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A New Analytical Approach for Solving Nonlinear Boundary Value Problems Arising in Nonlinear Phenomena

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Abstract. In this research a new analytical approach is used to solve nonlinear boundary value problems (BVPs) of higher order occurring in nonlinear phenomena. It converts a complex nonlinear problem into zeroth order and first order problem. It consists of initial guess, auxiliary functions (containing unknown convergence controlling parameters) and a homotopy. The unknown parameters are determined by minimizing the residual. Many methods which are explained in this paper are used to determine these parameters. Here Galerkin's method is used for this purpose. It is applied to solve non-linear BVPs of fourth and fifth order. The results are compared with the already existing methods e.g., Galerkin's Method with Quintic B-splines, Differential Transform Method (DTM), and Optimal Homotopy Asymptotic Method (OHAM). It gives efficient and accurate first-order approximate solution. The results achieved by this technique are in excellent concurrence with the exact solution and hence proved that this method is effective and easy to apply.

1. Introduction

Different problems in engineering and science can be formulated in terms of boundary value problems. They have a significant contribution in todays' modern fields of science and technology which take place from steady state solutions of transient problems. They are used in the mathematical modeling of different entities such as visco-elastic flows, hydrodynamic stability problems, non-Newtonian fluids, and convection of heat etc [1]. The physical situations like deformation of elastic beams with simply supported ends in an equilibrium state, visco elastic and inelastic flows, transverse vibrations of hinged beams, plate bending on an elastic foundation, and the deflection of a plate are modeled by fourth order BVPs. The BVPs of fifth order appear in the model of visco-elastic flows. [2]. Many Numerical methods, semi numerical method, Perturbation and Analytical techniques are used to solve such problems. Researchers have introduced

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many other methods based on Homotopy Perturbation Method (HPM) [3, 4], for example, OHAM [5–9, 11], and Optimal Homotopy Perturbation Method(OHPM)[12] to get the approximate solution of the nonlinear BVPs. The relevant work can also be seen in [15–17]. But it is still quite problematic and need new techniques for finding the approximate solutions. Inspired and aggravated by the continuing research in this area, we apply a new analytical approach, (OHAM-2)[5] for solving the nonlinear BVPs of order four and five as given in [9, 11, 13, 14]. It consists of few steps and converges to almost exact solution. The applied method is simple in learning and easy to apply. Math type and mathematica 7.0 is used for calculations as well as numerical simulations.

2. Explanation of the applied method

Fundamental Concept of OHAM:

Consider the following boundary value problem

$$\Upsilon(\mu(s)) + f(s) = \Phi(\mu(s)) + f(s) + \Psi(\mu(s)) = 0, \;\; \beta\left(\mu(s), \tfrac{d\mu(s)}{ds}\right) = 0. \tag{1}$$

Where $f(s)$ is a known function, p is an embedding parameter, β is a boundary operator,s is independent variable, and $\mu(s)$ is an undetermined function. Also Υ is a general operator,Φ is linear operator, and Ψ is nonlinear operator. In this method we define a homotopy: $H(\nu(s,p,c_i)) : \Omega \times [0,1] \rightarrow R$which satisfies

$$\begin{aligned} H(\nu(s,p,c_i)) &= (1-p)(\Phi(\nu(s,p,c_i) + f(s)) \\ &= H(s,p,c_i)(\Phi(\nu(s,p,c_i) + f(s) + \Psi(\nu(s,p,c_i)). \end{aligned} \tag{2}$$

Here $s \in R,\Omega$ is the domain of interest, $H(s,p,c_i)$ is an auxiliary function which is non-zero for $p \neq 0$,$H(s,0,c_i) = 0$, and $\nu(s,p,c_i)$ is an undetermined function. Clearly, when $p = 0$ then:

$$\nu(s,0,c_i) \;=\; \mu_0(s,c_i). \tag{3}$$

and when $p = 1$ then

$$\nu(s,1,c_i) \;=\; \mu(s,c_i). \tag{4}$$

Therefore, the solution $\nu(s,p,c_i)$ changes from $\mu_0(s)$ to $\mu(s)$ as p changes from 0 to 1. Now the initial guess $\mu_0(s)$ is calculated from Eq.(2) for $p = 0$ and we have:

$$\Phi(\mu_0(s)) \;+ f(s) \;= 0\;,\; \beta(\mu_0(s), \frac{d\mu_0(s)}{ds}) \;=\; 0. \tag{5}$$

