satisfying that $d(x, x^*) \ll b$ but $c - d(fx, fx^*) \notin \mathrm{int}Q$. Fix any $b \gg 0$, then $0 \ll \frac{b}{n}$ for all $n \in \mathbb{N}$. Hence for $\frac{b}{n}$, there exists $x_n \in X$ such that $d(x_n, x^*) \ll \frac{b}{n}$ but $c - d(fx_n, fx^*) \notin \mathrm{int}Q$ for all $n \in \mathbb{N}$. For any $a \in \mathcal{A}$ with $0 \ll a$, since $\frac{b}{n} \to 0$(as $n \to \infty$), there exists $N \in \mathbb{N}$ such that $\frac{b}{n} \ll a$ for all $n > N$, hence $d(x_n, x^*) \ll a$. This shows that $x_n \to x^*$ as $n \to \infty$. But $c - d(fx_n, fx^*) \notin \mathrm{int}Q$ for all n shows that fx_n is not convergent to fx^*. This is a contradiction, hence f is continuous at x^*.

Xu and Radenović [10] obtained the following Kannan type and Chatterjea type fixed point theorems on cone metric spaces over Banach algebras, which are the generalizations of Theorem 1.1 and Theorem 1.5 respectively.

Theorem 2.1([10]) Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a mapping. Suppose that there exists $\alpha \in P$ with $r(\alpha) < \frac{1}{2}$ such that for each $x, y \in X$,

$$d(fx, fy) \le \alpha\, [d(y, fy) + d(x, fx)].$$

Then f has a unique fixed point.

Theorem 2.2([10]) Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a mapping. Suppose that there exists $\alpha \in P$ with $r(\alpha) < \frac{1}{2}$ such that for each $x, y \in X$,

$$d(fx, fy) \le \alpha\, [d(x, fy) + d(y, fx)].$$

Then f has a unique fixed point.

3. Common fixed points and fixed points

Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, P a solid cone in $\mathcal{A}$ and $f, g : X \to X$ two mappings. We say that f and g satisfy the mixed contractive condition if the following holds

$$d(fgx, gfy) \le \alpha\, d(gx, fy),\ \forall\, x, y \in X,$$

where $\alpha \in P$ with $r(\alpha) < 1$.

Remark 3.1 If $\alpha \in [0, 1)$, then the above concept is the contractive version of the concept of mixed expansive condition defined in [22].

At first, we give a unique common fixed point theorem for two mappings satisfying Kannan type mixed contractive conditions.

Theorem 3.1 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exist $\{\alpha, \beta, \gamma\} \subset P$ with $r(\alpha) + r(\beta) + r(\gamma) < 1$ which commute such that for each $x, y \in X$, $x \ne y$,

$$d(fgx, gfy) \le \alpha\, d(fy, gfy) + \beta\, d(gx, fgx) + \gamma\, d(fy, gx) \tag{3.1}$$

If fX or gX is complete, then f and g have a unique common fixed point.

Proof Take any element $x_0 \in X$ and construct a sequence $\{x_n\}_0^\infty$ satisfying

$$x_{2n+1} = fx_{2n},\ x_{2n+2} = gx_{2n+1},\ n = 0, 1, \cdots. \tag{3.2}$$

If there exists n such that $x_{2n} = x_{2n+1}$, then $d(x_{2n}, x_{2n+1}) = 0$. By (3.1),

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) = d(fgx_{2n-1}, gfx_{2n}) \\ \le\ & \alpha\, d(fx_{2n}, gfx_{2n}) + \beta\, d(gx_{2n-1}, fgx_{2n-1}) + \gamma\, (fx_{2n}, gx_{2n-1}) \\ =\ & \alpha\, d(x_{2n+1}, x_{2n+2}) + \beta\, d(x_{2n}, x_{2n+1}) + \gamma\, (x_{2n+1}, x_{2n}) \\ =\ & \alpha\, d(x_{2n+1}, x_{2n+2}), \end{aligned}$$

hence

$$(e - \alpha) d(x_{2n+1}, x_{2n+2}) \le 0. \tag{3.3}$$

But $r(\alpha) < 1$ implies $(e-\alpha)^{-1} \geq 0$ by Proposition 2.6, so $d(x_{2n+1}, x_{2n+2}) = 0$ by (3.3) and Proposition 2.5, hence $x_{2n} = x_{2n+1} = x_{2n+2}$, therefore x_{2n} is a common fixed point of f and g. Similarly, if there exists n such that $x_{2n+1} = x_{2n+2}$, then we can prove that x_{2n+1} is a common fixed point of f and g. So we assume that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2\cdots$.

