Filomat 32:7 (2018), 2465–2481
https://doi.org/10.2298/FIL1807465C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and Multiplicity of Nontrivial Solutions for Nonlinear Schrödinger Equations with Unbounded Potentials

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Abstract. We investigate the existence of nontrivial solutions and multiple solutions for the following class of elliptic equations

$$\begin{cases} -\Delta u + V(x)u = K(x)f(u), \ x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $V(x)$ and $K(x)$ are both unbounded potential functions and f is a function with a super-quadratic growth. Firstly, we prove the existence of infinitely many solutions with compact embedding and by means of symmetric mountain pass theorem. Moreover, we prove the existence of nontrivial solutions without compact embedding in weighted Sobolev spaces and by means of mountain pass theorem. Our results extend and generalize some existing results.

1. Introduction and preliminaries

This article is concerned with a class of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V(x)u = K(x)f(u), \ x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1}$$

where $N \geq 3$, $V(x)$ and $K(x)$ are both unbounded potentials, f is a function with a super-quadratic growth.

Problem (1) stems from Schrödinger equation, which has found a great deal of interest last years because not only it is important in applications but it provides a good model for developing mathematical methods. There are two cases in studying the existence of solutions for Schrödinger equation. One is $u \in H^1(\mathbb{R}^N)$, and the other is $u \in D^{1,2}(\mathbb{R}^N)$. For the case of $u \in H^1(\mathbb{R}^N)$, the problem

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{2}$$

2010 *Mathematics Subject Classification*. 35J60, 35J20
Keywords. Schrödinger equations, super-quadratic growth, unbounded potentials, variational method
Received: 05 June 2017; Revised: 22 March 2018; Accepted: 24 March 2018
Communicated by Marko Nedeljkov
This work was supported by National Natural Science Foundation of China (Grant No. 11461043 and 11601525), and supported partly by the Provincial Natural Science Foundation of Jiangxi, China (20161BAB201009) and the Outstanding Youth Scientist Foundation Plan of Jiangxi (No. 20171BCB23004), Yunnan Local Colleges Applied Basic Research Projects (No. 2017FH001-011), and Hunan Provincial Innovation Foundation For Postgraduate (Grant No. CX2016B037).
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has been studied by a number of authors (see [9, 10, 28–30]). With the aid of variational methods, the existence and multiplicity of nontrivial solutions for (2) have been extensively investigated in the literature over the past several decades. Some related literature proved the existence of multiple solutions via fountain theorem. See, e.g., [7, 8, 12–16, 19, 24–27, 32–34, 36, 39, 41, 43, 43–46] and the references quoted in them. Based on this, Tang [31] gave some more weaker conditions and studied the existence of infinitely many solutions for equation (2) via symmetric mountain pass theorem with sign-changing potential. Using Tang's methods, some authors studied the existence of infinitely many solutions for various equations and systems. See, e.g., [22, 35, 37, 40, 42] and the references quoted in them. These results generalized and extended some existing results. To our best knowledge, for the case of $u \in D^{1,2}(\mathbb{R}^N)$, the following nonlinear elliptic equations

$$\begin{cases} -\Delta u + V(x)u = K(x)f(u), \ x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \\ u(x) > 0 \end{cases} \tag{3}$$

were studied. More concretely, there are many papers to consider Problem (3) with the potential V vanishing at infinity. An important class of Problems associated with (3) is the zero mass case, that is

$$\lim_{|x|\to\infty} V(x) = 0.$$

In [5], Ambrosetti et al. studied the existence of ground state solutions and established concentration behavior of ground state solutions by assuming that V, K satisfy the following conditions:

$$\frac{a_1}{1+|x|^\alpha} \le V(x) \le a_2 \ \text{ and } \ 0 < K(x) \le \frac{a_3}{1+|x|^\beta}, \ \ \forall\, x \in \mathbb{R}^N$$

and

$$\frac{N+2}{N-2} - \frac{4\beta}{\alpha(N-2)} < q, \text{ if } 0<\beta<\alpha, \quad \text{or} \quad 1<q, \text{ if } \beta \ge \alpha.$$

Since the work of [5], there are many papers on problem (1) with potential $V(x)$ vanishing at infinity, see, for example, [1, 2, 4, 5, 20]. However, in those papers, $f(s)$ is always supposed to be the form of s^q with $q \in (1, \frac{N+2}{N-2})$ and $-\Delta$ is replaced by $-\varepsilon^2\Delta$, and the authors were more interested in the semi-classical states to (1) for $\varepsilon > 0$ small, less interested in the existence of solutions, because under that situation it is trivial to find a mountain pass type solution in a suitable weighted Sobolev space with the help of the compactness of Sobolev embedding. But until the paper, Alves et al. [2] appeared it assumes that f has a subcritical growth and V is a nonnegative potential, which can vanish at infinity, that is, $V(x) \to 0$ as $|x| \to \infty$, and $K(x) = 1$. Recently, Alves et al. [1] consider the following a new class of vanishing potentials, which are much weaker than [2, 5, 20]:

(I) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K(x) \in L^\infty(\mathbb{R}^N)$.

(II) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \le R$, for all n and some $R > 0$, then we have

$$\lim_{r\to+\infty} \int_{A_n \cap B_R^c(0)} K(x)dx = 0, \ \text{uniformly in } n \in \mathbb{N}.$$

(III) One of the following conditions holds:

$$\frac{K(x)}{V(x)} \in L^\infty(\mathbb{R}^N) \tag{4}$$

or there exists an $\alpha_0 \in (2, 2^*)$ such that

$$\frac{K(x)}{|V(x)|^{(2^*-\alpha_0)/(2^*-2)}} \to 0 \text{ as } |x| \to \infty. \tag{5}$$

According to (I)-(III), the authors [1] used a Hardy-type inequality involving V and K, together with a version of mountain pass theorem with Cerami condition. For purpose of seeking a ground state solution by the above compactness conditions about $V(x)$ and $K(x)$, the authors made the following assumptions on nonlinear term f:

(A_1) $\limsup\limits_{s\to 0} \frac{f(s)}{s} = 0$ if (4) holds
or
$\limsup_{s\to 0^+} \frac{|f(s)|}{s^{p-1}} < +\infty$ if (5) holds.