Now consider the auxiliary function $H(s,p,c_i)$ as follows:

$$H(s,p,c_i) = p\; H_1(s,c_i) + p^2\; H_2(s,c_i) + \;\ldots, \tag{6}$$

where the auxiliary functions $H_i(s,c_j), i = 1,2,\ldots$ depend upon s and also on $c_j, j = 1,2,\ldots,s$.
Expand,$\nu(s,p,c_i)$ in Taylors series about p as follows:

$$\nu(s,p,c_i) = \mu_0(s) + \sum_{k=1}^{\infty} \mu_k(s,c_1,c_2,\ldots,c_k)p^k. \tag{7}$$

Now put Eq.(40) in Eq.(2) and compare the coefficients of the same powers of p to achieve linear equations as follows

Zeroth order problem:

$$\Phi(\mu_0(s)) \;+ f(s) \;= 0\;,\; \beta(\mu_0(s), \frac{d\mu_0(s)}{ds}) \;=\; 0. \tag{8}$$

First order problem:

$$\Phi(\mu_1(s)) = c_1\Psi_0(\mu_0(s)),\ \beta(\mu_1(s), \frac{d\mu_1(s)}{ds}) = 0. \tag{9}$$

The required governing equations for $\mu_k(s)$ are given by:

$$\Phi(\mu_k(s)) - \Phi(\mu_{k-1}(s)) = c_k\,\Psi_0(\mu_0(s)) + \textstyle\sum_{i=1}^{k-1} c_i[\Phi(\mu_{k-i}(s)) + \Psi_{k-i}(\mu_0(s), \mu_1(s), ..., \mu_{k-1}(s))],$$
$$k = 2, 3, ..., \beta(\mu_k, \tfrac{d\mu_k}{ds}) = 0, \tag{10}$$

where $\Psi_m(\mu_0(s), \mu_1(s), \mu_2(s), \ldots, \mu_m(s))$ is the coefficient of p^m when $\Psi(\nu(s, p, c_i))$ expand about the embedding parameter p:

$$\Psi(\nu(s, p, c_i) = \Psi(\mu_0(s)) + \sum_{m=1}^{\infty} \Psi_m(s, \mu_0(s), \mu_1(s), \ldots, \mu_m(s))p^m. \tag{11}$$

. The series Eq.(11) depends on auxiliary parameters $c_1, c_2, \ldots, c_m$. If at $p = 1$, it is convergent then one has:

$$\nu(s, 1, c_i) = \mu(s, c_i) = \mu_0(s) + \sum_{k=1}^{\infty} \mu_k(s, c_1, c_2, \ldots, c_k). \tag{12}$$

. The m^{th}-order approximate solution become as :

$$\mu(s, c_1, c_2, \ldots, c_m) = \widetilde{\mu}(s, c_1, c_2, \ldots, c_m) = \mu_0(s) + \sum_{i=1}^{m} \mu_i(s, c_1, c_2, \ldots, c_i). \tag{13}$$

Use Eq.(13) in Eq.(1) to achieve the residual as follow:

$$R(s, c_1, c_2, \ldots, c_m) = \Upsilon(\widetilde{\mu}(s, c_1, c_2, \ldots, c_m) + f(s)$$
$$= \Phi(\widetilde{\mu}(s, c_1, c_2, \ldots, c_m) + \Psi(\widetilde{\mu}(s, c_1, c_2, \ldots, c_m) + f(s). \tag{14}$$