For any fixed n,

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) = d(gfx_{2n}, fgx_{2n-1}) \\ \leq\ & \alpha\, d(fx_{2n}, gfx_{2n}) + \beta\, d(gx_{2n-1}, fgx_{2n-1}) + \gamma\, d(fx_{2n}, gx_{2n-1}) \\ =\ & \alpha\, d(x_{2n+1}, x_{2n+2}) + (\beta+\gamma) d(x_{2n}, x_{2n+1}) \end{aligned}$$

and

$$\begin{aligned} & d(x_{2n+2}, x_{2n+3}) = d(gfx_{2n}, fgx_{2n+1}) \\ \leq\ & \alpha\, d(fx_{2n}, gfx_{2n}) + \beta\, d(gx_{2n+1}, fgx_{2n+1}) + \gamma\, d(fx_{2n}, gx_{2n+1}) \\ =\ & (\alpha+\gamma)\, d(x_{2n+1}, x_{2n+2}) + \beta\, d(x_{2n+2}, x_{2n+3}). \end{aligned}$$

Hence we have the following results respectively:

$$d(x_{2n+1}, x_{2n+2}) \leq K_1 d(x_{2n}, x_{2n+1}) \tag{3.4}$$

and

$$d(x_{2n+2}, x_{2n+3}) \leq K_2 d(x_{2n+1}, x_{2n+2}), \tag{3.5}$$

where $K_1 = (e-\alpha)^{-1}(\beta+\gamma)$, $K_2 = (e-\beta)^{-1}(\alpha+\gamma)$. Let $K = K_1K_2$.

Since $\{\alpha, \beta, \gamma\}$ commute and $(e-\beta)^{-1} = \sum_{i=0}^{\infty} \beta^i$ and $(e-\alpha)^{-1} = \sum_{i=0}^{\infty} \alpha^i$, so $\{\alpha, \beta, \gamma\, K_1, K_2\}$ also commute. Hence by Lemma 2.5 and Lemma 2.7,

$$r(K) \leq r(K_1)r(K_2) \leq \frac{r(\beta)+r(\gamma)}{1-r(\alpha)}\,\frac{r(\alpha)+r(\gamma)}{1-r(\beta)} < 1. \tag{3.6}$$

Using mathematical induction, we obtain

$$d(x_{2n+1}, x_{2n+2}) \leq K_1\, d(x_{2n}, x_{2n+1}) \leq K_1K_2 d(x_{2n-1}, x_{2n}) \leq \cdots \leq K^n K_1 d(x_0, x_1) \tag{3.7}$$

and

$$d(x_{2n+2}, x_{2n+3}) \leq K_2 d(x_{2n+1}, x_{2n+2}) \leq K^{n+1} d(x_0, x_1). \tag{3.8}$$

For any $p, q \in \mathbb{N}$ with $p < q$,

$$d(x_{2p+1}, x_{2q+1}) \leq \sum_{i=2p+1}^{2q} d(x_i, x_{i+1}) \leq \Big(K_1 \sum_{i=p}^{q-1} K^i + \sum_{i=p+1}^{q} K^i\Big) d(x_0, x_1) \leq (e-K)^{-1} K^p (K_1+K) d(x_0, x_1). \tag{3.9}$$

