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Analytic Core and Quasi-Nilpotent Part of Linear Relations in Banach Spaces

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Abstract. In this paper, we investigate the notion of analytic core and quasi-nilpotent part of a linear relation. Furthermore, we are interested in studying the set of Generalized Kato linear relations to give some of their properties in connection with the analytic core and the quasi-nilpotent part.We finish by giving a perturbation result for this set of linear relations.

1. Introduction

Notice that throughout this paper $(X, \|\ \|)$ denotes a complex Banach space. A linear relation T is any mapping having domain $D(T)$ a nonempty set of X, and taking values in $\mathcal{P}(X)\backslash\emptyset$ (the collection of nonempty subsets of X) such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all non zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in D(T)$. If $x \notin D(T)$ then $Tx = \emptyset$. With this convention we have $D(T) = \{u \in X : T(u) \neq \emptyset\}$. The set of all linear relations in X is denoted by $LR(X)$. A linear relation T is uniquely defined by its graph $G(T) = \{(u, v) \in X \times X : u \in D(T), v \in T(u)\}$. The inverse of T is the relation T^{-1} given by:

$$G(T^{-1}) = \{(v, u) \in X \times X : (u, v) \in G(T)\}.$$

We denotes by $R(T) = \bigcup\limits_{x \in D(T)} Tx$, $T^{-1}(0) = \ker(T) := \{x \in X : (x, 0) \in G(T)\}$, $T(0) := \{x \in X : (0, x) \in G(T)\}$ the range, the kernel and the multivalued part of T respectively. Note that if $T(0) = \{0\}$, then T is said to be an operator or single-valued. The class of linear bounded operators defined on all X is denoted by $B(X)$. If $G(T)$ is closed, then T is said to be closed. Let Q_T denote the quotient map from X onto $X/\overline{T(0)}$ so we can easy show that $Q_T T$ is single valued and so we can define the norm of T by $\|T\| := \|Q_T T\|$. T is said to be continuous if $\|T\| < \infty$, bounded if T is continuous and everywhere defined, and T is called open if T^{-1} is continuous equivalently if $\gamma(T) > 0$ where

$$\gamma(T) = \begin{cases} +\infty & \text{if } D(T) \subset \overline{\ker(T)} \\ \inf\left\{\frac{\|Tx\|}{dis(x, \ker(T))}; x \in D(T)\backslash\overline{\ker(T)}\right\} & \text{otherwise.} \end{cases}$$

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We denote By $CR(X)$, $BR(X)$ and $CBR(X)$ the sets of closed, bounded, and closed bounded linear relations respectively. The adjoint T^* of a linear relation T is defined by

$$G(T^*) = G(-T^{-1})^{\perp}.$$

This means that $(y^*, x^*) \in G(T^*)$ if and only if $y^*y - x^*x = 0$ for all $(x, y) \in G(T)$. The adjoint T^* of a linear relation T is always closed with domain

$$D(T^*) = \{y^* \in X^* : \ y^*T \text{ is continuous and single-valued}\}.$$

For $T, S \in LR(X)$ the linear relations ST and $T + S$ are defined by

$$G(ST) := \{(x, z); (x, y) \in G(T), (y, z) \in G(S) \text{ for some } y \in X\},$$

$$G(T + S) := \{(x, y + z); (x, y) \in G(T), (y, z) \in G(S)\}.$$

The resolvent set of T is the set

$$\rho(T) = \{\lambda \in \mathbb{C} \text{ such that } (\lambda I - T)^{-1} \text{ is everywhere defined and single valued }\},$$

and the spectrum of T,

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

Let U and V be nonempty subsets of a Banach space. We define the distance between U and V by the formula

$$dis(U, V) = \inf\{\|u - v\|, \ u \in U \ and \ v \in V\}.$$

We shall also write $dis(x, V)$ for the distance between $\{x\}$ and V.

Notation 1.1. *We denote by:*
$R^{\infty}(T) = \bigcap_{n\in\mathbb{N}} R(T^n)$.
$N^{\infty}(T) = \bigcup_{n\in\mathbb{N}} \ker(T^n)$.

Lemma 1.2. *[7, Lemma 2.7] Let $T \in CR(X)$. The following statements are equivalent:*

1. $\ker(T) \subseteq R(T^n)$, *for all nonnegative integer n;*
2. $\ker(T^m) \subseteq R(T)$, *for all nonnegative integer m;*
3. $\ker(T^m) \subseteq R(T^n)$, *for all nonnegative integers n and m.*

Definition 1.3. *[2, Definition 10] Let $T \in CR(X)$. We say that T is s-regular if $R(T)$ is closed and T verifies one of the equivalent conditions of Lemma 1.2.*

The notion of analytic core and quasi-nilpotent part of a linear operator was mentioned and studied by M. Mbekhta [8, 9], A. Pietro [1] and A. Tajmouati [10]. For a linear operator T, the authors have defined the analytic core by $K(T) = \{x \in X : \exists (x_n) \subset X \text{ and } a > 0 \text{ such that } x_0 = x; Tx_n = x_{n-1} \forall n \geq 1 \text{ and } \|x_n\| \leq a^n\|x\|\}$ and the quasi-nilpotent part by $H_0(T) = \{x \in X; \ \lim_{n\mapsto+\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$. In [8], M. Mbekhta has made the connection between these sets and the class of linear operators which have the general Kato decomposition in Hilbert spaces. In after time, W. Bouamama in [4] has generalized this result for linear operators defined in Banach spaces. Our work aims to generalize this result for linear relations defined in Banach spaces. For a linear relation $T \in LR(X)$ we define its analytic core by the set $K(T) = \{x \in X$ that there exist $a > 0$ and a sequence $(u_n) \subset X$ that checks $x = u_0$, $u_n \in Tu_{n+1}$ and $dis(u_n, T(0)) \leq a^n dis(x, T(0))$, $\forall n \in \mathbb{N}$ $\}$ and its quasi-nilpotent part by the set $H_0(T) = \Big\{x \in X$, that there exists a sequence $(x_n) \subset X$ which verify $x = x_0, \ x_{n+1} \in Tx_n$

and $\lim_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = 0\Big\}$.
An outline of the paper is as follows. In section 2 we define the analytic core and algebraic core of linear relations and we gather some of their properties . The quasi-nilpotent part of a linear relation is introduced and studied in section 3. Furthermore, in section 4, we give some properties of these notions for Generalized Kato linear relations in relation with its kernels and its ranges. We finish our paper by giving a perturbation result for Generalized Kato linear relations in section 5.

2. Analytic and Algebraic core of linear relations

The main of this section is to define the analytic and algebraic core of a linear relation and to give some of their properties.

Definition 2.1. *Let $T \in LR(X)$. The algebraic core $C(T)$ of T is defined to be the greatest subspace M of X for which $T(M) = M$. It's clear that $C(T) \subseteq R^\infty(T)$.*

Proposition 2.2. *Let $T \in LR(X)$. These assertions are equivalent :*

1. *$x \in C(T)$;*
2. *There exists a sequence $(u_n) \subset X$ such that $x = u_0$ and $u_n \in Tu_{n+1}$ for all $n \in \mathbb{N}$.*

Proof. Let M denotes the set of all $x \in X$ for which there exists a sequence $(u_n) \subset X$ such that $x = u_0$ and $u_n \in Tu_{n+1}$ for all $n \in \mathbb{N}$.
Firstly, let's showing that $C(T) \subset M$. Let $x \in C(T)$. From the equality $T(C(T)) = C(T)$, we obtain that there is an element $u_1 \in C(T)$ such that $x \in Tu_1$. Since $u_1 \in C(T)$, the same equality implies that there is $u_2 \in C(T)$ such that $u_1 \in Tu_2$. By repeating this process we can find a sequence $(u_n) \subset X$ which verify $x = u_0$ and $u_n \in Tu_{n+1}$, for all $n \in \mathbb{N}$ and therefore $C(T) \subseteq M$.
Conversely, to show that $M \subseteq C(T)$ it suffices to prove that $T(M) = M$. Let $y \in T(M)$ so there exists $x \in M$ such that $y \in Tx$. Since $x \in M$ then, there exists a sequence (u_n), $n \in \mathbb{N}$, for which $x = u_0$ and $u_n \in Tu_{n+1}$.
Now, we define the sequence (w_n) as