If $R = 0$ then the exact solution will be$\widetilde{\mu}$. When $R \neq 0$, especially in nonlinear problems, then we find the optimal convergence control parameters (Auxiliary parameters)$c_i, i = 1, 2, \ldots$.
Now to find the above values we first construct the functional,

$$\zeta(c_1, c_2, \ldots, c_m) = \int_a^b R^2(s, c_1, c_2,\ \ldots, c_m)ds, \tag{15}$$

and then minimizing it, we have

$$\frac{\partial\zeta}{\partial c_1} = \frac{\partial\zeta}{\partial c_2} = \ \ldots = \frac{\partial\zeta}{\partial c_m} = 0, \tag{16}$$

$$\int_a^b R\frac{\partial\widetilde{\mu}}{\partial c_1}ds = 0,\ \int_a^b R\frac{\partial\widetilde{\mu}}{\partial c_2}ds = 0, \ldots. \tag{17}$$

The m^{th}-order approximate solution is calculated if a and b are in the domain of the problem and also if the auxiliary parameters are known. Many methods like Galerkin's method and method of Least Squares are used to find the values of auxiliary parameters. Besides these methods, Marinca et al. [5, 7, 12] also reported some other methods for this purpose e.g. the collocation method and Ritz's method.

First Version of OHAM:

We put $m = 2$ in Eq.(13) to achieve the 1^{st} version of OHAM (OHAM-1). In this case 2^{nd} order approximate solution of 2^{nd} order becomes as:

$$\widetilde{\mu}(s, 1, c_i) = \mu_0(s) + \mu_1(s, c_i) + \mu_2(s, c_i), \tag{18}$$

where the terms μ_0, μ_1 and μ_2 are achieved from the following equations: Eq.(19),Eq.(20) and Eq.(21) respectively

$$\Phi(\mu_0(s)) + f(s) = 0, \quad \beta(\mu_0, \frac{d\mu_0}{ds}) = 0, \tag{19}$$

$$\Phi(\mu_1(s,c_i)) = H_1(s,c_i)\Psi_0(\mu_0(s)), \quad \beta(\mu_1, \frac{d\mu_1}{ds}) = 0, \tag{20}$$

$$\begin{aligned}\Phi(\mu_2(s,c_i)) - \Phi(\mu_1(s,c_i)) = \quad & H_1(s,c_i)[\Phi(\mu_1(s,c_i)) + \Psi_1(\mu_0(s),\mu_1(s,c_i))] \\ & +H_2^*(s,c_i)\Psi_0(\mu_0(s)), \ \beta(\mu_2, \tfrac{d\mu_2}{ds}) = 0.\end{aligned}$$

Taking into account Eq.(20) in the last we can write

$$\Phi(\mu_2(s,c_i)) - \Phi(\mu_1(s,c_i)) = H_1(s,c_i)\Psi_1(\mu_0(s),\mu_1(s,c_i)) + H_2(s,c_i)\Psi_0(\mu_0(s)), \quad \beta(\mu_2, \tfrac{d\mu_2}{ds}) = 0, \tag{21}$$

where $H_2(s,c_i) = H_1^2(s,c_i) + H_2^*(s,c_i)$

New Version of OHAM (OHAM-2)[6]: We use Fundamental concept of OHAM to develop new form of OHAM. Consider the same BVP as above:

$$\Upsilon(\mu(s)) + f(s) = \Phi(\mu(s)) + f(s) + \Psi(\mu(s)) = 0, \ \beta\left(\mu(s), \frac{d\mu(s)}{ds}\right) = 0, \tag{22}$$

where Υ,$f(s)$,Φ,$\mu(s)$,Ψ, and β have the same meaning as above. Let $\mu_0(s)$ be an initial guess of $\mu(s)$ such that

$$\Phi(\mu_0(s)) + f(s) = 0, \ \beta\left(\mu_0(s), \ \frac{d\mu_0(s)}{ds}\right) = 0. \tag{23}$$