Similarly,

$$d(x_{2p}, x_{2q+1}) \leq \sum_{i=2p}^{2q} d(x_i, x_{i+1}) \leq \Big(\sum_{i=p}^{q} K^i + K_1 \sum_{i=p}^{q-1} K^i\Big) d(x_0, x_1) \leq (e-K)^{-1} K^p (e+K_1) d(x_0, x_1); \tag{3.10}$$

$$d(x_{2p}, x_{2q}) \leq \sum_{i=2p}^{2q-1} d(x_i, x_{i+1}) \leq \Big(\sum_{i=p}^{q-1} K^i + K_1 \sum_{i=p}^{q-1} K^i\Big) d(x_0, x_1) \leq (e-K)^{-1} K^p (e+K_1) d(x_0, x_1); \tag{3.11}$$

$$d(x_{2p+1}, x_{2q}) \leq \sum_{i=2p+1}^{2q-1} d(x_i, x_{i+1}) \leq \Big(K_1 \sum_{i=p}^{q-1} K^i + \sum_{i=p+1}^{q-1} K^i\Big) d(x_0, x_1) \leq (e-K)^{-1} K^p (K_1+K) d(x_0, x_1). \tag{3.12}$$

Since $\| K^n \| \to 0$ as $n \to \infty$ by Lemma 2.4 and (3.6), we have

$$\| (e-K)^{-1} K^p (K_1 + K) d(x_0, x_1) \| \to 0 \text{ as } p \to \infty \tag{3.13}$$

and

$$\| (e-K)^{-1} K^p (e + K_1) d(x_0, x_1) \| \to 0 \text{ as } p \to \infty. \tag{3.14}$$

Therefore by Lemma 2.3, for any $0 \ll c$ there exists N such that

$$(e-K)^{-1} K^p (K_1 + K) d(x_0, x_1) \ll c, \ \forall\, p > N \tag{3.15}$$

and

$$(e-K)^{-1} K^p (e + K_1) d(x_0, x_1) \ll c, \ \forall\, p > N. \tag{3.16}$$

Combining (3.9)-(3.12) and (3.15)-(3.16), we show that there is a $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$ for all $n > m > n_0$. Hence $\{x_n\}$ is a Cauchy sequence.

Suppose that fX is complete. Since $x_{2n+1} \in fX$ and $\{x_n\}$ is a Cauchy sequence, there exist $u \in fX$ and $v \in X$ such that $x_{2n+1} \to u = fv$ as $n \to \infty$. Also we obtain $x_{2n+2} \to u$ as $n \to \infty$ since $d(x_{2n+2}, u) \leq d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, u)$. By (3.1),

$$\begin{aligned} & d(u, gu) = d(u, gfv) \\ \leq\ & d(u, x_{2n+3}) + d(x_{2n+3}, gfv) \\ =\ & d(u, x_{2n+3}) + d(fgx_{2n+1}, gfv) \\ \leq\ & d(u, x_{2n+3}) + \alpha\, d(fv, gfv) + \beta\, d(gx_{2n+1}, fgx_{2n+1}) + \gamma\, d(fv, gx_{2n+1}) \\ =\ & d(u, x_{2n+3}) + \alpha\, d(u, gu) + \beta\, d(x_{2n+2}, x_{2n+3}) + \gamma\, d(u, x_{2n+2}), \end{aligned}$$

hence

$$(e-\alpha) d(u, gu) \leq (e+\beta) d(u, x_{2n+3}) + (\beta+\gamma) d(u, x_{2n+2}),$$

i.e.,

$$d(u, gu) \leq (e-\alpha)^{-1}(e+\beta) d(u, x_{2n+3}) + (e-\alpha)^{-1}(\beta+\gamma) d(u, x_{2n+2}).$$

Since x_n converges to u, by Proposition 2.4 and Proposition 2.7, for each $c \gg 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$(e-\alpha)^{-1}(e+\beta) d(u, x_{2n+3}) \ll \frac{c}{2},\ (e-\alpha)^{-1}(\beta+\gamma) d(u, x_{2n+2}) \ll \frac{c}{2},$$

hence for all $n > N$,

$$d(u, gu) \ll c.$$

Therefore, by Lemma 2.2, we have

$$u = gu.$$

By (3.1) again,

$$d(u, fu) = d(fgu, gfv) \leq \alpha d(fv, gfv) + \beta d(gu, fgu) + \gamma d(fv, gu) = \beta d(u, fu),$$

hence

$$(e-\beta) d(u, fu) \leq 0.$$

Therefore $d(u, fu) = 0$, i.e., $u = fu$ by Proposition 2.5 and Proposition 2.6. This show that u is the common fixed point f and g.