$$w_0 := y \quad \text{ and } \ w_n := u_{n-1}, \forall \ n \in \mathbb{N}^*.$$

Then

$$w_n = u_{n-1} \in Tu_n = Tw_{n+1} \ \text{ and } \ w_0 = y \in Tu_0 = Tw_1.$$

Hence $y \in M$ and therefore $T(M) \subseteq M$.
On the other hand, to prove the opposite inclusion, let consider an element $x \in M$. So, there is a sequence (u_n), $n \in \mathbb{N}$, for which $x = u_0$ and $u_n \in Tu_{n+1}$, for all $n \in \mathbb{N}$. Since $x = u_0 \in Tu_1$ it suffices to prove that $u_1 \in M$. Let consider the sequence (w_n) defined by

$$w_0 := u_1 \quad \text{ and } \ w_n := u_{n+1}, \forall \ n \in \mathbb{N}.$$

Then

$$w_n = u_{n+1} \in Tu_{n+2} = Tw_{n+1} \text{ for all } n \in \mathbb{N}.$$

Hence $u_1 \in M$. Therefore $M \subseteq T(M)$, and hence $T(M) = M$. □

Lemma 2.3. *Let $T \in LR(X)$. If there exists $m \in \mathbb{N}$ such that*

$$ker(T) \cap R(T^m) = ker(T) \cap R(T^{m+k}) \ \ \forall k \ \in \mathbb{N}^*,$$

then

$$C(T) = R^\infty(T).$$

Proof. The first inclusion ($C(T) \subseteq R^\infty(T)$) is evident. For the opposite inclusion, we shall prove that $T(R^\infty(T)) = R^\infty(T)$. We have $T(R^\infty(T)) \subseteq R^\infty(T)$, so that all we need is to show the inverse inclusion. Let $y \in R^\infty(T)$. Then $y \in R^m(T)$, for all $m \in \mathbb{N}$, so there exists $x_k \in X$ such that $y \in T^{m+k}x_k$, for every $k \in \mathbb{N}^*$ and then there exists $\alpha_k \in T^{m+k-1}x_k$ such that $y \in T\alpha_k$. If we set

$$z_k := \alpha_1 - \alpha_k$$

then $z_k \in R(T^m)$ and since $0 \in T(\alpha_1 - \alpha_k) = Tz_k$ we also have $z_k \in ker(T)$.
Thus $z_k \in ker(T) \cap R(T^m) = ker(T) \cap R(T^{m+k})$ and from the equality

$$ker(T) \cap R(T^{m+k}) = ker(T) \cap R(T^{m+k-1})$$

it follows that $z_k \in R(T^{m+k-1})$. This implies that

$$\alpha_1 = z_k + \alpha_k \in R(T^{m+k-1}) \forall\, k \geq 1.$$

So $\alpha_1 \in R^{\infty}(T)$ and then we deduce that

$$y \in T\alpha_1 \subset T(R^{\infty}(T)).$$

So $R^{\infty}(T) \subseteq T(R^{\infty}(T))$. □

As a consequence of Lemma 2.3, we have the following corollary.

Corollary 2.4. *Let $T \in LR(X)$. Suppose that one of the following conditions holds:*

1. $\dim \ker(T) < \infty$;
2. $\ker(T) \subset R(T^n)$, $\forall n \in \mathbb{N}$.

Then $C(T) = R^{\infty}(T)$.

Definition 2.5. *The analytical core of a linear relation T is the set defined by $K(T) = \{x \in X$ that there exist $a > 0$ and a sequence $(u_n) \subset X$ such that $x = u_0$, $u_n \in Tu_{n+1}$ and $dis(u_n, T(0)) \leq a^n dis(x, T(0))$, $\forall n \in \mathbb{N}\}$.*

Remark 2.6. *It's clear that $T(0) \subset K(T)$.*

The following proposition is considered to show some properties of this set.

Proposition 2.7. *Let $T \in CR(X)$.*

1. *$K(T)$ is a subspace of X which is not necessary closed;*
2. *$T(K(T)) = K(T)$;*
3. *$K(T) \subseteq C(T)$.*

Proof.

1. Let $x, y \in K(T)$ and showing that $x + y \in K(T)$. Since x and y are in $K(T)$ so there exist two positive real a and b, and two sequences (u_n) and (v_n) such that :

$$x = u_0 \text{ , } u_n \in Tu_{n+1} \text{ and } dis(u_n, T(0)) \leq a^n dis(x, T(0)) \text{ , } \forall\, n \in \mathbb{N};$$

$$y = v_0 \text{ , } v_n \in Tv_{n+1} \text{ and } dis(v_n, T(0)) \leq b^n dis(y, T(0)) \text{ , } \forall\, n \in \mathbb{N}.$$

If $x+y \in T(0)$, then $x+y \in K(T)$ since $T(0) \subset K(T)$. If not, we consider the sequence $(w_n) = (u_n+v_n)$ then: $x + y = u_0 + v_0 = w_0$ and $w_n = u_n + v_n \in Tu_n + Tv_n = T(u_n + v_n)$, and

$$\begin{aligned} dis(w_n, T(0)) = dis(u_n + v_n, T(0)) &\leq dis(u_n, T(0)) + dis(v_n, T(0)) \\ &\leq a^n dis(x, T(0)) + b^n dis(y, T(0)) \\ &\leq \delta^n \mu\, dis(x + y, T(0)) \end{aligned}$$

with $\delta = \max(a, b)$ and $\mu = \dfrac{dis(x, T(0)) + dis(y, T(0))}{dis(x + y, T(0))}$. So, we show that

$x + y \in K(T)$. Finally, if $x \in K(T)$ and α is a scalar it's easy to verify that $\alpha\, x \in K(T)$. So, we conclude that $K(T)$ is a subspace of X.

2. Let proving the inclusion $K(T) \subset T(K(T))$. Let $x \in K(T)$ then there exist $a > 0$ and $(u_n) \subset X$ such that:
$$x = u_0 \,;\, u_n \in Tu_{n+1} \text{ and } dis(u_n, T(0)) \leq a^n dis(x, T(0))\,,\, \forall n \in \mathbb{N}.$$
So it suffices de prove that $u_1 \in K(T)$. If $u_1 \in T(0)$, then there is nothing to prove. If not, we consider the sequence (w_n) defined by
$$w_0 := u_1 \quad \textit{and} \;\; w_n := u_{n+1}, \;\; w_n = u_{n+1} \in Tu_{n+2} = Tw_{n+1} \quad \forall \;\; n \in \mathbb{N}.$$
Besides, we have
$$\begin{array}{rcl} dis(w_n, T(0)) & = & dis(u_{n+1}, T(0)) \\ & \leq & a^{n+1} dis(x, T(0)) \\ & \leq & b^n \; dis(u_1, T(0)) \end{array}$$
with $b > \max(a, a^2 \dfrac{dis(x, T(0))}{dis(u_1, T(0))})$. Then $u_1 \in K(T)$ which permits us to deduce that $K(T) \subset T(K(T))$.
Moving to the other inclusion. Let x be in $K(T)$. So there exist $a > 0$, $(u_n) \subset X$ such that:
$$x = u_0;\, u_n \in Tu_{n+1} \text{ and } dis(u_n, T(0)) \leq a^n dis(x, T(0))\,,\, \forall n \in \mathbb{N}.$$
Now, let $y \in Tx$. We want to show that $y \in K(T)$. Let consider the sequence (w_n) such that:
$$w_0 := y \quad \textit{and} \;\; w_n := u_{n-1} \in Tu_n = Tw_{n+1}, \forall \;\; n \in \mathbb{N}^*.$$
We have
$$\begin{array}{rcl} dis(w_n, T(0)) & = & dis(u_{n-1}, T(0)) \\ & \leq & a^{n-1} dis(x, T(0)) \\ & \leq & b^n \; dis(u_1, T(0)) \end{array}$$
with $b > \max(a, \dfrac{dis(x, T(0))}{dis(u_1, T(0))})$. Thus $y \in K(T)$ and therefore $T(K(T)) \subset K(T)$.
3. By seeing the second part of this proposition and the Definition 2.1, we can show that $K(T) \subseteq C(T)$. □

Lemma 2.8. *Let $T \in CBR(X)$.*

1. *If F is a closed subspace of X such that $T(F) = F$ then $F \subseteq K(T)$;*
2. *If $C(T)$ is closed then $C(T) = K(T)$.*

Proof.