Let us consider the function $\nu(s,p,c_i)$ in the particular form as

$$\nu(s,p,c_i) = \mu_0(s) + p \ \mu_1(s,c_i), \tag{24}$$

where p represents an embedding parameter such that $0 \le p \le 1$. Now the 1^{st}-order approximate solution become as:

$$\widetilde{\mu}(s,c_i) = \mu(s,c_i) = \mu_0(s) + \ \mu_1(s,c_i), \ \beta\left[\widetilde{\mu}(s,c_i), \ \frac{d\widetilde{\mu}(s,c_i)}{ds}\right] = 0, \tag{25}$$

where $c_1, c_2, \ldots, c_s$ are auxiliary parameters which will be calculated later. Now we define a family of equations as:

$$\begin{aligned}&H[\Phi(\nu(s,p,c_i)) + f(s), \ H(s,c_i), \ \Psi(\nu(s;p,c_i)] \\ &= \Phi(\mu_0(s)) + f(s) + p[\Phi(\mu_1(s,c_i)) - H(s,c_i)\Psi(\mu_0(s))],\end{aligned} \tag{26}$$

which satisfies the properties:

$$H[\Phi(\nu(s,0,c_i)) + f(s), H(s,c_i), \ \Psi(\nu(s;0,c_i)] = \Phi(\mu_0(s)) + f(s) = 0, \tag{27}$$

$$H[\Phi(\nu(s,1,c_i)) + f(s), H(s,c_i), \Psi(\nu(s;1,c_i)] = H(s,c_i)[\Phi(\widetilde{\mu}(s,c_i)) + f(s) + \Psi(\widetilde{\mu}(s,c_i))] = 0, \tag{28}$$

where $H(s,c_i) \neq 0$ is an auxiliary function and the terms in p^2 are omitted. From Eq.(24) and Eq.(25) one gets

$$\nu(s,0,c_i) = \mu_0(s), \ \nu(s,1,c_i) = \widetilde{\mu}(s,c_i). \tag{29}$$

Now compare the coefficients of p^0 and p^1 in Eq.(26) we get the required equation of $\mu_0(s)$ specified by Eq.(23) and the equation of the 1^{st} order approximation $\mu_1(s,c_i)$, i.e.

$$\Phi(\mu_1(s,c_i)) = H(s,c_i)\Psi(\mu_0(s)), \ \beta\left[\mu_1(s,c_i), \ \tfrac{d\mu_1(s,c_i)}{ds}\right] = 0, \quad i = 1, \ 2, \ \ldots, \ s. \tag{30}$$

Generally, the nonlinear operator may be written as:

$$\Psi(\mu_0(s)) = \Sigma_{i=1}^{m} h_i(s) g_i(s) \tag{31}$$

where $h_i(s)$ and $g_i(s)$ are known functions which are depended upon the functions $\mu_0(s)$ and nonlinear operator Ψ. m is also a known as an integer. Since, Eq.(30) is non-homogeneous linear therefore it has two solutions; one is the solution of corresponding homogeneous equation and the other one is some particular solutions of the non-homogeneous equation. So, the solution of Eq.(30) is the sum of the above mentioned two solutions but in exceptional cases, only particular solutions may be selected readily. Now suppose the unknown function $\mu_1(s, c_j)$ in the form

$$\begin{aligned} &\mu_1(s, c_j)) = \Sigma_{i=1}^{m} H_i(s, h_j(s), c_j) g_i(s), \\ &\beta\left[\mu_1(s, c_j), \tfrac{d\mu_1(s,c_j)}{ds}\right] = 0, \quad j = 1,\ 2,\ \ldots,\ s. \end{aligned} \tag{32}$$

or

$$\begin{aligned} &\mu_1(s, c_j)) = \Sigma_{i=1}^{m} H_i(s, g_j(s), c_j) h_i(s), \\ &\beta\left[\mu_1(s, c_j), \tfrac{d\mu_1(s,c_i)}{ds}\right] = 0, \ j = 1,\ 2,\ \ldots,\ s. \end{aligned} \tag{33}$$