Suppose that u^* is also common fixed point of f and g but $u^* \neq u$, then by (3.1),

$$d(u, u^*) = d(fgu, gfu^*) \leq \alpha\, d(fu^*, gfu^*) + \beta\, d(gu, fgu) + \gamma\, d(fu^*, gu) = \gamma\, d(u.u^*),$$

hence

$$(e-\gamma) d(u, u^*) \leq 0.$$

Therefore $u = u^*$ by Proposition 2.5 and Proposition 2.6. This contradiction shows that u is the unique common fixed point of f and g. Similarly, we obtain the same result for the case that gX is complete.

Using Theorem 3.1, we obtain the following result.

Corollary 3.1 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exist $\{\alpha, \beta, \gamma\} \subset P$ with $r(\alpha) + r(\beta) + r(\gamma) < 1$ which commute. If (3.1) holds and f or g is onto, then f and g have a unique common fixed point.

Using Theorem 3.1, we can obtain the following unique common fixed point theorems for two mappings satisfying weak Kannan type mixed (see Theorem 1.2) and Kannan type mixed (see Theorem 1.1) contractive conditions.

Theorem 3.2 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exists $\gamma \in P$ with $r(\gamma) < \frac{1}{3}$ such that for each $x, y \in X$, $x \neq y$,

$$d(fgx, gfy) \preceq \gamma\,[d(fy, gfy) + d(gx, fgx) + d(fy, gx)].$$

If fX or gX is complete, then f and g have a unique common fixed point.

Proof Let $\alpha = \beta = \gamma$ in Theorem 3.1, then the conclusion follows from Theorem 3.1.

Theorem 3.3 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exists $\alpha \in P$ with $r(\alpha) < \frac{1}{2}$ such that for each $x, y \in X$, $x \neq y$,

$$d(fgx, gfy) \preceq \alpha\,[d(fy, gfy) + d(gx, fgx)].$$

If fX or gX is complete, then f and g have a unique common fixed point.

Proof Let $\alpha = \beta, \gamma = 0$ in Theorem 3.1, then the conclusion follows from Theorem 3.1.

Using Corollary 3.1, we obtain the next two fixed point theorems, which are the variant forms of Kannan type- Chatterjea type fixed point theorems(i.e., Theorem 2.1 and Theorem 2.2) and Banach contraction principle.

Theorem 3.4 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a mapping. Suppose that there exists $\gamma \in P$ with $r(\gamma) < \frac{1}{2}$ such that for each $x, y \in X$, $x \neq y$,

$$d(fx, fy) \preceq \gamma\,[d(x, fx) + d(fy, x)].$$

Then f has a unique fixed point.

Proof Let $\alpha = 0, \beta = \gamma$ and $g = 1_X$ in (3.1), then the conclusion follows from Corollary 3.1.

Theorem 3.5 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a mapping. Suppose that there exists $\beta \in P$ with $r(\beta) < 1$ [or $\gamma \in P$ with $r(\gamma) < 1$] such that for each $x, y \in X$, $x \neq y$,

$$d(fx, fy) \preceq \beta\, d(x, fx)\, [or\ d(fx, fy) \preceq \gamma\, d(x, fy)].$$

Then f has a unique fixed point.

Proof Let $\alpha = \gamma = 0$[or $\alpha = \beta = 0$] and $g = 1_X$ in (3.1), then the conclusion follows from Corollary 3.1.

Now, we give the continuous and non-surjective version of Corollary 3.1.