(A_2) f has a quasicritical growth, that is,

$$\limsup_{s\to+\infty} \frac{|f(s)|}{s^{2^*-1}} = 0.$$

(A_3) $s^{-1}f(s)$ is a non-decreasing function in $(0,+\infty)$ and its primitive F is superquadratic at infinity, that is,

$$\lim_{s\to+\infty} \frac{|F(s)|}{s^2} = +\infty. \tag{6}$$

In [1], the author emphasized that related to condition (A_3), (6) was first used in the papers of Liu and Wang [17], and Liu, Wang and Zhang [18] and that it is weaker than the well-known AR-conditions(see [3]):
(A_3') There exists $\theta > 2$ such that

$$0 < \theta F(s) \le sf(s) \quad \forall\ s > 0.$$

In [1], the authors gave the following results.

Theorem 1.1. [1] *Suppose that (I)-(III),and (A_1)-(A_3) are satisfied. Then problem (3) has a positive ground state solution.*

Remark 1.2. *(1) It is not hard to find that Theorem 1.1 somewhat improve the results of [2, 4, 5].*
(2) In [1, 2, 5], we can see that $V(x)$ and $K(x)$ are bounded.

Remark 1.3. *In reviewing the literature mentioned above, they put forward the concept of vanishing potential so that they can overcome the lack of the compactness of Sobolev embedding.*

Motivated by all results mentioned above, it is very natural for us to pose the following interesting questions:

(i) If $K(x) \in L^\infty(\mathbb{R}^N)$ but $V(x) \notin L^\infty(\mathbb{R}^N)$, (1) admits infinitely many nontrivial solutions when $f(x,u)$ satisfies some suitable assumptions?

(ii) If $K(x) \notin L^\infty(\mathbb{R}^N)$ and $V(x) \notin L^\infty(\mathbb{R}^N)$, namely, $V(x)$ and $K(x)$ can tend to ∞ as $|x| \to +\infty$, (1) admits one nontrivial solution when $f(x,u)$ satisfies some suitable assumptions? Whether Problem (1) exists a nontrivial solution without compactness of Sobolev embedding?

As is known, there are few results on such above questions in current literature. Actually, this is one of the motivations for us to study the existence of infinitely many solutions and nontrivial solutions of (3).

Next, we first answer question (i): we prove the existence of infinitely many solutions for problem (1) with compact embedding by using Tang's methods in [31]. In our mind, we must need compact embedding to prove the boundedness of $(C)_c$-sequence for Problem (1). In order to get compact embedding, we need to enhance some conditions for potentials $K(x)$ and $V(x)$. But our conditions about the following nonlinear term f is weaker than (A_2)-(A_3) which are given by Theorem 1.1. Now, we consider problem (1) with unbounded potentials, and establish the existence of infinitely many solutions by symmetric mountain pass theorem in [6, 23]. Before proving our results, we need to make the following assumptions on V, K and f:

$(VK1)$ $V, K \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \ge \min V(x) \ge 1$, $K(x) \ge \min K(x) \ge 0$, $K(x) \not\equiv 0$ and $K(x) \in L^\infty(\mathbb{R}^N)$;

$(VK2)$

$$\lim_{|x|\to\infty}\frac{K(x)}{V^{\theta}(x)}=0,\ \forall\ 0<\theta<1;$$

(f_1) $f\in C(\mathbb{R},\mathbb{R})$, and there exist constants $c_2, c_3>0$ and $p\in(2,2^*)$ such that

$$|f(u)|\le c_2|u|+c_3|u|^{p-1},\ \forall\ u\in\mathbb{R};$$

(f_2) $\lim\limits_{|u|\to\infty}\frac{|F(u)|}{|u|^2}=\infty$ and there exists $r_0\ge 0$ such that

$$F(u)\ge 0,\ \ \forall\ u\in\mathbb{R},\ |u|\ge r_0;$$

(f_3) $\mathcal{F}(x,u):=\frac{1}{2}uf(u)-F(u)\ge 0$, and there exist $c_0>0$ and $\kappa>\max\{1,N/2\}$ such that

$$|F(u)|^{\kappa}\le c_0|u|^{2\kappa}\mathcal{F}(u),\ \forall\ u\in\mathbb{R},\ |u|\ge r_0;$$

(f_4) there exist $\mu>2$ and $\varrho>0$ such that

$$\mu F(u)\le uf(u)+\varrho u^2,\ \forall\ u\in\mathbb{R};$$

(f_5) there exist $\mu>2$ and $r_1>0$ such that

$$\mu F(u)\le uf(u),\ \forall\ u\in\mathbb{R},\ |u|\ge r_0;$$

(f_6) $f(-u)=-f(u),\ \forall\ u\in\mathbb{R}$.
The below functions are typical example of functions that verify $(VK1)$ and $(VK2)$:

Example 1.4. *Let*

$$K(x)=2\ \text{ and }\ V(x)=(|x|+1)^{\frac{1}{\theta}}\ \ \forall\ 0<\theta<1.$$

It is easy to check that $\lim\limits_{|x|\to\infty}\frac{K(x)}{V^{\theta}(x)}=0$, $K(x)\not\equiv 0$, $K(x)\in L^{\infty}(\mathbb{R}^N)$, $V(x)\ge\min V(x)\ge 1$ *and* $K(x)\ge\min K(x)\ge 0$ *for all* $0<\theta<1$.

Next, we intend to state the results of infinitely many solutions.

Theorem 1.5. *Suppose that* $(VK1)$-$(VK2)$, (f_1), (f_2), (f_3) *and* (f_6) *are satisfied. Then problem* (1) *has infinitely many nontrivial solutions.*

Theorem 1.6. *Suppose that* $(VK1)$-$(VK2)$, (f_1), (f_2), (f_4) *and* (f_6) *are satisfied. Then problem* (1) *has infinitely many nontrivial solutions.*

It is easy to check that (f_1) and (f_5) imply (f_4). Thus, we have the following corollary.

Corollary 1.7. *Suppose that* $(VK1)$-$(VK2)$, (f_1), (f_2), (f_5) *and* (f_6) *are satisfied. Then problem* (1) *has infinitely many nontrivial solutions.*

Secondly, we answer question (ii): we establish the existence of a nontrivial solution via mountain pass theorem in [3]. More precisely, we make the following assumptions:
$(VK3)$ $V,K\in C(\mathbb{R}^N,\mathbb{R})$; $V(x)\ge\min V(x)\ge 0$, $K(x)\ge\min K(x)\ge 0$, $K(x)\not\equiv 0$ and $V(x)$ satisfies

$$\lim_{|x|\to\infty}V(x)=+\infty.$$

$(VK4)$ For any $\theta\in(0,1)$ and $x\in\mathbb{R}^N$, there exists a constant $M>0$ such that $K(x)\le MV^{\theta}(x)$.
(F_1) $f\in C(\mathbb{R},\mathbb{R})$, and there exist a constant $c_1>0$ and $p\in(2,2^*)$ such that

$$|f(u)|\le c_1(1+|u|^{p-1}),\ \forall\ u\in\mathbb{R};$$

(F_2) $f(u) = o(|u|)$ as $|u| \to 0$;
(F_3) There exists $\mu > 2$ such that $0 < \mu F(u) \le f(u)u$.
There are many functions that verify $(VK3)$ and $(VK4)$:

Example 1.8. *For $\theta \in (0,1)$, let*

$$K(x) = \ln(1+|x|) \text{ and } V(x) = |x|^{\frac{1}{\theta}}.$$

If $|x| = 0$, then we infer that for any $M > 0$, $0 = K(x) = V(x) = 0 \le MV^{\theta}(x)$. If $|x| \ne 0$, then it is easy to check that there exists a constant $M > 0$ such that $K(x) \le MV^{\theta}(x)$ for all $0 < \theta < 1$. Obviously,

$$\lim_{|x|\to\infty} K(x) = +\infty, \quad \lim_{|x|\to\infty} V(x) = +\infty \quad \text{and} \quad K(x) \not\equiv 0.$$

Remark 1.9. *It follows from (F_3) that $\lim_{|u|\to\infty} \frac{F(u)}{|u|^2} = +\infty$ and $F(u) > 0$. Compared to (I)-(III) and $(VK3)$-$(VK4)$, we infer that $(VK3)$-$(VK4)$ are much weaker.*

Next, we are ready to state the result of nontrivial solutions.

Theorem 1.10. *Suppose that $(VK3)$-$(VK4)$ and (F_1)-(F_3) are satisfied. Then problem* (1) *has a nontrivial solution.*

Remark 1.11. *(1) In [1, 2, 4, 5], they all studied the existence of ground state solutions for* (1)*. But, in this paper, we discuss the existence of infinitely many solutions and nontrivial solutions for* (1)*.*

(2) Generally speaking, in [1, 2, 5], $K(x) \in L^{\infty}(\mathbb{R}^N)$, which shows that $K(x)$ is essential bounded. But in $(VK3)$ and $(VK4)$, $K(x)$ can tend to ∞ as $|x| \to \infty$.

(3) By $(VK1)$-$(VK2)$ and $(VK3)$-$(VK4)$, we know that the results are proved by permitting $V(x) \to \infty$ as $|x| \to \infty$. Hence, our results are different from [1] and replenish the results of [1].

Remark 1.12. *On the one hand, in order to prove Theorem 1.5 and 1.6, we give $(VK1)$ and $(VK2)$, which are used to get compact embedding. On the other hand, in Theorem 1.7, we prove the existence of nontrivial solutions without compact embedding involving $(VK3)$ and $(VK4)$, which are somewhat weaker than $(VK1)$ and $(VK2)$. Based on these two facts that we can not prove the existence of infinitely many solutions with $(VK3)$ and $(VK4)$, because it lacks compact embedding.*

Remark 1.13. *By (f_1) and (f_2)* or *(F_1) and (F_2), then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$|f(u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \text{ and } |F(u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p.$$

This paper is organized as follows. In the next section, we present variational framework. In section 3, we prove Theorem 1.5 and 1.6 by $(VK1)$ and $(VK2)$. In section 4, we prove Theorem 1.10 by $(VK3)$ and $(VK4)$.

2. Variational framework

In this section, we present some weighted Sobolev spaces. To this end, we define the space

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\}$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx$$

and the inner product

$$(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) dx.$$

Define the weighted Lebesgue space by

$$L^q_K(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^q dx < +\infty \right\}$$

endowed with the norm

$$\|u\|^q_{L^q_K(\mathbb{R}^N)} = \int_{\mathbb{R}^N} K(x)|u|^q dx.$$

E and $L^q_K(\mathbb{R}^N)$ are particular cases of weighted space and are discussed in [21]. By the above definitions, we can get $E \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ for $N \geq 3$.

Now, we define the following energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^N} K(x)F(u)dx \tag{7}$$

for all $u \in E$, that is,

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)dx, \quad \forall\ u \in E.$$

By the conditions on f, the integral $\int_{\mathbb{R}^N} K(x)F(u)dx$ is well defined.

3. Existence of infinitely many solutions

In this section, we prove the existence of infinitely many solutions for (1). In order to prove our results, the following two lemmas discuss the continuous and compact embedding $E \hookrightarrow L^q_K(\mathbb{R}^N)$ for all $q \in [2, 2^*)$.

Lemma 3.1 *Assume that* $(VK1)$-$(VK2)$ *hold. Then E is continuously embedded in* $L^q_K(\mathbb{R}^N)$ *for all* $q \in [2, 2^*)$.

Proof. Since $\frac{K(x)}{V^\theta(x)} \to 0$ as $|x| \to \infty$ and $0 < \frac{K(x)}{V(x)} \leq \frac{K(x)}{V^\theta(x)}$. Hence, $\frac{K(x)}{V(x)} \to 0$ as $|x| \to \infty$. By the continuous of $V(x)$ and $K(x)$, there exists $M > 0$ such that $K(x) \leq MV^\theta(x) \leq MV(x)$ for all $x \in \mathbb{R}^N$ and $0 < \theta < 1$. If $q = 2$, the the proof is trivial. Fix $q \in (2, 2^*)$, choose $\sigma = \frac{2^*-q}{2^*-2}$, then $q = 2\sigma + (1-\sigma)2^*$ and $0 < \sigma < 1$. Hence we can get the following inequality

$$\begin{aligned}
\|u\|^q_{q,K} &= \int_{\mathbb{R}^N} K(x)|u|^q dx \\
&= \int_{\mathbb{R}^N} K(x)|u|^{2\sigma}|u|^{(1-\sigma)2^*} dx \\
&\leq \left(\int_{\mathbb{R}^N} K(x)^{\frac{1}{\sigma}} u^2 dx\right)^\sigma \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{1-\sigma} \\
&\leq \left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\sigma}\right)\left(\int_{\mathbb{R}^N} V(x)u^2 dx\right)^\sigma \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{1-\sigma} \\
&\leq CM\left(\int_{\mathbb{R}^N} V(x)u^2 dx\right)^\sigma \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{\frac{(1-\sigma)2^*}{2}} \\
&\leq CM\left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{\sigma + \frac{(1-\sigma)2^*}{2}} \\
&= CM\left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{\frac{q}{2}} \\
&= CM\|u\|^q.
\end{aligned}$$

It follows that $E \hookrightarrow L^q_K(\mathbb{R}^N)$ is continuous embedding. □

Lemma 3.2 *Assume that* $(VK1)$-$(VK2)$ *hold. Then E is compactly embedded in* $L^q_K(\mathbb{R}^N)$ *for all* $q \in [2, 2^*)$.