Where $H_i(s, h_j, c_j)$ consist of linear combinations of some functions h_i, some terms which are given by corresponding homogeneous equation and several undetermined parameters c_j for $j = 1, 2, ..., s$. Also$m$ is an arbitrary integer number. Now, if h_1 is a polynomial function such as $h_1 = s^3$, then $H_1(s, h_1, c_j)$ is a combination of polynomials, $H_1(s, h_1, c_j) = c_1 s + c_2 s^3 + c_3 s^7 + \ldots$. If h_1 is a trigonometric function i.e. if $h_1 = Sin(\gamma s)$, then $H_1(s, h_1, c_j) = c_1 Sin(\gamma s) + c_2 Cos(\gamma s) + c_3 Sin(2\gamma s) + \ldots$, Similarly, when h_1 is a logarithmic function i.e. $h_1 = \ell n(s)$ then $H_1(s, h_1, c_j) = c_1 \ell n(s) + c_2 s \ell n(s) + c_3 s^2 \ell n(2s) + \ldots$. Where H_i and m can be defined in many ways. The solution $\mu_1(s, c_j)$ specified by Eq.(32) is not complete solution of Eq.(30), but $\widetilde{\mu}(s, c_i)$ given by Eq.(25) is the solution of Eq.(22). The same considerations can be made for the Eq.(33), where h_i and g_i are interchangeable. Now in the last putting the values of μ_0 and $\mu_1(s, c_i)$ in Eq.(25)after finding the optimal values of auxiliary parameters $c_i, i = 1, 2, 3, \ldots, s$ to achieve complete solution of Eq.(22).

Application of method

In this section high accuracy of third alternative of OHAM is shown over the existing methods in the literature. The proposed technique is applied to some non linear BVPs of different orders. As a result, we see that this method gives best approximation and takes very less time to produce good results.

Model 1.Consider non-linear boundary value problem of order four as solved in [11, 13]:

$$\frac{d^4\mu}{ds^4} = Sin(s) + Sin^2(s) - (\frac{d^2\mu}{ds^2})^2, \mu(0) = 0,\ \mu'(0) = 1,\ \mu(1) = Sin(1),\ \mu'(1) = Cos(1),\ 0 \le\ s\ \le 1. \tag{34}$$

Where $\mu(s) = Sin(s)$ is the exact solution. To apply the third alternative of OHAM, we first find the initial guess $\mu_0(s)$ from the following as:
Zeroth order problem:

$$-Sin(s) - Sin^2(s) + (\mu_0)''''(s) = 0, \mu_0(0) = 0,\ \mu_0'(0) = 1,\ \mu_0(1) = Sin(1),\ \mu_0'(1) = Cos(1), \tag{35}$$

which gives,

$$\mu_0(s) = \frac{1}{48}(3-8s^2+4s^3+s^4+9s^2(Cos(1))^2-6s^3(Cos(1))^2-3(Cos(s))^2+6s^2Cos(1)Sin)(1)-6s^3Cos(1)Sin(1)+48Sin(s)). \tag{36}$$

Now, since $\Psi(\mu(s)) = -Sin(s) - Sin^2(s) + (\mu''(s))^2$ which gives $\Psi(\mu_0(s)) = -Sin(s) - Sin^2(s) + (\mu_0''(s))^2$. Therefore according to the Eq.(5.14) and Eq.(5.15) on page 392 and 393 in [6] we choose the auxiliary function as:

$$H(s, c_i) = H_1(s, h_1, c_i) = c_1 Sin(s) + c_2 Cos(s) + c_3 Sin(2s), \tag{37}$$

and put in Eq.(5.13) we get: First order problem:

$$\mu_1''''(s, c_i) = (c_1 Sin(s) + c_2 Cos(s) + c_3 Sin(2s))\Psi(\mu_0(s)), \mu_1(0) = 0,\ \mu_1'(0) = 0,\ \mu_1(1) = 0,\ \mu_1'(1) = 0. \tag{38}$$

Now solve the above equations and using Galerkin's method to get:

$$c_1 = 0.850097,\ c_2 = -0.0276344,\ c_3 = 0.128225, \tag{39}$$

put the above values of $\mu_0(s), \mu_1(s, c_i)$, and $p = 1$ in Eq.(5.6) on page (392)in [6] to achieve the 1^{st} order approximate solution as:

$$\begin{aligned}\mu(s) = 1.s + 7.005678758470035 \times 10^{-6}s^2 - 0.166702s^3 - 2.5225495034330193 \times 10^{-7}s^4 + 0.00857287s^5 \\ -0.000401879s^6 - 0.0000584755s^7 + 0.000107857s^8 - 0.0000135431s^9 - 0.0000510225s^{10} \\ +8.701874084073485 \times 10^{-7}s^{11} + 0.0000102883s^{12} + 4.3424879690708535 \times 10^{-7}s^{13} \\ -1.3215505387360188 \times 10^{-6}s^{14}.\end{aligned} \tag{40}$$