Theorem 3.6 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two continuous mappings. If that there exist $\{\alpha, \beta, \gamma\} \subset P$ with $r(\alpha) + r(\beta) + r(\gamma) < 1$ which commute such that (3.1) holds. Then f and g have a unique common fixed point.

Proof Following the proof of Theorem 3.1, we have a sequence $\{x_n\}$ satisfying (3.2) such that $\{x_n\}$ is a Cauchy sequence, hence there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. By Lemma 2.10 and the continuity of f and g, we have

$$x_{2n+1} = fx_{2n} \to fu,\ x_{2n+2} = gx_{2n+1} \to gu.$$

Therefore, $fu = u = gu$ by Lemma 2.9. The rest is similar to the proof of Theorem 3.1.

Using Theorem 3.6, we can give two unique fixed point theorems, i.e., generalized Kannan type fixed point theorem and generalized Banach contractive principle.

Theorem 3.7 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a continuous mapping. Suppose that there exists $\alpha \in P$ with $r(\alpha) < \frac{1}{2}$ such that for each $x, y \in X$, $x \neq y$,

$$d(f^2x, f^2y) \le \alpha\,[d(fy, f^2y) + d(fx, f^2x)].$$

Then f has a unique fixed point.

Proof Let $f = g, \alpha = \beta, \gamma = 0$ in Theorem 3.6, then the conclusion follows from Theorem 3.6.

Theorem 3.8 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a continuous mapping. Suppose that there exists $\gamma \in P$ with $r(\gamma) < 1$ such that for each $x, y \in X$, $x \neq y$,

$$d(f^2x, f^2y) \le \gamma\, d(fx, fy).$$

Then f has a unique fixed point.

Proof Let $f = g, \alpha = \beta = 0$ in Theorem 3.6, then the conclusion follows from Theorem 3.6.

Remark 3.2 Obviously, the contractive condition in Theorem 3.7 is weaker than Kannan type contractive condition(see Theorem 1.1 and Theorem 2.1). In fact, the contractive condition in Theorem 3.7 can be written as

$$d(f(fx), f(fy)) \le \gamma\,[d(fy, f(fy)) + d(fx, f(fx))],\ \forall\, x, y \in X,\ x \neq y. \tag{3.17}$$

(3.17) is equivalent to the following

$$d(fx, fy) \le \gamma\,[d(y, fy) + d(x, fx)],\ \forall\, x, y \in f(X). \tag{3.18}$$

Similarly, the contractive condition in Theorem 3.8 can be written as

$$d(fx, fy) \le \gamma\, d(x, y),\ \forall\, x, y \in f(X). \tag{3.19}$$

Hence the contractive condition in Theorem 3.8 is weaker than Banach contractive condition.

Example 3.1 Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\| x \| = \| x \|_\infty + \| x' \|_\infty$ for $x \in \mathcal{A}$. Define multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with unit $e = 1$. The set $P = \{x \in \mathcal{A} : x \ge 0\}$ is not normal(see[10, 23]).

Let $X = \{a, b, c\}$ and define $d : X \times X \to \mathcal{A}$ as follows: for each $t \in [0, 1]$ and $x \in X$,

$$d(a, b)(t) = d(b, a)(t) = e^t, d(a, c)(t) = d(c, a)(t) = 3e^t, d(b, c)(t) = d(c, b)(t) = 2e^t, d(x, x)(t) = 0.$$

Then (X, d) is a complete cone metric space over a Banach algebra $\mathcal{A}$ without normality.

Define a mapping $f : X \to X$ by $fa = a, fb = c, fc = a$. Let $\alpha \in P$ be $\alpha(t) = \frac{1}{5}t + \frac{1}{4}$ for all $t \in [0, 1]$. It is easy to prove that $r(\alpha) = \frac{9}{20} < \frac{1}{2}$.

$f^2x = a$ for all $x \in X$ implies that $d(f^2x, f^2y) = 0$ for all $x, y \in X$, hence the contractive condition in Theorem 3.7 is satisfies. Therefore f has a unique fixed point a.