1. If F is a closed subspace of X and $T(F) = F$ then the restriction $T_0 : F \longrightarrow F$ is an open map (See [6, Theorem III.4.2]). Therefore, by [6, Proposition II.3.2] we see that $\gamma(T_0) > 0$. Now let $x \in F$. So, there exists $u \in F$ such that $x \in Tu$. On the other hand we have
$$dis(u, \ker(T_0)) \leq \frac{1}{\gamma(T_0)} \parallel T_0 u \parallel .$$
Let $\delta > \dfrac{1}{\gamma(T_0)}$. Then there exists $y \in \ker(T) \cap F$ such that
$$\parallel u - y \parallel \leq \delta \; dis(x, T(0)).$$
Take $u_1 = u - y$. We have $x \in Tu_1$ and
$$dis(u_1, T(0)) \leq \parallel u_1 \parallel \leq \delta \; dis(x, T(0)).$$
By the same reason there exists $u_2 \in F$ such that $u_1 \in Tu_2$ and
$$\begin{array}{rcl} dis(u_2, T(0)) & \leq & \delta dis(u_1, T(0)) \\ & \leq & \delta^2 dis(x, T(0)). \end{array}$$
By repeating this process we find a sequence $(u_n) \subset X$ such that :
$$x = u_0 \,;\, u_n \in Tu_{n+1} \text{ and } dis(u_n, T(0)) \leq \delta^n dis(x, T(0))\,,\, \forall n \in \mathbb{N}.$$
This implies that $x \in K(T)$ and therefore $F \subseteq K(T)$.

2. Suppose that $C(T)$ is closed. Since $C(T) = T(C(T))$ then the first part of Lemma 2.8 shows that $C(T) \subseteq K(T)$, and hence, by using the third part of Proposition 2.7, we get $C(T) = K(T)$. □

Theorem 2.9. *Let $T \in CBR(X)$.*

1. *If T is s-regular then $K(T) = R^{\infty}(T)$ and it is closed;*
2. *If T is a quasi-nilpotent operator (that is $\lim\limits_{n\mapsto\infty} \|T^n\|^{\frac{1}{n}} = 0$) then $K(T) = \{0\}$.*

Proof.

1. Through [2, Theorem 21], if T is an s-regular relation, then for all $n \in \mathbb{N}$, T^n is s-regular and so $R(T^n)$ is closed for every $n \in \mathbb{N}$. So, by using Corollary 2.4, we get:
$$C(T) = R^{\infty}(T) = \bigcap_{n\geq 0} R(T^n) \text{ is closed.}$$
Using the second part of Lemma 2.8 we conclude that $K(T)$ coincides with $C(T)$.
2. See [4, Lemma1.4]. □

Now, we will show some properties of the algebraic and analytic core of linear relations which verify the stabilization criteria that is $T^n(0) = T(0),\ \forall n \in \mathbb{N}^*$.

Proposition 2.10. *Let X be a Banach space and let T and $S \in BR(X)$, verifing the stabilization criteria, $T(0) = S(0)$ and $TS = ST$. Then we have:*

1. *$K(TS) \subseteq K(T) \cap K(S)$;*
2. *$C(TS) \subseteq C(T) \cap C(S)$.*

Proof.

1. We want to show that $K(TS) \subseteq K(T)$. Let $x \in K(TS)$. So, there exist $a > 0$ and a sequence $(u_n) \subset X$ such that:
$$x = u_0,\ u_n \in TSu_{n+1};\ \forall n \in \mathbb{N} \textit{ and } dis(u_n, TS(0)) \leq a^n dis(x, TS(0)).$$
Take $y_0 = x$. We have $x \in TSu_1$ so there exists $y_1 \in Su_1$ such that $y_0 \in Ty_1$. $y_1 \in Su_1 \subset S(TS)u_2 = TS^2u_2$ so there exists $y_2 \in S^2u_2$ such that $y_1 \in Ty_2$. By induction we construct a new sequence y_n such that $x = y_0$, $y_n \in Ty_{n+1}$, and $y_n \in S^n u_n$ for all $n \in \mathbb{N}$. Now, we will show that there exists $\gamma > 0$, for which $dis(y_n, T(0)) \leq \gamma^n dis(x, T(0))$. We have
$$\begin{aligned} dis(y_n, T(0)) &= dis(y_n, S(0)) \\ &= dis(y_n, S^n(0)) \\ &= \|S^n u_n\|. \end{aligned}$$
On the other hand, if we take $\alpha \in S(0) = S^n(0) = TS(0)$ then we have $S^n(u_n - \alpha) = S^n u_n - S^n\alpha \subset S^n u_n - S^n(0) = S^n u_n$. Thus
$$\begin{aligned} dis(y_n, T(0)) &= \|S^n(u_n - \alpha)\| \\ &\leq \|S^n\|\|u_n - \alpha\|. \end{aligned}$$
This is true for all $\alpha \in TS(0)$. So
$$\begin{aligned} dis(y_n, T(0)) &\leq \|S^n\| \inf_{\alpha\in TS(0)} \|u_n - \alpha\| \\ &\leq \|S\|^n dis(u_n, TS(0)) \\ &\leq \|S\|^n a^n dis(x, T(0)) \\ &\leq \gamma^n dis(x, T(0)),\ \textit{with } \gamma = a\|S\|. \end{aligned}$$
Therefore, $x \in K(T)$.

2. It is obvious. □

Corollary 2.11. *Let $T \in BR(X)$ which checks the stabilization criteria. Then we have :*

1. $K(T^n) = K(T)$;
2. $C(T^n) = C(T)$.

Proof.

1. From Proposition 2.10 we have $K(T^n) \subseteq K(T)$. So, we need only to prove the reverse inclusion. Let $x \in K(T)$. So there exist $a > 0$ and a sequence $(u_p) \subset X$ such that $x = u_0$, $u_p \in Tu_{p+1}$ and $dis(u_p, T(0)) \leq a^p dis(x, T(0))$. Let v_p be the sequence in X defined by $v_p = u_{pn}$. So we have

$$v_p \in Tu_{pn+1} \subset T^2 u_{pn+2} \subset .... \subset T^n u_{n(p+1)} = T^n v_{p+1}.$$

Besides, we have

$$\begin{aligned} dis(v_p, T^n(0)) = dis(v_p, T(0)) &= dis(u_{pn}, T(0)) \\ &\leq a^{pn} dis(x, T(0)) \\ &\leq (a^n)^p dis(x, T^n(0)). \end{aligned}$$

Then $x \in K(T^n)$ which completes the proof.

2. Arguing as in (1) we can prove this equality. □

Lemma 2.12. *Let A, B, C and D be in $BR(X)$ which verify the stabilization criteria. Suppose that CA=AC, CB=BC, DA=AD, DB=BD, AB=BA, $A(0) = B(0)$, $I \subset AC + BD$ and*

$$D(0) + C(0) \subseteq A(0) \subseteq \ker(D) \cap \ker(C)$$

and let $u, v \in X$ such that $Au = Bv$. Then there exists $w \in X$ for which we have

$$u \in Bw, \ v \in Aw$$

and

$$dis(w, AB(0)) \leq (\|C\| + \|D\|) \max\Big(dis(u, AB(0)), dis(v, AB(0))\Big).$$

Proof. Let prove the existence of $w \in X$ such that $v \in Aw$ and $u \in Bw$. We have $I \subset AC+BD$ so $u \in ACu+BDu = CAu+BDu = CBv+BDu = B(Du+Cv)$ and $v \in ACv+BDv = ACv+DBv = ACv+DAu = A(Du+Cv)$ so there exists $w_1, w_2 \in Du + Cv$ such that $u \in Bw_1$, $v \in Aw_2$. As $w_1, w_2 \in Du + Cv$, then $w_1 - w_2 \in D(0) + C(0) \subseteq A(0)$ so there exists $\alpha \in A(0)$ such that $w_1 = w_2 + \alpha$. Thus, $u \in B(w_2 + \alpha)$ and, since $A(0) \subseteq \ker(A)$, we have $v \in A(w_2 + \alpha)$. Take $w = w_2 + \alpha$, then it is easy to see that

$$u \in Bw \ , \ v \in Aw.$$

Now, let consider the relation:

$$\begin{aligned} D\hat{+}C : X \times X &\to X \\ (x, y) &\mapsto Dx + Cy. \end{aligned}$$

It is obvious that $D\hat{+}C$ is bounded and $\|D\hat{+}C\| \leq \|D\| + \|C\|$.
Taken $(\alpha_1, \alpha_2) \in \ker(D) \times \ker(C)$ so, since

$$w_2 \in Du + Cv = D\hat{+}C(u, v) = D\hat{+}C(u + \alpha_1, v + \alpha_2)$$

we can write

$$\begin{aligned} dis(w_2, D\hat{+}C(0)) &= \Big\|D\hat{+}C(u + \alpha_1, v + \alpha_2)\Big\| \\ &\leq \|D\hat{+}C\| \max\Big(\|u + \alpha_1\|, \|v + \alpha_2\|\Big). \end{aligned}$$