Proof. From Lemma 3.1, we have $\frac{K(x)}{V(x)} \to 0$ as $|x| \to \infty$. Hence for any $\varepsilon > 0$, there exists $R > 0$ such that

$$K(x) \le \varepsilon V(x), \quad \forall\, |x| > R.$$

Let $\{u_n\} \in E$ be be a bounded sequence of E. Going if necessary to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup 0 \text{ in } E,\\ &u_n \to 0 \text{ in } L^q_{K,loc}(\mathbb{R}^N) \text{ for } 2 \le p < 2^*. \end{aligned} \tag{8}$$

Next, we claim that

$$u_n \to 0 \text{ strongly in } L^2_K(\mathbb{R}^N). \tag{9}$$

Set

$$B_R(0) = \left\{x \in \mathbb{R}^N : |x| \le R\right\},$$

then

$$\int_{\mathbb{R}^N \setminus B_R} K(x) u_n^2(x) dx < \varepsilon \int_{\mathbb{R}^N} V(x)|u_n(x)|^2 dx \le \varepsilon \|u_n\|^2 \tag{10}$$

Hence, for any $\varepsilon > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)|u_n(x)|^2 dx &= \int_{B_R} K(x)|u_n(x)|^2 dx + \int_{\mathbb{R}^N \setminus B_R} K(x)|u_n(x)|^2 dx\\ &< \varepsilon(1 + \|u_n\|^2), \end{aligned} \tag{11}$$

from which (10) holds. Since $|s|^q/|s|^2 \to 0$ as $s \to 0$ and $|s|^q/|s|^{2^*} \to 0$ as $s \to \infty$, then for any $\varepsilon > 0$, there exists $C > 0$ such that

$$K(x)|s|^q \le \varepsilon C(K(x)|s|^2 + |s|^{2^*}) + CK(x)|s|^2, \text{ for all } s \in \mathbb{R}. \tag{12}$$

To prove the lemma for general exponent q, we use an interpolation argument. Let $u_n \rightharpoonup 0$ in E, we have just proved that $u_n \to 0$ in $L^q_K(\mathbb{R}^N)$. That is

$$\int_{\mathbb{R}^N} K(x)|u_n(x)|^q dx \to 0.$$

Since the embedding $E \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous and $\{u_n\}$ is bounded in E, we also have $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$. For any $q \in (2, 2^*)$, there exists a $\tau \in (0, 1)$ such that $\tau = \frac{2^*-q}{2^*-2}$ i.e. $q = 2\tau + 2^*(1-\tau)$. From Hölder's inequality, (12), (VK2) and $E \hookrightarrow L^{2^*}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} K(x)|u_n(x)|^q dx \le \varepsilon C \int_{\mathbb{R}^N} (K(x)|u_n|^2 + |u_n|^{2^*}) dx + C \int_{\mathbb{R}^N} K(x)|u_n|^2 dx \to 0$$

implying that

$$u_n \to 0 \text{ in } L^q_K(\mathbb{R}^N).$$

This completes the proof. □

By the conditions on f in Theorem 1.5 and Theorem 1.6, the functional $J \in C^1(E, \mathbb{R})$ and its Gateaux derivate is given by

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx - \int_{\mathbb{R}^N} K(x) f(u) v dx, \quad \forall\ u, v \in E,$$

that is,

$$\langle J'(u), v\rangle = (u, v) - \int_{\mathbb{R}^N} K(x) f(u) v dx, \quad \forall\ u, v \in E.$$

A sequence $\{u_n\} \subset E$ is said to be a $(C)_c$-sequence if $J(u) \to c$ and $\|J'(u)\|(1 + \|u_n\|) \to 0$. J is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence. To prove our results, we state the following symmetric mountain pass theorem.

Lemma 3.3 [6, 23] *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $J \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*
(J_1) $J(0) = 0$, $J(-u) = J(u)$ for all $u \in X$;
(J_2) there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Z} \geq \alpha$;
(J_3) for any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $J(u) \leq 0$ on $\widetilde{X} \backslash B_R$;
then J possesses an unbounded sequence of critical values.

Lemma 3.4 *Suppose that $(VK1)$, $(VK2)$, (f_1), (f_2) and (f_3) are satisfied. Then any $\{u_n\} \subset E$ satisfying*

$$J(u_n) \to c > 0, \qquad \langle J'(u_n), u_n\rangle \to 0 \tag{13}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, assume that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\|v_n\|_{L^s_K} \leq \gamma_s \|v_n\| = \gamma_s$ for $2 \leq s < 2^*$. For n large enough, we have

$$c + 1 \geq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle = \int_{\mathbb{R}^N} K(x) \mathcal{F}(u_n) dx. \tag{14}$$

It follows from (7) and (13) that

$$\frac{1}{2} \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} K(x) \frac{|F(u_n)|}{\|u_n\|^2} dx. \tag{15}$$

For $0 < a < b$, let

$$\Omega_n(a, b) = \left\{x \in \mathbb{R}^N : a \leq |u_n| < b\right\}.$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, then by Lemma 3.2, $v_n \to v$ in $L^s_K(\mathbb{R}^N)$, $2 \leq s < 2^*$, and $v_n \to v$ a.e. on $\mathbb{R}^N$.

If $v = 0$, then $v_n \to 0$ in $L^s_K(\mathbb{R}^N)$ for all $s \in [2, 2^*)$, and $v_n \to 0$ a.e. in $\mathbb{R}^N$. By (f_1) and Remark 1.13, we know that

$$\begin{aligned}\int_{\Omega_n(0,r_0)} K(x) \frac{|F(u_n)|}{|u_n|^2} |v_n|^2 dx &\leq \left(\frac{c_2}{2} + \frac{c_3}{p} r_0^{p-2}\right) \int_{\Omega_n(0,r_0)} K(x)|v_n|^2 dx \\ &\leq \left(\frac{c_2}{2} + \frac{c_3}{p} r_0^{p-2}\right) \int_{\mathbb{R}^N} K(x)|v_n|^2 dx \to 0.\end{aligned} \tag{16}$$