On the other hand, if f satisfies Kannan type contractive condition, then there exists $\alpha' \in P$ with $r(\alpha') < \frac{1}{2}$ satisfies

$$d(fx, fy) \le \alpha'\,(d(x, fx) + d(y, fy)), \forall\, x, y \in X,$$

especially,

$$d(fa, fb) \le \alpha'\,(d(a, fa) + d(b, fb)).$$

hence for all $t \in [0, 1]$,

$$d(fa, fb)(t) \le [\alpha'\,(d(a, fa) + d(b, fb))](t),$$

but

$$d(fa, fb)(t) = d(a, c)(t) = 3e^t,\ [\alpha'\,(d(a, fa) + d(b, fb))](t) = [\alpha'\, d(b, c)](t) = 2\alpha'(t)e^t,\ \forall\ t \in [0, 1],$$

hence

$$3e^t = d(fa, fb)(t) \le [\alpha'\,(d(a, fa) + d(b, fb))](t) = 2\alpha'(t)e^t,$$

so

$$\alpha'(t) \geq \frac{3}{2},\ \forall\, t \in [0,1].$$

Therefore

$$r(\alpha') = \lim_{n\to\infty}\{\|\ (\alpha'(t))^n\ \|_\infty + \|\ ((\alpha'(t))^n)'\ \|_\infty\}^{\frac{1}{n}} \geq \lim_{n\to\infty}\{\|\ (\alpha'(t))^n\ \|_\infty\}^{\frac{1}{n}} \geq \frac{3}{2},$$

which is a contradiction. This shows that f does not satisfy the Kannan contractive condition, hence Theorem 3.7 is a generalization of Kannan type fixed point theorem(i.e., Theorem 2.1).

Next, we will discuss unique common fixed point problems for two mappings with Chatterjea type mixed contractive conditions.

Theorem 3.9 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exist $\{\alpha, \beta, \gamma\} \subset P$ with $2\max\{r(\alpha), r(\beta)\} + r(\gamma) < 1$ which commute such that for each $x, y \in X$, $x \neq y$,

$$d(fgx, gfy) \leq \alpha\, d(fy, fgx) + \beta\, d(gx, gfy) + \gamma\, d(fy, gx). \tag{3.20}$$

If fX or gX is complete, then f and g have a unique common fixed point.

Proof Consider the sequence $\{x_n\}$ satisfying (3.2).

If there exists n such that $x_{2n} = x_{2n+1}$, then $d(x_{2n}, x_{2n+1}) = 0$, hence by (3.20),

$$\begin{aligned}
& d(x_{2n+1}, x_{2n+2}) = d(fgx_{2n-1}, gfx_{2n}) \\
\leq\ & \alpha\, d(fx_{2n}, fgx_{2n-1}) + \beta\, d(gx_{2n-1}, gfx_{2n}) + \gamma\, d(fx_{2n}, gx_{2n-1}) \\
=\ & \beta\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma\, d(x_{2n+1}, x_{2n}) \\
=\ & \beta\, d(x_{2n+1}, x_{2n+2}),
\end{aligned}$$

hence

$$(e - \beta)d(x_{2n+1}, x_{2n+2}) \leq 0. \tag{3.21}$$

But $r(\beta) < 1$ implies that $(e - \beta)$ is invertible and $(e - \beta)^{-1} \geq 0$, hence from (3.21), we obtain

$$x_{2n+2} = x_{2n+1} = x_{2n}.$$

Hence it is easy to check that x_{2n} is a common fixed point of f and g. Similarly, if there exists n such that $x_{2n+1} = x_{2n+2}$, then x_{2n+1} is a common fixed point of f and g. So we can assume that $x_n \neq x_{n+1}$, $n = 0, 1, 2 \cdots$.