This is true for all $\alpha_1 \in \ker(D)$ and $\alpha_2 \in \ker(C)$. So, in particular for $\alpha_1 \in A(0)$ and $\alpha_2 \in B(0)$, then by passage to the inf we obtain

$$\begin{aligned} dis(w_2, D\hat{+}C(0)) &\leq \|D\hat{+}C\| \max\Big(\inf_{\alpha_1 \in A(0)} \|u+\alpha_1\|, \inf_{\alpha_2 \in B(0)} \|v+\alpha_2\|\Big) \\ &\leq (\|D\| + \|C\|) \max\Big(dis(u, A(0)), dis(v, B(0))\Big). \end{aligned}$$

Finally we can conclude that we have

$$\begin{aligned} dis(w, A(0)) = dis(w_2, A(0)) &\leq dis(w_2, D\hat{+}C(0)) \\ &\leq (\|D\| + \|C\|) \max\Big(dis(u, A(0)), dis(v, B(0))\Big) \\ &\leq (\|D\| + \|C\|) \max\Big(dis(u, AB(0)), dis(v, AB(0))\Big). \end{aligned}$$

Which ends the proof. $\square$

Theorem 2.13. *Let A,B,C and D be in BR(X). Under the conditions of Lemma 2.12 we have :*

1. $K(AB) = K(A) \cap K(B)$;
2. $C(AB) = C(A) \cap C(B)$.

Proof.

1. By using Proposition 2.10, we have $K(AB) \subseteq K(A) \cap K(B)$, so we need only to prove the reverse inclusion. Let $x \in K(A) \cap K(B)$, so there exist two sequences $(x_{i,0}) \subset X$, $(x_{0,j}) \subset X$ and a positive integer δ such that:

$$x = x_{0,0},\ x_{i,0} \in Ax_{i+1,0} \text{ and } dis(x_{i,0}, A(0)) \leq \delta^i dis(x, A(0)).$$

$$x = x_{0,0},\ x_{0,j} \in Bx_{0,j+1} \text{ and } dis(x_{0,j}, B(0)) \leq \delta^j dis(x, B(0)).$$

$x_{0,0} \in Ax_{1,0} \cap Bx_{0,1}$ so $Ax_{1,0} \cap Bx_{0,1} \neq \emptyset$ and then $Ax_{1,0} = Bx_{0,1}$. By Lemma 2.12, there exists $x_{1,1} \in X$ satisfying $x_{0,1} \in Ax_{1,1}$, $x_{1,0} \in Bx_{1,1}$ and

$$dis(x_{1,1}, AB(0)) \leq (\|C\| + \|D\|) \max\Big(dis(x_{0,1}, AB(0)), dis(x_{1,0}, AB(0))\Big).$$

We have $x_{0,1} \in Ax_{1,1} \cap Bx_{0,2}$. Thus, by Lemma 2.12, there exists $x_{1,2} \in X$ such that $x_{0,2} \in Ax_{1,2}$, $x_{1,1} \in Bx_{1,2}$ and

$$dis(x_{1,2}, AB(0)) \leq (\|C\| + \|D\|) \max\Big(dis(x_{0,2}, AB(0)), dis(x_{1,1}, AB(0))\Big).$$

Again we have $x_{1,0} \in Ax_{2,0} \cap Bx_{1,1}$. Thus, there exists $x_{2,1} \in X$ such that $x_{1,1} \in Ax_{2,1}$, $x_{2,0} \in Bx_{2,1}$ and

$$dis(x_{2,1}, AB(0)) \leq (\|C\| + \|D\|) \max\Big(dis(x_{1,1}, AB(0)), dis(x_{2,0}, AB(0))\Big).$$

Also we have $x_{1,1} \in Ax_{2,1} \cap Bx_{1,2}$. Consequently, by Lemma 2.12, there exists $x_{2,2} \in X$ such that $x_{1,2} \in Ax_{2,2}$, $x_{2,1} \in Bx_{2,2}$ and

$$dis(x_{2,2}, AB(0)) \leq (\|C\| + \|D\|) \max\Big(dis(x_{1,2}, AB(0)), dis(x_{2,1}, AB(0))\Big).$$

By repeating this process, we construct a sequence $(x_{i,j})_{i,j \in \mathbb{N}}$ such that

$$x_{i-1,j} \in Ax_{i,j}, x_{i,j-1} \in Bx_{i,j}, \ \ and$$

$$dis(x_{i,j}, AB(0)) \leq \Big(\|D\| + \|C\|\Big) \max\Big(dis(x_{i-1,j}, A(0)), dis(x_{i,j-1}, B(0))\Big).$$

By induction we get

$$dis(x_{i,j}, AB(0)) \leq \Big[\delta \ \max\Big(\|D\| + \|C\|, 1\Big)\Big]^{i+j} dis(x, AB(0)).$$

Now, let denote by $y_i = x_{i,i}$ so we obtain $x = y_0$, $y_i \in ABy_{i+1}$ and

$$\begin{aligned} dis(y_i, AB(0)) &= dis(x_{i,i}, AB(0)) \\ &\leq \Big[\delta \ \max\Big(\|D\| + \|C\|, 1\Big)\Big]^{2i} dis(x, AB(0)) \\ &\leq \gamma^i dis(x, AB(0)) \end{aligned}$$

with $\gamma = \Big[\delta \ \max\Big(\|D\| + \|C\|, 1\Big)\Big]^2$. This provides that $x \in K(AB)$ so, the inclusion $K(A) \cap K(B) \subseteq K(AB)$ is done and we get the equality desired.

2. Using Lemma 2.12 and arguing as in (1) we get the result. □

3. Quasi-nilpotent part of linear relations

The main of this section is to define the quasi-nilpotent part of a linear relation and to give some of its properties.

Definition 3.1. *Let T be a linear relation. Then, we define its quasi-nilpotent part by* $H_0(T) = \Big\{x \in X,$ *that there exists a sequence* $(x_n) \subset X$ *such that* $x = x_0,\ \ x_{n+1} \in Tx_n$ *and* $\lim\limits_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = 0\Big\}$.

Lemma 3.2. *Let* $T \in BR(X)$. *We have*

1. $H_0(T)$ *is a subspace of X which is not necessary closed;*
2. $\forall j \in \mathbb{N}, \ \ \ker(T^j) \subset H_0(T)$;
3. $x \in H_0(T) \Leftrightarrow Tx \cap H_0(T) \neq \emptyset$.

Proof.

1. Obvious.
2. This proof is given by induction. The case $j = 0$ is obvious. Assume that until the order j we have $\ker(T^j) \subset H_0(T)$, and let prove that $\ker(T^{j+1}) \subset H_0(T)$. Let $x \in \ker(T^{j+1})$. So, $0 \in T^{j+1}x = T^j(Tx)$ and this implies that there exists $y \in Tx$ such that $0 \in T^j y$ and so $y \in \ker(T^j) \subset H_0(T)$. Thus, there exits a sequence $(y_n) \subset X$ such that $y_{n+1} \in Ty_n$ and $\lim\limits_{n\mapsto\infty} \|y_n\|^{\frac{1}{n}} = 0$. Now take the sequence x_n defined by $x_0 = x$ and $x_n = y_{n-1}$ for all $n \in \mathbb{N}^*$. We have

$$\lim_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = \lim_{n\mapsto\infty} \|y_{n-1}\|^{\frac{1}{n}} = 0$$

so $x \in H_0(T)$.
3. Let prove the first inclusion. Let $x \in H_0(T)$. So, there exists a sequence (x_n) such that $x = x_0, x_{n+1} \in Tx_n$, and $\lim\limits_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = 0$. We have $x_1 \in Tx$. Let consider the sequence (y_n) such that $x_1 = y_0$ and $y_n = x_{n+1}$. We get $y_n \in Ty_{n-1}$ $\forall n \geq 1$ and $\lim\limits_{n\mapsto\infty} \|y_n\|^{\frac{1}{n}} = 0$. So $x_1 \in H_0(T)$ thus $Tx \cap H_0(T) \neq \emptyset$.
Let prove the reverse inclusion. Suppose that $Tx \cap H_0(T) \neq \emptyset$. Let $y \in Tx \cap H_0(T)$. So, there exists a sequence (y_n) such that

$$y = y_0\ ,\ y_{n+1} \in Ty_n \ \textit{and} \ \|y_n\|^{\frac{1}{n}} \to 0 \ \textit{when} \ n \mapsto \infty.$$

Let consider the new sequence (x_n) defined by $x = x_0$, $x_n = y_{n-1}$ $\forall$ $n \geq 1$. So, we have $x_{n+1} = y_n \in Ty_{n-1} = Tx_n$ and $\lim\limits_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = \lim\limits_{n\mapsto\infty} \|y_{n-1}\|^{\frac{1}{n}} = 0$. Thus $x \in H_0(T)$. □

Proposition 3.3. *Let $T \in CBR(X)$ be such that $\rho(T) \neq \emptyset$. Then we have :*

1. $H_0(T) \subseteq K(T^*)^\top$ *and* $K(T) \subseteq H_0(T^*)^\top$;
2. *If T is s-regular then we have*

$$\overline{N^\infty(T)} = \overline{H_0(T)} = K(T^*)^\top \text{ and } K(T) = H_0(T^*)^\top;$$

3. *If T is s-regular then* $\overline{H_0(T)} \subseteq K(T)$.