Let $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > \max\{1, N/2\}$, we obtain $2\kappa' \in (2, 2^*)$. Hence, from (f_3) and (14), we have

$$\begin{aligned}\int_{\Omega_n(r_0,\infty)} K(x)\frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2dx &\le \left[\int_{\Omega_n(r_0,\infty)} K(x)\left(\frac{|F(u_n)|}{|u_n|^2}\right)^{\kappa} dx\right]^{\frac{1}{\kappa}} \left[\int_{\Omega_n(r_0,\infty)} K(x)|v_n|^{2\kappa'}dx\right]^{\frac{1}{\kappa'}} \\ &\le c_0^{\frac{1}{\kappa}}\left[\int_{\Omega_n(r_0,\infty)} K(x)\mathcal{F}(u_n)dx\right]^{\frac{1}{\kappa}}\left(\int_{\Omega_n(r_0,\infty)} K(x)|v_n|^{2\kappa'}dx\right)^{\frac{1}{\kappa'}} \\ &\le [c_0(c+1)]^{\frac{1}{\kappa}}\left(\int_{\Omega_n(r_0,\infty)} K(x)|v_n|^{2\kappa'}dx\right)^{\frac{1}{\kappa'}} \to 0.\end{aligned} \tag{17}$$

From (16) and (17), we have

$$\int_{\mathbb{R}^N}\frac{|F(u_n)|}{\|u_n\|^2}dx = \int_{\Omega_n(0,r_0)} K(x)\frac{|F(u_n)|}{|u_n|^2}|v_n|^2dx + \int_{\Omega_n(r_0,\infty)} K(x)\frac{|F(u_n)|}{|u_n|^2}|v_n|^2dx \to 0,$$

which contradicts (15).

Now, we consider the case $v \ne 0$. Set $A := \left\{x \in \mathbb{R}^N : v(x) \ne 0\right\}$. Thus $meas(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty}|u_n(x)| = \infty$. Hence $A \subset \Omega_n(r_0,\infty)$ for large $n \in \mathbb{N}$, which implies that $\chi_{\Omega_n(r_0,\infty)} = 1$ for large n, where χ_{Ω_n} denotes the characteristic function on Ω. Since $v_n \to v$ a.e. in $\mathbb{R}^N$, we have $\chi_{\Omega_n}(x) = 1$ a.e. in A. It follows from (7), (f_3) and Fatou's Lemma that

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty}\frac{J(u_n)}{\|u_n\|^2} \\ &= \lim_{n\to\infty}\left[\frac{1}{2} - \int_{\Omega_n(0,r_0)} K(x)\frac{F(u_n)}{u_n^2}v_n^2dx - \int_{\Omega_n(r_0,\infty)} K(x)\frac{F(u_n)}{u_n^2}v_n^2dx\right] \\ &\le \limsup_{n\to\infty}\left[\frac{1}{2} + \left(\frac{c_2}{2} + \frac{c_3}{p}r_0^{p-2}\right)\int_{\mathbb{R}^N} K(x)v_n^2dx - \int_{\Omega_n(r_0,\infty)} K(x)\frac{F(u_n)}{u_n^2}v_n^2dx\right] \\ &\le \limsup_{n\to\infty}\left[\frac{1}{2} + \left(\frac{c_2}{2} + \frac{c_3}{p}r_0^{p-2}\right)\gamma_2^2 - \int_{\Omega_n(r_0,\infty)} K(x)\frac{F(u_n)}{u_n^2}v_n^2dx\right] \\ &\le \frac{1}{2} + \left(\frac{c_2}{2} + \frac{c_3}{p}r_0^{p-2}\right)\gamma_2^2 - \liminf_{n\to\infty}\int_{\Omega_n(r_0,\infty)} K(x)\frac{F(u_n)}{u_n^2}v_n^2dx \\ &= \frac{1}{2} + \left(\frac{c_2}{2} + \frac{c_3}{p}r_0^{p-2}\right)\gamma_2^2 - \liminf_{n\to\infty}\int_{\mathbb{R}^N} K(x)\frac{|F(u_n)|}{u_n^2}\chi_{\Omega_n(r_0,\infty)}(x)v_n^2dx \\ &\le \frac{1}{2} + \left(\frac{c_2}{2} + \frac{c_3}{p}r_0^{p-2}\right)\gamma_2^2 - \int_{\mathbb{R}^N}\liminf_{n\to\infty} K(x)\frac{F(u_n)}{u_n^2}[\chi_{\Omega_n(r_0,\infty)}(x)]v_n^2dx \\ &= -\infty,\end{aligned} \tag{18}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. This completes the proof. □

Lemma 3.5 *Suppose that* $(VK1)$, $(VK2)$, (f_1), (f_2) *and* (f_3) *are satisfied. Then any* $\{u_n\} \subset E$ *satisfying* (13) *has a convergent subsequence in E.*

Proof. By Lemma 3.4, it can conclude that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. From Lemma 3.2, we have $u_n \to u$ in $L^s_K(\mathbb{R}^N)$ for all $2 \le s < 2^*$. Hence, together with Remark 1.13, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^N} K(x)(f(u_n)-f(u))(u_n-u)dx\right| &\le \int_{\mathbb{R}^N} K(x)(|f(u_n)|+|f(u)|)|u_n-u|dx\\ &\le \int_{\mathbb{R}^N} K(x)(\varepsilon|u|+C_\varepsilon|u|^{p-1}+\varepsilon|u_n|+C_\varepsilon|u_n|^{p-1})|u_n-u|dx\\ &\le \varepsilon C + C_\varepsilon\left(\int_{\mathbb{R}^N} K(x)|u_n|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N} K(x)|u_n-u|^p\right)^{\frac{1}{p}}\\ &\to 0,\ \text{ as }\ n\to\infty.\end{aligned} \tag{19}$$

Observe that

$$\|u_n-u\|^2 = \langle J'(u_n)-J'(u), u_n-u\rangle + \int_{\mathbb{R}^N} K(x)(f(u_n)-f(u))(u_n-u)dx. \tag{20}$$

It is clear that

$$\langle J'(u_n)-J'(u), u_n-u\rangle \to 0,\ \text{ as }\ n\to\infty. \tag{21}$$

From (19), (20) and (21), we have $\|u_n-u\|\to 0$, $n\to\infty$. □

Lemma 3.6 *Suppose that* $(VK1)$, $(VK2)$, (f_1), (f_2) *and* (f_4) *are satisfied. Then any* $\{u_n\}\subset E$ *satisfying* (13) *has a convergent subsequence in* E.