Model 2. Consider non-linear boundary value problem of order Five [9]:

$$\frac{d^5\mu(s)}{ds^5} = 1/32\mu^3\ e^{-s}, \mu(0) = 1, \mu'(0) = 1/2, \mu''(0) = 1/4, \mu(1) = e^{1/2}, \mu'(1) = 1/2e^{1/2}, \quad 0 \le s \le 1, \tag{41}$$

with exact solution $\mu(s) = e^{s/2}$. Now we use the third alternative of OHAM: Let $\mu_0(s)$ be the initial guess then
Zeroth order problem:

$$\mu_0'''''(s) = 0,\ \mu_0(0) = 1,\ \mu_0'(0) = 1/2, \mu_0''(0) = 1/4,\ \mu_0(1) = e^{1/2},\ \mu_0'(1) = 1/2e^{1/2}, \tag{42}$$

which gives

$$\mu_0(s) = 1. + 0.5s + 0.125s^2 + 0.0205244s^3 + 0.00319682s^4. \tag{43}$$

Since, $\Psi(\mu(s)) = -1/32\mu(s)^3 e^{-s}$, therefore $\Psi(\mu_0(s)) = -1/32\mu_0^3(s)e^{-s}$. Now we choose the auxiliary function as

$$H(s, c_i) = e^s(c_1 + c_2 s + c_3 s^2 + c_4 s^3 + c_5 s^4), \tag{44}$$

and solve the equation given below to get $\mu_1(s, c_i)$.
First order problem:

$$\mu_1'''''(s, c_i) = e^s\left(c_1 + c_2 s + c_3 s^2 + c_4 s^3 + c_5 s^4\right)\Psi(\mu_0(s)), \mu_1(0) = 0,\ \mu_1'(0) = 0,\ \mu_1''(0) = 0,\ \mu_1(1) = 0,\ \mu_1'(1) = 0. \tag{45}$$

$$c_1 = -0.999779,\ c_2 = 0.997536,\ c_3 = -0.490143,\ c_4 = 0.147591,\ c_5 = -0.0231079. \tag{46}$$

put the above values of $\mu_0(s), \mu_1(s, c_i)$, and $p = 1$ in Eq.(5.6) on page (392)in [6] to achieve the 1^{st} order approximate solution as:

$$\begin{aligned}\mu(s) = 1. + 0.5s + 0.125s^2 + 0.0208333s^3 + 0.00260419s^4 + 0.000260359s^5 + 0.0000217939s^6 \\ +1.4706107555298904 \times 10^{-6}s^7 + 1.2483440484747375 \times 10^{-7}s^8 + 9.681857366777685 \times 10^{-9}s^9 \\ -3.7132926569796935 \times 10^{-9}s^{10} - 5.817591715198018 \times 10^{-10}s^{11} + 1.6855346987092359 \times 10^{-10}s^{12} \\ +1.0758247621938015 \times 10^{-10}s^{13} + 3.344725434548341 \times 10^{-11}s^{14}.\end{aligned} \tag{47}$$

3. Figures:

Here the obtained results are illustrated graphically.

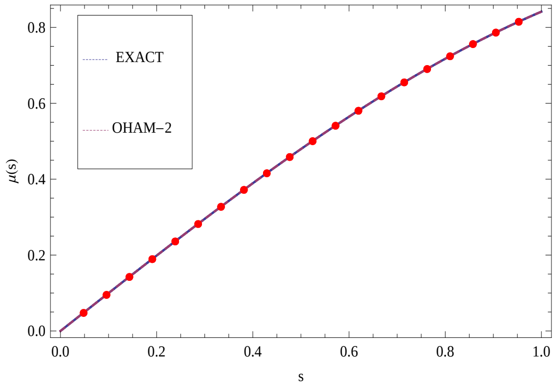

Figure 1: Shows comparison of the solution achieved by OHAM-2 with that of exact solution as well as with the results achieved by Methods in [13] for model 1. It shows that the results achieved by the applied OHAM-2 are far better.