Proof.

1. Consider an element $x \in H_0(T)$ and $f \in K(T^*)$. From the definition of $K(T^*)$ we know that there exists $\gamma > 0$ and a sequence (g_n), such that

$$g_0 = f \ , \ g_n = T^* g_{n+1} \text{ and } \|g_n\| \leq \gamma^n \|f\|$$

for every $n \in \mathbb{N}$. These equalities entail that $f = (T^*)^n g_n$ so that, by [? , Lemma 2.3]

$$f(x) = (T^*)^n g_n(x) = g_n(T^n x).$$

From the definition of $H_0(T)$ we know that there exists (x_n) such that

$$x = x_0 \ , \ x_{n+1} \in Tx_n \text{ and } \lim_{n \mapsto \infty} \|x_n\|^{\frac{1}{n}} = 0.$$

We remark that $x_n \in T^n x$, so we get

$$\begin{aligned} |f(x)| = |g_n(T^n x)| &\leq \|g_n\| \|T^n x\| \\ &\leq \gamma^n \|f\| dis(x_n, T^n(0)) \\ &\leq \gamma^n \|f\| \|x_n\|. \end{aligned}$$

Hence by taking the n-th root in this inequality we conclude that $f(x) = 0$ and therefore $H_0(T) \subseteq K(T^*)^\top$.

In a similar way we can prove the inclusion $K(T) \subseteq H_0(T^*)^\top$. Consider an element $x \in K(T)$ and $f \in H_0(T^*)$. From the definition of $K(T)$ and $H_0(T^*)$ we know that there exist $\delta > 0$, a sequence $(x_n) \subset X$ and a sequence $(g_n) \subset D(T^*)$ such that

$$x = x_0 \ , \ x_n \in Tx_{n+1} \text{ and } dis(x_n, T(0)) \leq \delta^n dis(x, T(0))$$

$$f = g_0 \ , \ g_{n+1} = T^* g_n \text{ and } \lim_{n \mapsto \infty} \|g_n\|^{\frac{1}{n}} = 0.$$

For every $\alpha \in T(0) = D(T^*)^\top$ we have $|f(x)| = \|(T^*)^n f x_n\| = \|g_n x_n\| = \|g_n(x_n - \alpha)\|$. This is true for all $\alpha \in T(0)$ so, by passage to the inf, we get

$$\begin{aligned} |f(x)| = \|g_n(x_n - \alpha)\| &\leq \|g_n\| \inf_{\alpha \in T(0)} \|x_n - \alpha\| \\ &\leq \|g_n\| dis(x_n, T(0)) \\ &\leq \delta^n \|g_n\| dis(x, T(0)) \\ &\leq \delta^n \|g_n\| \|x\|. \end{aligned}$$

By taking the n-th root in the last inequality we find that $f(x) = 0$ and therefore $K(T) \subseteq H_0(T^*)^\top$.

2. Notice that from [2, Theorem 13] we have T^* is s-regular and from [2, Theorem 21] T^n is s-regular so we have $R((T^*)^n)$ is closed. Using Lemma 3.2, the part (1) of this proposition and that $K(T^*)^\top$ is closed we get

$$\overline{N^\infty(T)} \subseteq \overline{H_0(T)} \subseteq \overline{K(T^*)^\top} = K(T^*)^\top.$$

Now, we need only to show that $K(T^*)^\top \subseteq \overline{N^\infty(T)}$. We have $\ker(T^n) \subseteq N^\infty(T)\ \ \forall n \in \mathbb{N}$ so $N^\infty(T)^\perp \subseteq \ker(T^n)^\perp = R((T^*)^n)$. Hence we get $N^\infty(T)^\perp \subseteq R^\infty(T^*) = K(T^*)$ and then it is easy to show that

$$K(T^*)^\top \subseteq N^\infty(T)^\perp)^\top = \overline{N^\infty}(T)$$

which ends the proof of the first equality.
From (1) we have $K(T) \subseteq H_0(T^*)^\top$. On the other hand we have $N^\infty(T^*) \subseteq K(T^*)$ so $K(T^*)^\top \subseteq N^\infty(T^*)^\top$ and $N^\infty(T^*)^\top \subseteq \ker((T^*)^n)^\top = R(T^n)\ \forall n \in \mathbb{N}$. Since T is an s-regular linear relation then by Theorem 2.9 we conclude that $N^\infty(T^*)^\top \subseteq R^\infty(T) = K(T)$. Finally we obtain

$$N^\infty(T^*)^\top \subseteq K(T) \subseteq H_0(T^*)^\top = N^\infty(T^*)^\top$$

and therefore $K(T) = H_0(T^*)^\top$.

3. The semi-regularity of T entails that $N^\infty(T) \subseteq R^\infty(T) = K(T)$ where the last equality follows from Theorem 2.9. Consequently from part (2) it follows that

$$\overline{H_0(T)} = \overline{N^\infty(T)} \subseteq \overline{K(T)} = K(T)$$

since $K(T)$ is closed by Theorem 2.9. □

Theorem 3.4. *Let $T \in CBR(X)$ be such that $\rho(T) \neq \emptyset$. If T is an s-regular linear relation then $H_0(T) \subseteq T(H_0(T))$.*

Proof. Let $x \in H_0(T)$. So, there exists a sequence (x_n) such that $x = x_0, x_{n+1} \in Tx_n$, and $\lim\limits_{n\mapsto\infty} \|x_n\|^{\frac{1}{n}} = 0$. By the third part of Proposition 3.3 we get that $x \in K(T) = T(K(T))$. So, there exists $y \in K(T)$ for which $x \in Ty$. Now let (y_n) be the sequence defined by: $y_0 = y$; $y_n = x_{n-1}\ \forall\ n \geq 1$. We have, $\forall\ n \in \mathbb{N}$, $y_{n+1} \in Ty_n$ and $\lim\limits_{n\mapsto\infty} \|y_n\|^{\frac{1}{n}} = \lim\limits_{n\mapsto\infty} \|x_{n-1}\|^{\frac{1}{n}} = 0$ which confirms that $y \in H_0(T)$. Therefore $H_0(T) \subseteq T(H_0(T))$. □

Corollary 3.5. *Let $T \in CBR(X)$ be such that $\rho(T) \neq \emptyset$. If T is an s-regular linear relation then:*

1. $\overline{H_0(T)} \subseteq T(\overline{H_0(T)})$;
2. $Tx \cap \overline{H_0(T)} \neq \emptyset \Rightarrow x \in \overline{H_0(T)}$;
3. *If moreover we have $\overline{H_0(T)} + T(0)$ is closed, then*

$$x \in \overline{H_0(T)} \Rightarrow Tx \cap \overline{H_0(T)} \neq \emptyset.$$

Proof.