Proof. Firstly, we prove that $\{u_n\}$ is bounded in E. To prove the boundedness of $\{u_n\}$, arguing by contradiction, assume that $\|u_n\|\to\infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\|=1$ and $\|v_n\|_{L^s_K}\le\gamma_s\|v_n\|=\gamma_s$ for $2\le s<2^*$. Form (7), (13), (f_4) and Gateaux derivate on J, we have

$$\begin{aligned}c+1 &\ge J(u_n)+\frac{1}{\mu}\langle J'(u_n)-J'(u), u_n-u\rangle\\ &= \frac{\mu-2}{2\mu}\|u_n\|^2+\int_{\mathbb{R}^N} K(x)\left[\frac{1}{\mu}f(u_n)u_n-F(u_n)\right]dx\\ &\ge \frac{\mu-2}{2\mu}\|u_n\|^2-\frac{\varrho}{\mu}\|u_n\|^2_{L^2_K(\mathbb{R}^N)},\quad \text{for enough large } n\in\mathbb{N},\end{aligned} \tag{22}$$

which implies

$$1\le\frac{2\varrho}{\mu-2}\limsup_{n\to\infty}\|v_n\|^2_{L^2_K(\mathbb{R}^N)}. \tag{23}$$

Passing to a subsequence, we may assume that $v_n\rightharpoonup v$ in E, then by Lemma 3.2, $v_n\to v$ in $L^s_K(\mathbb{R}^N)$, $2\le s<2^*$, and $v_n\to v$ a.e. on $\mathbb{R}^N$. Hence, it follows from (23) that $v\ne 0$. By a similar fashion as (18), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest proof is the same as that in Lemma 3.5. □

Lemma 3.7 *Suppose that* $(VK1)$, $(VK2)$, (f_1) *and* (f_2) *are satisfied. Then for any* $\widetilde{E}\subset E$, *there holds*

$$J(u)\to-\infty,\quad \|u\|\to\infty,\quad u\in\widetilde{E}. \tag{24}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\}\subset\widetilde{E}$ with $\|u_n\|\to\infty$, there is $M>0$ such that $J(u_n)\ge -M$ for all $n\in\mathbb{N}$. Set $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$. Passing to a subsequence, we may assume that $v_n\rightharpoonup v$ in E. Since $\widetilde{E}$ is finite dimensional, then $v_n\to v\in\widetilde{E}$ in E, $v_n\to v$ a.e. on $\mathbb{R}^N$, and so $\|v\|=1$. Hence, we can conclude a contradiction by a similar fashion as (18). □

Corollary 3.8 *Suppose that* (VK1), (VK2), (f_1) *and* (f_2) *are satisfied. Then for any* $\widetilde{E} \subset E$, *there exists* $R = R(\widetilde{E}) > 0$, such that

$$J(u_n) \le 0, \quad \|u\| \ge R, \quad \forall\ u \in \widetilde{E}.$$

Let $\{e_j\}$ is a total orthonormal basis of E and define $X_j = \mathbb{R}e_j$,

$$Y_k = \bigoplus_{j=1}^{k} X_j,\ Z_k = \bigoplus_{j=k+1}^{\infty} X_j,\ \forall\ k \in \mathbb{Z}. \tag{25}$$

Lemma 3.9 *Suppose that* (VK1) and (VK2) *are satisfied. Then for* $2 \le s < 2^*$, we have

$$\beta_k(s) := \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^s_K(\mathbb{R})} \to 0, \quad k \to \infty.$$

Proof. It is clear that $0 < \beta_{k+1} \le \beta_k$, so that $\beta_k \to \beta \ge 0 (k \to \infty)$. For every $k \in \mathbb{N}$, there exists $u_k \in Z_k$ such that $|u_k|_{L^2_K(\mathbb{R})} > \frac{\beta_k}{2}$ and $\|u_k\| = 1$. For any $v \in E$, writing $v = \Sigma_{j=1}^{\infty} c_j e_j$, we have, by the Cauchy-Schwartz inequality,

$$|(u_k, v)| = |(u_k, \Sigma_{j=1}^{\infty} c_j e_j)| = |(u_k, \Sigma_{j=k}^{\infty} c_j e_j)| \le \|u_k\| \|\Sigma_{j=k}^{\infty} c_j e_j\| = (\Sigma_{j=k}^{\infty} c_j^2)^{\frac{1}{2}} \to 0$$

as $k \to \infty$, which implies that $u_k \rightharpoonup 0$ in E. By Lemma 3.2, the compact embedding of $E \hookrightarrow L^s_K(\mathbb{R}^N)$ $(2 \le s < 2^*)$ implies that $u_k \to 0$ in $L^s_K(\mathbb{R}^N)$. Hence, letting $k \to \infty$, we get $\beta = 0$, which completes the proof.□

By Lemma 3.9, we can choose an integer $m \ge 1$ such that

$$\|u\|^2_{L^2_K(\mathbb{R}^N)} \le \frac{1}{2c_2}\|u\|^2, \qquad \|u\|^p_{L^p_K(\mathbb{R}^N)} \le \frac{p}{4c_3}\|u\|^p, \quad \forall\ u \in Z_m. \tag{26}$$

Lemma 3.10 *Suppose that* $(VK1)$, $(VK2)$ and (f_1) *are satisfied. Then there exists constant* $\rho, \alpha > 0$ such that

$$J|_{\partial B_R \cap Z_m} \ge \alpha.$$

Proof. From Remark 1.13 and (26), for $u \in Z_m$, choosing $\rho := \|u\| = \frac{1}{2}$, we get

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\|u\|^2_{L^2_K(\mathbb{R}^N)} - \frac{c_2}{p}\|u\|^p_{L^p_K(\mathbb{R}^N)} \\ &\ge \frac{1}{4}\left(\|u\|^2 - \|u\|^p\right) \\ &= \frac{2^{p-2}-1}{2^{p+2}} := \alpha > 0. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.5. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. By Lemma 3.4, 3.5, 3.10 and Corollary 3.8, all conditions of Lemma 3.3 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial solutions.□

Proof of Theorem 1.6. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. By Lemmas 3.6 and Corollary 3.8, all conditions of Lemma 3.3 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial solutions. □

4. Existence of nontrivial solutions

In this section, we prove the existence of a nontrivial solution for the problem (1). Next, the following lemma discuss the continuous embedding $E \hookrightarrow L^q_K(\mathbb{R}^N)$ for all $q \in [2, 2^*)$. Moreover, under $(VK3)$ and $(VK4)$, we can not prove that $E \hookrightarrow L^q_K(\mathbb{R}^N)$ is compact embedding for all $q \in [2, 2^*)$.