For any fixed n, by (3.20),

$$\begin{aligned}
& d(x_{2n+2}, x_{2n+3}) = d(gfx_{2n}, fgx_{2n+1}) \\
\leq\ & \alpha\, d(fx_{2n}, fgx_{2n+1}) + \beta\, d(gx_{2n+1}, gfx_{2n}) + \gamma\, d(fx_{2n}, gx_{2n+1}) \\
=\ & (\alpha + \gamma)\, d(x_{2n+1}, x_{2n+2}) + \alpha\, d(x_{2n+2}, x_{2n+3}).
\end{aligned}$$

Hence using $r(\alpha) < 1$, we can obtain

$$d(x_{2n+2}, x_{2n+3}) \leq (e - \alpha)^{-1}(\alpha + \gamma)\, d(x_{2n+1}, x_{2n+2}). \tag{3.22}$$

Similarly,

$$\begin{aligned}
& d(x_{2n+1}, x_{2n+2}) = d(gfx_{2n}, fgx_{2n-1}) \\
\leq\ & \alpha\, d(fx_{2n}, fgx_{2n-1}) + \beta\, d(gx_{2n-1}, gfx_{2n}) + \gamma\, d(fx_{2n}, gx_{2n-1}) \\
=\ & \beta\, d(x_{2n+1}, x_{2n+2}) + (\beta + \gamma)d(x_{2n}, x_{2n+1}).
\end{aligned}$$

Hence

$$d(x_{2n+1}, x_{2n+2}) \leq (e - \beta)^{-1}(\beta + \gamma)\, d(x_{2n}, x_{2n+1}). \tag{3.23}$$

Let $L_1 = (e-\alpha)^{-1}(\alpha+\gamma)$, $L_2 = (e-\beta)^{-1}(\beta+\gamma)$ and $L = L_1L_2$, then $\{\alpha.\beta, \gamma, L_1, L_2\}$ commute, hence by Lemma 2.5 and Lemma 2.7,

$$r(L) \leq r(L_1)r(L_2) \leq \frac{r(\alpha)+r(\gamma)}{1-r(\alpha)}\,\frac{r(\beta)+r(\gamma)}{1-r(\beta)} < 1.$$

Modifying the proof of Theorem 3.1 and using (3.22), (3.23) and $r(L) < 1$, we can prove $\{x_n\}$ is a Cauchy sequence.

Suppose that fX is complete. Since $x_{2n+1} \in fX$ and $\{x_n\}$ is a Cauchy sequence, there exist $u \in fX$ and $v \in X$ such that $x_{2n+1} \to u = fv$ as $n \to \infty$. Also we obtain $x_{2n+2} \to u$ as $n \to \infty$ since $d(x_{2n+2}, u) \leq d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, u)$. By (3.20),

$$\begin{aligned}
&d(u, gu) = d(u, gfv)\\
\leq\ & d(u, x_{2n+3}) + d(x_{2n+3}, gfv)\\
\leq\ & d(u, x_{2n+3}) + d(fgx_{2n+1}, gfv)\\
\leq\ & d(u, x_{2n+3}) + \alpha\, d(fv, fgx_{2n+1}) + \beta\, d(gx_{2n+1}, gfv) + \gamma\, d(fv, gx_{2n+1})\\
=\ & d(u, x_{2n+3}) + \alpha\, d(u, x_{2n+3}) + \beta\, d(x_{2n+2}, gu) + \gamma\, d(u, x_{2n+2})\\
\leq\ & (e+\alpha)d(u, x_{2n+3}) + \beta[d(u, x_{2n+2}) + d(u, gu)] + \gamma\, d(u, x_{2n+2}),
\end{aligned}$$

hence

$$d(u, gu) \leq (e-\beta)^{-1}(e+\alpha)d(u, x_{2n+3}) + (e-\beta)^{-1}(\beta+\gamma)d(u, x_{2n+2}).$$

Since x_n converges to u, so for each $c \gg 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$(e-\beta)^{-1}(e+\alpha)d(u, x_{2n+3}) \ll \frac{c}{2},\ (e-\beta)^{-1}(\beta+\gamma)d(u, x_{2n+2}) \ll \frac{c}{2},$$

hence for all $n > N$,

$$d(u, gu) \ll c,$$

that is,

$$u = gu.$$

By (3.20) again,

$$d(u, fu) = d(fgu, gfv) \leq \alpha d(fv, fgu) + \beta d(gu, gfv) + \gamma d(fv, gu) = \alpha d(u, fu),$$

i.e.,

$$(e-\alpha)d(u, fu) \leq 0.$$

Hence $fu = u = gu$, i.e., u is a common fixed point of f and g.