1. Let $y \in \overline{H_0(T)}$. There exists a sequence $y_n \in H_0(T)$ such that $y_n \to y$ and from Theorem 3.4, there exits a sequence $x_n \in H_0(T)$ such that $y_n \in Tx_n$. Since $\overline{H_0(T)} \subset R(T)$, then there exists $x \in X$ such that $y \in Tx$. Since T and $R(T)$ are closed then $\gamma(T) > 0$ and $\|T^{-1}\| = \frac{1}{\gamma(T)} < \infty$. On another hand, since $x_n \in T^{-1}y_n$ then $T^{-1}y_n = x_n + T^{-1}(0)$, so $dis(x_n - x, T^{-1}(0)) = \|T^{-1}y_n - T^{-1}y\| \leq \|T^{-1}\|\|y_n - y\| \to 0$. So we conclude that there exists $\alpha_n \subset \ker(T)$ such that $\|x_n - x - \alpha_n\| \to 0$. Thus $x = \lim_{+\infty} x_n - \alpha_n$ and $x_n - \alpha_n \in H_0(T)$. Therefore $x \in \overline{H_0(T)}$ and so $\overline{H_0(T)} \subseteq T(\overline{H_0(T)})$.
2. Let $y \in Tx \cap \overline{H_0(T)}$. So, from the first part of this theorem , $y \in T(\overline{H_0(T)})$ which implies that there exists $z \in \overline{H_0(T)}$ such that $y \in Tz$. Then $x - z \in \ker(T) \subset \overline{H_0(T)}$ which provides that $x \in \overline{H_0(T)}$.
3. In order to prove this implication, we claim that $\overline{H_0(T)} + T(0) = \overline{H_0(T) + T(0)}$. In fact, since $\overline{H_0(T)} \subseteq \overline{H_0(T) + T(0)}$ and $T(0) \subseteq \overline{H_0(T) + T(0)}$, we have $\overline{H_0(T)} + T(0) \subseteq \overline{H_0(T) + T(0)}$. In the other hand we have $H_0(T) + T(0) \subseteq \overline{H_0(T)} + T(0)$ so $\overline{H_0(T) + T(0)} \subseteq \overline{\overline{H_0(T)} + T(0)} = \overline{H_0(T)} + T(0)$.
Now, let $x \in \overline{H_0(T)}$. So, there exists $(x_n) \in H_0(T)$ such that $x_n \to x$. the third part of Lemma 3.2 involve that $Tx_n \cap H_0(T) \neq \emptyset$. Let $y_n \in Tx_n \cap H_0(T)$ and $y \in Tx$. So, $y_n - y \in T(x_n - x)$. Then, $dis(y_n - y, T(0)) = \|Tx_n - Tx\| \leq \|T\|\|x_n - x\| \to 0$. So, there exists $\alpha_n \in T(0)$ such that $\|y_n - \alpha_n - y\| \to 0$. Thus $y_n - \alpha_n \to y$. So, $y \in \overline{H_0(T) + T(0)} = \overline{H_0(T)} + T(0)$. Then, there exist $\bar{y} \in \overline{H_0(T)}$ and $\alpha_1 \in T(0)$ such that $y = \bar{y} + \alpha_1$. We conclude that $\bar{y} = y - \alpha_1 \in Tx - T(0) = T(x)$ which implies that $\bar{y} \in Tx \cap \overline{H_0(T)}$. So, $Tx \cap \overline{H_0(T)} \neq \emptyset$ which ends the proof. □

4. Generalized Kato linear relation

For $T, S \in LR(X)$, the linear relations $T \hat{+} S$ and $T \oplus S$ are defined by

$$G(T \hat{+} S) := \{(x+u, y+v); (x,y) \in G(T), (u,v) \in G(S)\},$$

the last sum is direct when $G(T) \cap G(S) = \{(0,0)\}$ and we write $T \oplus S$. If M is a subspace of X, we denote by T_M the linear relation in M defined by

$$G(T_M) := G(T) \cap (M \times M).$$

Note that $R(T_M) \subset M$ and $D(T_M) \subset M$ by definition.

Definition 4.1. *[3, Definition 2.1] Let M and N be two subspaces of a Banach space X such that $X = M \oplus N$ (that is $X = M + N$ and $M \cap N = 0$). We say that $T \in LR(X)$ is completely reduced by the pair (M, N), denoted as $(M, N) \in Red(T)$, if $T = T_M \oplus T_N$.*

Definition 4.2. *[3, Definition 3.1] Let $T \in CBR(X)$. T is said to be a Generalized Kato linear relation, and we write $T \in GKR(X)$, if there exist tow closed subspaces M and N of X that verify the following conditions:*

1. *$X = M \oplus N$.*
2. *$T = T_M \oplus T_N$ where*
   - *$T_M := G(T) \cap (M \times M)$ and T_M is an s-regular linear relation.*
   - *$T_N := G(T) \cap (N \times N)$ and T_N is a quasi-nilpotent operator.*

The pair (M, N) is called generalized Kato decomposition of T and we write (M, N) is a GKD of T. If we assume in the above definition that T_N is a nilpotent operator, that is $(T_N)^d = 0$ for some $d \in \mathbb{N}$, then T is said to be of Kato type linear relation of degree d.

Proposition 4.3. *If (M, N) is a GKD of T, then $(N^\perp, M^\perp)$ is a GKD of T^*.*

Proof. See Proof of [3, Theorem 3.2]. □

The next two lemmas are considered to study the analytic core of a generalized Kato linear relation and a Kato type linear relation.

Lemma 4.4. *Let M, N be two closed subspaces such that $X = M \oplus N$. Let P denotes the bounded projection from X onto N with kernel M. So, for all $x \in X$, we have:*

$$\|P(x)\| \leq \|P\|\, d(x, M).$$

Proof. Since P is continuous, then, for all $y \in N$ and $z \in M$ we have: $\|P(y-z)\| \leq \|P\|\|y-z\|$. So, $\|y-z\| \geq \dfrac{1}{\|P\|}$ $\forall\, y \in N,\ z \in M$ *with* $\|y\| = 1$.
Now, let $x \in X$. So, there exist $x_1 \in N$ and $x_2 \in M$ such that $x = x_1 + x_2$ and $P(x) = x_1$.

- If $P(x) = 0$, then $x = x_2 \in M$ so $d(x, M) = 0$ and we get $\|P(x)\| \leq \|P\|\, d(x, M)$.
- If $P(x) \neq 0$, then for all $y \in M$ we have:

$$\left\| \frac{P(x)}{\|P(x)\|} - y \right\| \geq \frac{1}{\|P\|}.$$

So,

$$\Big\| P(x) - y\|P(x)\| \Big\| \geq \frac{\|P(x)\|}{\|P\|}$$

and then

$$\|P(x)\| \leq \|P\|\Big\|x - x_2 - y\|P(x)\|\Big\|.$$

This is true for all $y \in M$, so we obtain

$$\|P(x)\| \leq \|P\|\|x - t\|, \quad \forall t \in M.$$

Thus

$$\|P(x)\| \leq \|P\| \inf_{t \in M} \|x - t\|.$$

Finally

$$\|P(x)\| \leq \|P\|\, d(x, M), \quad \forall x \in X.$$

□

Lemma 4.5. *Let $T \in LR(X)$ and let M and N be two closed subspaces such that $(M, N) \in Red(T)$. If $x \in H_0(T)$ and $x = \alpha + \beta$, $\alpha \in M$ and $\beta \in N$, then $\alpha \in H_0(T_M)$.*

Proof. Let $x \in H_0(T)$. So, there exists a sequence $(x_n) \subset X$ such that $x_0 = x$, $x_{n+1} \in Tx_n$ and $\|x_n\|^{\frac{1}{n}} \to 0$. Let take the continuous projection P from X onto M with kernel N. We have $\alpha = Px$ and we consider the sequence $(Px_n) \subset M$. We have, $\alpha = Px = Px_0$ and $Px_{n+1} \in PTx_n = P(T_M x_{nM} + T_N x_{nN}) = T_M x_{nM} = T_M Px_n$. Furthermore, $\|Px_n\|^{\frac{1}{n}} \leq \|P\|^{\frac{1}{n}}\|x_n\|^{\frac{1}{n}} \to 0$. Thus, $\alpha \in H_0(T_M)$. □

Lemma 4.6. *Let $T \in LR(X)$ and let M and N be two closed subspaces such that $(M, N) \in Red(T)$ and $T(M) \subset M$. Then :*

1. $H_0(T_M) = H_0(T) \cap M$;
2. $\overline{H_0(T)} \cap M = \overline{H_0(T_M)}$.

Proof.

1. Let $x \in H_0(T_M)$. So there exists a sequence $(x_n) \subset M$ such that $x = x_0$, $x_{n+1} \in T_M(x_n) \subset T(x_n)$ and $\lim\limits_{n \mapsto \infty} \|x_n\|^{\frac{1}{n}} = 0$. Since the sequence $(x_n) \subset M \subset X$ then $x \in H_0(T)$ and so $x \in H_0(T) \cap M$. Conversely, let $x \in H_0(T) \cap M$. So, there exists a sequence $(x_n) \subset X$ such that $x = x_0$, $x_{n+1} \in T(x_n)$ and $\lim\limits_{n \mapsto \infty} \|x_n\|^{\frac{1}{n}} = 0$. Since $x \in M$ and $T(M) \subseteq M$ we conclude that the sequence $(x_n) \subset M$ so that $x \in H_0(T_M)$.
2. We have $\overline{H_0(T_M)} = \overline{H_0(T) \cap M} \subseteq \overline{H_0(T)} \cap M$. To prove the inverse inclusion, let x be in $\overline{H_0(T)} \cap M$. So there exists a sequence $x_n \in H_0(T)$ such that $x_n \to x$. We have $x = Px = \lim\limits_{n \mapsto +\infty} Px_n$ where P denotes the bounded projection from X onto M with kernel N. By using Lemma 4.5 we get $\big(P(x_n)\big) \subset H_0(T_M)$. Thus $x \in \overline{H_0(T_M)}$ and we get the desired equality. □

In the next theorem we bring together some properties of the Generalized Kato linear relations.