Lemma 4.1 *Assume that* $(VK3)$-$(VK4)$ hold. *Then E is continuously embedded in* $L^q_K(\mathbb{R}^N)$ *for all* $q \in [2, 2^*)$.
Proof. By condition $(VK4)$, we have $K(x) \leq MV^\theta(x)$. Hence, if $q = 2$, the the proof is trivial. Fix $q \in (2, 2^*)$, choose $\sigma = \frac{2^*-q}{2^*-2}$, then $q = 2\sigma + (1-\sigma)2^*$ and $0 < \sigma < 1$, which implies that $K(x) \leq MV^\sigma(x)$. Hence we can get the following inequality

$$\begin{aligned}
\|u\|^q_{q,K} &= \int_{\mathbb{R}^N} K(x)|u|^q dx \\
&= \int_{\mathbb{R}^N} K(x)|u|^{2\sigma}|u|^{(1-\sigma)2^*} dx \\
&\leq \left(\int_{\mathbb{R}^N} (K(x))^{\frac{1}{\sigma}} u^2 dx\right)^\sigma \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{1-\sigma} \\
&\leq M\left(\int_{\mathbb{R}^N} V(x)u^2 dx\right)^\sigma \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{1-\sigma} \\
&\leq CM\left(\int_{\mathbb{R}^N} V(x)u^2 dx\right)^\sigma \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{\frac{(1-\sigma)2^*}{2}} \\
&\leq CM\left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{\sigma + \frac{(1-\sigma)2^*}{2}} \\
&= CM\left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{\frac{q}{2}} \\
&= CM\|u\|^q.
\end{aligned}$$

It follows that $E \hookrightarrow L^q_K(\mathbb{R}^N)$ is continuous embedding. □

Lemma 4.2 *Suppose that* $(VK3)$, (F_1) *and* (F_2) *holds. Then* $J \in C^1(E, \mathbb{R})$ *and* $J' : E \to E^*$

$$\langle J'(u), v\rangle = (u, v) - \int_{\mathbb{R}^N} K(x) f(u) v dx$$

is weakly sequentially continuous for $u, v \in E$.
Proof. For convenience, let

$$\ell(u) = \int_{\mathbb{R}^N} K(x)F(u)dx.$$

For any $u, v \in E$ and $0 < |t| < 1$, by mean value theorem and Remark 1.13, there is a $\tau \in (0, 1)$ such that

$$\begin{aligned}
\frac{|F(u + tv) - F(u)|}{|t|} &\leq |f(u + \tau tv)v| \\
&\leq \varepsilon|u + \tau tv||v| + C_\varepsilon|u + \tau tv|^{p-1}|v| \\
&\leq \varepsilon|u||v| + \varepsilon|v|^2 + C_\varepsilon|u + \tau tv|^{p-1}|v| \\
&\leq \varepsilon|u||v| + \varepsilon|v|^2 + 2^{p-1}C_\varepsilon(|u|^{p-1}|v| + |v|^p).
\end{aligned}$$

By Hölder's inequality, we get

$$\varepsilon|u||v| + \varepsilon|v|^2 + 2^{p-1}C_\varepsilon(|u|^{p-1}|v| + |v|^p) \in L^1(\mathbb{R}^N).$$

Hence, by the Lebesgue's Dominated Theorem, we have

$$\langle \ell'(u), v\rangle = \int_{\mathbb{R}^N} K(x)f(u)v dx, \quad \forall\, u, v \in E.$$

Next, we prove that $\ell' : E \to E^*$ is weakly sequentially continuous. Suppose that $u_n \rightharpoonup u$ in E. By Lemma 4.1 implies that $u_n \to u$ in $L^q_{K,loc}$ for any $q \in [2, 2^*)$ and $u_n \to u$ for a.e. $x \in \mathbb{R}^N$. Thus, by (F_1) and (F_2), it follows that for any $\varphi \in C_0^\infty$,

$$\langle \ell'(u_n), \varphi\rangle = \int_{\mathbb{R}^N} K(x)f(u_n)\varphi dx \to \int_{\mathbb{R}^N} K(x)f(u)\varphi dx = \langle \ell'(u), \varphi\rangle \tag{27}$$

Since C_0^∞ is dense in E, for any $w \in E$ we take $\varphi_n \in C_0^\infty$ such that $\|\varphi_n - \omega\| \to 0$. Note that $|\langle \ell'(u_n) - \ell'(u), \varphi_n\rangle| \to 0$ as $n \to \infty$ by (27). Indeed, by (F_1) and (F_2), we have

$$\begin{aligned}
&|\langle \ell'(u_n) - \ell'(u), \omega\rangle| \\
&\le \left|\langle \ell'(u_n), \omega\rangle - \langle \ell'(u_n), \varphi_n\rangle + \langle \ell'(u_n), \varphi_n\rangle - \langle \ell'(u), \varphi_n\rangle + \langle \ell'(u), \varphi_n\rangle - \langle \ell'(u), \omega\rangle\right| \\
&\le \left|\langle \ell'(u_n) - \ell'(u), \varphi_n\rangle\right| + \left|\langle \ell'(u_n) - \ell'(u), \omega - \varphi_n\rangle\right| \\
&\le \left|\langle \ell'(u_n) - \ell'(u), \varphi_n\rangle\right| + c_1 \int_{\mathbb{R}^N} K(x)(|u_n|^{p-1} + |u|^{p-1})|\omega - \varphi_n| \\
&\le \left|\langle \ell'(u_n) - \ell'(u), \varphi_n\rangle\right| + c_2\|\omega - \varphi_n\| \to 0.
\end{aligned}$$

Therefore, we have shown that ℓ is weakly sequentially continuous. It follows that $J \in C^1(E, \mathbb{R})$ *and* $J' : E \to E^*$ is weakly sequentially continuous. □

In order to prove our results in the rest of paper, we need to use mountain pass theorem, which is introduced by Ambrosetti-Rabinoeitz [3]. Next, we prove that all conditions of mountain pass theorem [3] are satisfied.

Lemma 4.4 *Assume that $(VK3)$, $(VK4)$ and (F_1)-(F_3) are satisfied. Then all conditions of the above mountain pass theorem are satisfied.*

Proof. It is easy to see that $J(0) = 0$. By $(VK3)$, set $K(x) \equiv 0$ in a domain Ω and $K(x) \ne 0$ in $R^N \backslash \Omega$. On the one hand, for fixed $u_0 \in E$, we have

$$\begin{aligned}
J(tu_0) &\le \frac{t^2}{2}\|u_0\|^2 - \int_{\mathbb{R}^N} K(x)F(tu_0)dx \\
&= \frac{t^2}{2}\|u_0\|^2 - \int_{\Omega} K(x)F(tu_0)dx - \int_{\mathbb{R}^N\backslash\Omega} K(x)F(tu_0)dx \\
&= \frac{t^2}{2}\|u_0\|^2 - \int_{\mathbb{R}^N\backslash\Omega} K(x)F(tu_0)dx,
\end{aligned}$$

which implies that

$$\frac{J(tu_0)}{t^2} \le \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}^N\backslash\Omega} K(x)\frac{F(tu_0)}{|tu_0|^2}u_0^2 dx, \tag{28}$$