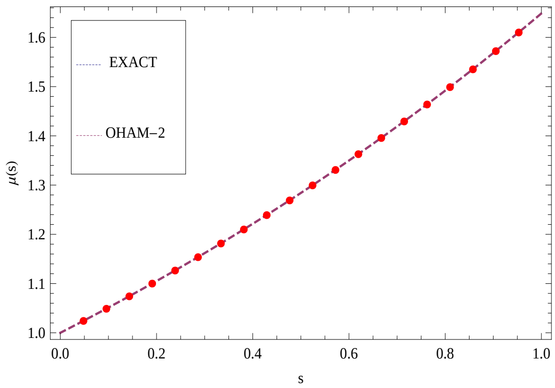

Figure 2: Indicates comparison of the solution obtained by OHAM-2 with the exact solution as well as with the results achieved by Methods in [9] for model 2. It shows that the results achieved by the applied OHAM-2 are more better.

4. Tables:

This section shows the comparison of the achieved results with already published work in the form of tables

s	EXACT	OHAM-2	E^* (B-splines)	E^* (OHAM)	E^* (OHAM-2)
0.0	0.0	0.0	$\cdots$	2.1×10^{-13}	0.0
0.1	0.0998334	0.0998335	9.7×10^{-7}	3.4×10^{-8}	-3.7×10^{-8}
0.2	0.198669	0.198669	4.1×10^{-6}	1.1×10^{-7}	-5.1×10^{-8}
0.3	0.29552	0.29552	8.0×10^{-6}	2.1×10^{-7}	-4.7×10^{-9}
0.4	0.389418	0.389418	1.0×10^{-5}	2.8×10^{-7}	4.0×10^{-8}
0.5	0.479426	0.479426	1.2×10^{-5}	3.2×10^{-7}	2.0×10^{-8}
0.6	0.564642	0.564643	1.4×10^{-5}	2.9×10^{-7}	-4.8×10^{-8}
0.7	0.644218	0.644218	1.2×10^{-5}	2.1×10^{-7}	-8.1×10^{-8}
0.8	0.717356	0.717356	7.7×10^{-6}	1.2×10^{-7}	-3.4×10^{-8}
0.9	0.783327	0.783327	4.5×10^{-6}	3.4×10^{-7}	2.1×10^{-8}
1.	0.841471	0.841471	$\cdots$	1.4×10^{-13}	4.2×10^{-9}

Table 1: Shows Comparison of the errors gained by method in [13], OHAM [11] and OHAM-2 for model 1, E^*=Exact-Approx.

s	EXACT	OHAM-2	E^* (DTM)	E^* (OHAM)	E^* (OHAM-2)
0.0	1.0	1.	0.0000	0.0000	0.0000
0.1	1.05127	1.05127	1.0×10^{-9}	-9.2×10^{-10}	1.4×10^{-12}
0.2	1.10517	1.10517	2.0×10^{-9}	-5.0×10^{-9}	5.0×10^{-12}
0.3	1.16183	1.16183	1.0×10^{-8}	-1.1×10^{-8}	7.5×10^{-12}
0.4	1.2214	1.2214	2.0×10^{-8}	-1.5×10^{-8}	7.9×10^{-12}
0.5	1.28403	1.28403	3.1×10^{-8}	-1.6×10^{-8}	6.7×10^{-12}
0.6	1.34986	1.34986	3.7×10^{-8}	-1.4×10^{-8}	4.2×10^{-12}
0.7	1.41907	1.41907	4.1×10^{-8}	-9.9×10^{-9}	7.9×10^{-13}
0.8	1.49182	1.49182	3.1×10^{-8}	-5.6×10^{-9}	-1.6×10^{-12}
0.9	1.56831	1.56831	1.4×10^{-8}	-1.1×10^{-9}	1.2×10^{-13}
1.0	1.64872	1.64872	0.0000	0.0000	9.4×10^{-12}

Table 2: Shows Comparison of the errors achieved by methods: DTM in [14], OHAM in [9]and OHAM-2 for model 2

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