Theorem 4.7. *Let $T \in GKR(X)$ with a GKD (M, N). We have :*

1. *$K(T)$ is closed and*
$$K(T) = K(T_M) = R^{\infty}(T_M);$$
2. $\ker(T_M) = K(T) \cap \ker T$;
3. $N \subseteq H_0(T)$;
4. $Tx \cap \overline{H_0(T)} \neq \emptyset \Rightarrow x \in \overline{H_0(T)}$;
5. *$R(T) + H_0(T)$ is closed and*
$$R(T) + H_0(T) = R(T) + \overline{H_0(T)}.$$

Proof.

1. We will first verify that $K(T) \subset M$. Let $x \in K(T)$, so there exist $a > 0$ and a sequence (u_n) such that $x = u_0$, $u_n \in Tu_{n+1}$ and $dis(u_n, T(0)) \leq a^n dis(x, T(0))$. Clearly $x \in T^n u_n$, for every $n \in \mathbb{N}$. Besides, from the decomposition $X = M \oplus N$ we know that $x = y + z$, $u_n = y_n + z_n$, with $y, y_n \in M$ and $z, z_n \in N$. Then $x \in T^n u_n = T^n_M y_n + T^n_N z_n$, hence, $y \in T^n_M y_n$ and $z = T^n_N z_n$, for all $n \in \mathbb{N}$. Let P denotes the projection from X onto N with kernel M. Since, by assumption, T_N is quasi-nilpotent, therefore, for $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|T^n_N\|^{\frac{1}{n}} < \varepsilon$ for all $n > n_0$. Now, by using Lemma 4.4, we obtain:
$$\begin{aligned}\|z\| = \|T^n_N z_n\| = \|T^n_N P u_n\| &\leq \|T^n_N\|\|Pu_n\| \\ &\leq \varepsilon^n \|P\|\, d(u_n, M) \\ &\leq \varepsilon^n \|P\|\, d(u_n, T(0)) \\ &\leq \varepsilon^n a^n \|P\|\, d(x, T(0)),\ \forall\, n > n_0.\end{aligned}$$
The last term converges to 0, since ε is arbitrary. Hence $z = 0$ and this implies that $x = y \in M$. It remains to prove that $K(T) \subset K(T_M)$. For this let $x \in K(T)$. So, there exist $b > 0$ and a sequence (x_n) such that $x = x_0$, $x_n \in Tx_{n+1}$ and $dis(x_n, T(0)) \leq b^n dis(x, T(0))$. Now let take the sequence (Px_n) where P denotes the projection from X onto M with kernel N. So, $P(x_0) = Px = x$, $Px_n \in P(Tx_{n+1}) = P[T_M Px_{n+1} + T_N(I - P)(x_{n+1})] = T_M(Px_{n+1})$. Besides we have:
$$\begin{aligned}dis(Px_n, T_M(0)) &= dis(x_n - (I-P)x_n, T(0)) \\ &\leq dis(x_n, T(0)) + dis((I-P)x_n, T(0)) \\ &\leq b^n\, dis(x, T(0)) + \|(I-P)x_n\| \\ &\leq b^n\, dis(x, T(0)) + \|I-P\|\, dis(x_n, M) \\ &\leq [b^n + b^n\|I-P\|]\, dis(x, T(0)).\end{aligned}$$
So $x \in K(T_M)$ and we get $K(T) \subset K(T_M)$. The last assertion is a consequence of Theorem 2.9, since the restriction T_M is s-regular.
2. From [2, Lemma 9] and part (1), we remark that
$$\ker(T_M) \subseteq R^\infty(T_M) = K(T_M) = K(T).$$
This implies that
$$K(T) \cap \ker(T) = K(T) \cap M \cap \ker(T) = K(T) \cap \ker(T_M) = \ker(T_M).$$
3. Let $x \in N$. So, $Tx = T_M(0) + T_N x$ thus $T_N x \in Tx$. Also, by seeing that $T^n = T^n_M \oplus T^n_N$ and $T^n_N x \in N$ we deduce that $T^{n+1}_N x \in T(T^n_N x)$. Let consider the sequence (x_n) defined by $x = x_0$; $x_n = T^n_N x$. So we have $x_{n+1} = T^{n+1}_N x \in T(T^n_N x)$ and $\|x_n\|^{\frac{1}{n}} = \|T^n_N x\|^{\frac{1}{n}} \mapsto 0$, since T_N is a quasi-nilpotent operator. Then $x \in H_0(T)$ and consequently $N \subseteq H_0(T)$.
4. Let x be in X such that $Tx \cap \overline{H_0(T)} \neq \emptyset$. Since $X = M \oplus N$ then there exist $u \in M$ and $v \in N$ such that $x = u + v$. Let $y \in Tx \cap \overline{H_0(T)}$ then $y \in T_M u + T_N v$ and $y = w + k$ with $w \in \overline{H_0(T_M)}$ and $k \in N$. So, by using the fact that $\overline{H_0(T_M)} \subset R(T_M)$, we deduce that $w \in T_M u$ and $k = T_N v$. By using Corollary 3.5 we prove that there exits $z \in \overline{H_0(T_M)}$ such that $u - z \in \ker(T_M) \subset \ker(T) \subset \overline{H_0(T)}$. Then $u \in \overline{H_0(T)}$ which implies that $x \in \overline{H_0(T)}$.
5. Let start by showing that $R(T) + H_0(T) = T(M) + N$. It's clear that $T(M) + N \subseteq R(T) + H_0(T)$, so we need only to prove the reverse inclusion. Since $N \subseteq H_0(T)$, then we get $H_0(T) = H_0(T) \cap M + N$. Using that $H_0(T) \cap M = H_0(T_M)$ and $H_0(T_M) \subseteq R(T_M)$ we get, $H_0(T) \subseteq T(M) + N$. On the other hand we have $R(T) = T_M(M) + T_N(N) \subseteq T(M) + N$. Consequently, $R(T) + H_0(T) \subseteq T(M) + N$.
Now let prove that $T(M) + N$ is closed. We have $X = M \oplus N$ so X is isomorph to $M \times N$ and so $T(M) + N$ is isomorph to $T(M) \times N$. Since T_M is s-regular we get $T(M) = R(T_M)$ is closed and thus $T(M) \times N$ is closed which confirm that $T(M) + N$ is closed. In other hand we have $R(T) + H_0(T) \subseteq R(T) + \overline{H_0(T)} \subseteq \overline{R(T) + H_0(T)} = R(T) + H_0(T)$. □

Corollary 4.8. *Assume that T is a linear relation of Kato type of degree d and let (M, N) be the associated decomposition. Then :*

1. $K(T) = R^{\infty}(T)$;
2. $\ker(T_M) = \ker T \cap R^{\infty}(T) = K(T) \cap R(T^n)$ *for all naturel* $n \geq d$;
3. *For every* $n \geq d$, $R(T) + \ker(T^n) = T(M) \oplus N$ *and closed.*

Proof.

1. We have $(T_N)^d = 0$, so for $n \geq d$ we have $R(T^n) = T^n(M) \oplus T^n(N) = T^n(M)$ and consequently, by using Theorem 2.9, $R^{\infty}(T) = R^{\infty}(T_M) = K(T)$.
2. Let $n \geq d$ then $R(T^n) = T^n(M)$ so by using part (2) of Theorem 4.7, we show that:
$$\ker(T_M) = \ker(T) \cap K(T) \subseteq \ker(T) \cap T^n(M) \subseteq \ker(T) \cap M = \ker(T_M).$$
Hence, for all $n \geq d$, we have $\ker(T_M) = \ker(T) \cap R(T^n)$.
3. Clearly, if $n \geq d$ we have, $N \subset \ker(T^n)$, so $T(M) \oplus N \subset R(T) + \ker T^n$.
On the other hand, by seeing that if $n \geq d$ then
$$\ker T^n = \ker(T_M)^n \oplus \ker(T_N)^n = \ker(T_M)^n \oplus N$$
and from the semi-regularity of T_M it follows that $\ker T^n \subset T(M) \oplus N$ and since $R(T) = T_M(M) \oplus T_N(N) \subseteq T(M) \oplus N$ we get the inverse inclusion and hence $R(T) + \ker(T^n) = T(M) \oplus N$ which is closed. □

5. Perturbation of a Generalized Kato linear relation

In order to give a perturbation result of a generalized Kato linear relation we need the following lemma.