In (28), letting $|t| \to \infty$, by Fatou's Lemma, then

$$\begin{aligned}
\limsup_{|t|\to\infty} \frac{J(tu_0)}{t^2} &\le \frac{1}{2}\|u_0\|^2 - \liminf_{|t|\to\infty} \int_{\mathbb{R}^N\backslash\Omega} K(x)\frac{F(tu_0)}{|tu_0|^2}u_0^2 dx \\
&\le \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}^N\backslash\Omega} K(x)\liminf_{|t|\to\infty} \frac{F(tu_0)}{|tu_0|^2}u_0^2 dx \\
&= -\infty.
\end{aligned}$$

On the other hand, by (28), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|^2_{L^2_K} - \frac{C_\varepsilon}{p}\|u\|^p_{L^p_K} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\gamma_2^2\|u\|^2 - \frac{C_\varepsilon}{p}\gamma_p^p\|u\|^p \end{aligned} \tag{29}$$

Set $\varepsilon = \frac{1}{2\gamma_2^2}$ and $\rho = \left[p/(6\gamma_p^p C_{\frac{1}{2\gamma_2^2}})\right]^{\frac{1}{p-2}}$ in (29), then we have

$$J(u) \geq \frac{1}{4}\|u\|^2 - \frac{C_{\frac{1}{2\gamma_2^2}}}{p}\gamma_p^p\|u\|^p = \frac{1}{12}\rho^2 > 0 \quad \text{for any } \|u\| = \rho.$$

This completes the proof. □

From Lemma 4.4 and mountain pass theorem, we can get the following lemma.

Lemma 4.5 *Suppose that* $(VK3)$, $(VK4)$ *and* (F_1)-(F_3) *are satisfied. Then there exists a sequence* $\{u_n\} \subset E$ *such that*

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0. \tag{30}$$

Lemma 4.6 *Suppose that* $(VK3)$, $(VK4)$ *and* (F_1)-(F_3) *are satisfied. Then any* $\{u_n\} \subset E$ *satisfying* (30) *is bounded in* E.

Proof. By (F_3) and (30), we can get

$$\begin{aligned} c + 1 &\geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + \frac{1}{\mu}\int_{\mathbb{R}^N} K(x)[f(u_n)(u_n) - \mu F(u_n)]dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E. This completes the proof. □

Proof of Theorem 1.10. By Lemma 4.6, passing to a subsequence if necessary, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. From the weakly sequentially continuous of J', we can get $J'(u_n) \to J'(u)$ as $n \to \infty$. Since $J'(u_n) \to 0$, by the uniqueness of limits, then we have that u is weakly solution of J.

Next, we show that $u \neq 0$. By contradiction, we can assume that $u = 0$. In order to achieve a contradiction, we remark that, at least for $n \gg 1$, by (30) and Hölder's inequality,

$$\begin{aligned} \frac{c}{2} &\leq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^N} K(x)\left[\frac{1}{2}f(u_n)u_n - F(u_n)\right]dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} K(x)f(u_n)u_n dx \\ &\leq \int_{\mathbb{R}^N}\left(\frac{\varepsilon}{2}K(x)|u_n|^2 + \frac{C_\varepsilon}{2}K(x)|u_n|^p\right)dx \\ &\leq \frac{\varepsilon}{2}\|u\|^2_{L^2_K} + \frac{C_\varepsilon}{2}\left(\int_{\mathbb{R}^N} K(x)|u|^{2(p-1)}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} K(x)|u_n|^2 dx\right)^{\frac{1}{2}}. \end{aligned} \tag{31}$$

Since $p \in (2, 2^*)$, then we have $2(p-1) \in (2, 2^*)$. Therefore, there exists $\gamma_{2(p-1)} > 0$ such that

$$\int_{\mathbb{R}^N} K(x)|u_n|^{2(p-1)}dx = \|u_n\|_{L_K^2}^{2(p-1)} \le \gamma_{2(p-1)}^{2(p-1)}\|u_n\|^{2(p-1)} \tag{32}$$

Combined (31) and (32), we have

$$\frac{c}{2} \le \frac{\varepsilon}{2}\|u_n\|_{L_K^2}^2 + \gamma_{2(p-1)}^{2(p-1)}\|u_n\|^{2(p-1)}\|u_n\|_{L_K^2}^2.$$

Choose

$$\varepsilon \le \frac{c}{2\left(\sup_m \|u_n\|_{L_K^2}\right)^2},$$

which implies that there exists a constant $C_1 > 0$ such that

$$\|u_n\|_{L_K^2} \ge \exp(C_1 \log \frac{c}{4}) := \sigma > 0.$$

Since $E \hookrightarrow L_K^q(\mathbb{R}^N)$ and $u_n \rightharpoonup 0$ in E, then $u_n \to 0$ in $L_{K,loc}^q(\mathbb{R}^N)$, where $q \in [2, 2^*)$. Therefore for any $R > 0$, there are some $n_0 = m_0(R)$ such that for any $n \ge n_0$,

$$\|u_n\|_{L^2(B_R(0))} \le \frac{\sigma}{2}.$$

By

$$\|u_n\|_{L^2(\mathbb{R}^N\setminus B_R(0))} \ge \|u_n\|_{L_K^2} + \|u_n\|_{L^2(B_R(0))} \ge \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$$

and $(VK2)$, then

$$\begin{aligned}
\frac{\sigma}{2} &\le \|u_n\|_{L^2(\mathbb{R}^N\setminus B_R(0))} \\
&\le \left(\int_{|x|\ge R} K(x)|u_n|^2dx\right)^{\frac{1}{2}} \\
&\le M^{\frac{1}{2}}\left(\int_{|x|\ge R} V^{\theta}(x)|u_n|^2dx\right)^{\frac{1}{2}} \\
&\le M^{\frac{1}{2}}\left(\int_{|x|\ge R} \frac{1}{V^{1-\theta}(x)}V(x)|u_n|^2dx\right)^{\frac{1}{2}} \\
&\le \frac{M^{\frac{1}{2}}(\sup_n \|u_n\|)}{\inf_{|x|\ge R} V^{\frac{1-\theta}{2}}(x)}.
\end{aligned}$$

Since $V(x) \to +\infty$ as $|x| \to \infty$ and $0 < \theta < 1$, we can achieve a contradiction, when $R \gg 1$. Therefore $u \ne 0$. u is a nontrivial solution of J. This completes the proof. □

Acknowledgment

The authors thank the editor and the referees for their valuable comments and suggestions.

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