If u^* is another fixed point of f and g, then by (3.20),

$$d(u, u^*) = d(fgu, gfu^*) \leq \alpha\, d(fu^*, fgu) + \beta\, d(gu, gfu^*) + \gamma\, d(fu^*, gu) \leq (\alpha+\beta+\gamma)\, d(u.u^*),$$

hence

$$[e - (\alpha+\beta+\gamma)]d(u.u^*) \leq 0.$$

Since $r(\alpha+\beta+\gamma) \leq 2\max\{r(\alpha), r(\beta)\} + r(\gamma) < 1$, $[e-(\alpha+\beta+\gamma)]$ is invertible and $[e-(\alpha+\beta+\gamma)]^{-1} \geq 0$, hence $d(u, u^*) = 0$. Therefore u is the unique common fixed point of f and g. Similarly, we can obtain the same result for the another case.

Using Theorem 3.9, we obtain the following conclusion

Corollary 3.2 Let (X, d) be a comlpete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exist $\{\alpha, \beta, \gamma\} \subset P$ with $2\max\{r(\alpha), r(\beta)\} + r(\gamma) < 1$ which commute. If (3.20) holds and f or g is onto, then f and g have a unique common fixed point.

Using Theorem 3.9, we obtain unique common fixed point theorems for two mappings satisfying weak Chatterjea-mixed type and Chatterjea-mixed type contractive conditions respectively.

Theorem 3.10 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exists $\gamma \in P$ with $r(\gamma) < \frac{1}{3}$. If for each $x, y \in X$, $x \neq y$,

$$d(fgx, gfy) \leq \gamma\, [d(fy, fgx) + d(gx, gfy) + d(fy, gx)].$$

and fX or gX is complete. Then f and g have a unique common fixed point.
Proof Let $\alpha = \beta = \gamma$ in Theorem 3.9, then the conclusion from Theorem 3.9.
Theorem 3.11 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two mappings. Suppose that there exists $\alpha \in P$ with $r(\alpha) < \frac{1}{2}$. If for each $x, y \in X$, $x \neq y$,

$$d(fgx, gfy) \leq \alpha\, [d(fy, fgx) + d(gx, gfy)].$$

and fX or gX is complete, then f and g have a unique common fixed point.
Proof Let $\alpha = \beta, \gamma = 0$ in Theorem 3.9, then the conclusion from Theorem 3.9.

Modifying the proof of Theorem 3.6, we obtain the continuous and non-surjective version of Corollary 3.2.
Theorem 3.12 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f, g : X \to X$ two continuous mappings. If there exist $\{\alpha, \beta, \gamma\} \subset P$ with $2\max\{r(\alpha), r(\beta)\} + r(\gamma) < 1$ which commute such that (3.20) holds. Then f and g have a unique common fixed point.

Using Theorem 3.12, we obtain the following unique fixed point theorem.
Theorem 3.13 (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and P is a solid cone in $\mathcal{A}$, $f : X \to X$ a continuous mapping. If there exist $\alpha \in P$ with $r(\alpha) < \frac{1}{2}$ such that for each $x, y \in X$, $x \neq y$,

$$d(f^2x, f^2y) \leq \alpha\, [d(fy, f^2x) + d(fx, f^2y)].$$

Then f has a unique fixed point.
Proof Let $f = g, \alpha = \beta, \gamma = 0$ in Theorem 3.12,
Remark 3.3 The contractive condition in Theorem 3.13 is weaker than Chatterjea contractive condition(see Theorem 1.5 and Theorem 2.2). Hence Theorem 3.13 is a new version and a generalization of Theorem 1.5 and Theorem 2.2.

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