Lemma 5.1. *Let $T \in GKR(X)$. Then there exists $\alpha > 0$ such that for all invertible operator $S \in B(X)$ which satisfy $TS = ST$ and $\|S\| < \alpha$ we have :*

1. $S(K(T)) = K(T)$;
2. $(T - S)(K(T)) = K(T)$;
3. $\overline{H_0(T)} \subseteq R(T - S)$;
4. $S(\overline{H_0(T)}) \subset \overline{H_0(T)}$;
5. $\ker(T - S) \subseteq K(T)$.

Proof.

1. Let prove that $S(K(T)) \subseteq K(T)$. We have S commutes with T so
$$S(K(T)) = S(T(K(T))) = T(S(K(T))).$$
Since $S(K(T))$ is closed , then by Lemma 2.8 we get $S(K(T)) \subseteq K(T)$. Now if we replace S by S^{-1} we get $S^{-1}(K(T)) \subseteq K(T)$ so $S(K(T)) = K(T)$.
2. Let $T_0 = T_{K(T)}$ and $S_0 = S_{K(T)}$. We remark that T_0 is surjective, closed and open. From [5, Corollary 1.4.3] there exists $\alpha > 0$ such that $T_0 - S_0$ is surjective when $\|S_0\| < \alpha$. So, if $\|S\| < \alpha$, then $(T - S)(K(T)) = (T_0 - S_0)(K(T)) = K(T)$.
3. Its clear that $N \subseteq \overline{H_0(T)}$ and so $\overline{H_0(T)} = \overline{H_0(T)} \cap (M \oplus N) \subseteq K(T) \oplus N$. From (2) we have $K(T) \subseteq R(T - S)$. So, it suffices to prove that $N \subseteq R(T - S)$. Let $x \in N \subset H_0(T)$. So, there exists a sequence $(x_n) \subset X$ such that $x = x_0$, $x_{n+1} \in Tx_n$ *and* $\|x_n\|^{\frac{1}{n}} \to 0$ $n \mapsto \infty$. By applying the Cauchy criteria, $\sum_{n \geq 0} S^{-n-1} x_n$ is

convergent since $\|S^{-n-1}x_n\|^{\frac{1}{n}} \leq \|(S^{-1})^{n+1}\|^{\frac{1}{n}}\|x_n\|^{\frac{1}{n}} \to 0 \; n \mapsto \infty$.
Take $y = -\sum_{n\geq 0} S^{-n-1}x_n$, thus, we have $x \in (T-S)y$. In fact,

$$\begin{aligned}
Q_{(T-S)}(T-S)y &= Q_{(T-S)}(T-S)\Big[-\sum_{n\geq 0} S^{-n-1}x_n\Big] \\
&= (Q_T T - Q_T S)\Big[-\sum_{n\geq 0} S^{-n-1}x_n\Big] \\
&= \sum_{n\geq 0} Q_T S^{-n}x_n - \sum_{n\geq 0} Q_T T S^{-n-1}x_n \\
&= \sum_{n\geq 0} Q_T S^{-n}x_n - \sum_{n\geq 0} Q_T S^{-n-1}Tx_n \\
&= \sum_{n\geq 0} Q_T S^{-n}x_n - \sum_{n\geq 0} Q_T S^{-n-1}[x_{n+1} + T(0)] \\
&= \sum_{n\geq 0} Q_T S^{-n}x_n - \sum_{n\geq 0} Q_T [S^{-n-1}x_{n+1} + T(0)] \\
&= \sum_{n\geq 0} Q_T S^{-n}x_n - \sum_{n\geq 0} Q_T S^{-n-1}x_{n+1} \\
&= Q_T x_0 + \sum_{n\geq 0} Q_T S^{-n}x_n - \sum_{n\geq 0} Q_T S^{-n}x_n \\
&= Q_T x_0 = Q_T x = Q_{T-S}x.
\end{aligned}$$

Thus $Q_{(T-S)}(T-S)y = Q_{(T-S)}x$ which implies that $x \in (T-S)y$ and therefore $N \subseteq R(T-S)$. Finally we have

$$\overline{H_0(T)} \subseteq K(T) \oplus N \subseteq R(T-S).$$

4. We start by proving that $S(H_0(T)) \subset H_0(T)$. Let $x \in H_0(T)$. Then, there exists a sequence (x_n) such that $x = x_0$, $x_{n+1} \in Tx_n$ and $\|x_n\|^{\frac{1}{n}} \to 0$. We consider the sequence (Sx_n). We have $Sx = Sx_0$, $Sx_{n+1} \in TSx_n$ and $\|Sx_n\|^{\frac{1}{n}} \to 0$. So, $Sx \in H_0(T)$. Let take $x \in \overline{H_0(T)}$ then there exists $(v_n) \subset H_0(T)$ such that $v_n \to x$. So, $Sx = \lim_{n\mapsto +\infty} Sv_n \in \overline{H_0(T)}$.
5. $\ker(T-S) \subset K(T)$. Indeed, let $x \in \ker(T-S)$. Then $Sx \in Tx$. Therefore, $x \in TS^{-1}x$. Take, $u_1 = S^{-1}x$. We have, $x \in Tu_1$, $u_1 \in \ker(T-S)$ and $u_1 \in TS^{-2}x$. So, by induction we construct a sequence (u_n) such that

$$x = u_0,\; u_n \in Tu_{n+1}\;, u_n = S^{-n}x \;\; \forall\, n \geq 0.$$

We have

$$\begin{aligned}
dis(u_n, T(0)) &= dis(S^{-n}x, S^{-n}T(0)) \\
&= \inf_{\gamma\in T(0)} \|S^{-n}x - S^{-n}\gamma\| \\
&\leq \|S^{-1}\|^n dis(x, T(0)).
\end{aligned}$$

So, $x \in K(T)$ and the claim is proved. □

Now, we give the main result of this section.

Theorem 5.2. *Let $T \in GKR(X)$. Then there exists $\gamma > 0$ such that for all $0 < |\lambda| < \gamma$ we have $T - \lambda I$ is an s-regular linear relation.*

Proof. Let α be the real given in Lemma 5.1. So, by using parts (2) and (5) of Lemma 5.1, for all $0 < |\lambda| < \alpha$ we have for all $n \in \mathbb{N}$,

$$\ker(T - \lambda I) \subseteq R(T - \lambda I)^n.$$

Now, let take the application defined by $\hat{T}: \hat{X} \to \hat{X}$; $\hat{x} \mapsto \hat{T}\hat{x} = \widehat{Tx}$ with $\hat{X} = X \setminus \overline{H_0(T)}$. Using the forth part of Theorem 4.7, we get $\hat{T}$ is injective. Also we have $\hat{T}$ is bounded. In fact,

$$\begin{aligned}
\|\widehat{Tx}\| &= \|\hat{y}\| \ \forall\ y \in Tx \\
&= dis(y, \overline{H_0(T)}) \\
&= \inf_{z \in \overline{H_0(T)}} \|y - z\| \\
&\leq \inf_{z \in T(\overline{H_0(T)})} \|y - z\| \Big(\textit{since}\ T(\overline{H_0(T)}) \subset \overline{H_0(T)} \Big) \\
&\leq \inf_{\alpha \in T(0)} \inf_{z \in T(\overline{H_0(T)})} \|y - z - \alpha\| \Big(\textit{since}\ T(0) \subset T(\overline{H_0(T)}) \Big) \\
&\leq \inf_{t \in \overline{H_0(T)}} \|T(x - t)\| \\
&\leq \|T\| \inf_{t \in \overline{H_0(T)}} \|x - t\| \\
&\leq \|T\| \|\hat{x}\|.
\end{aligned}$$

In an other hand $R(\hat{T}) = \Big(R(T) + \overline{H_0(T)}\Big) / \overline{H_0(T)}$ which is closed since by the fifth part of Theorem 4.7 $R(T) + \overline{H_0(T)}$ is closed. So $\hat{T}$ is an s-regular linear relation. Now, according to [2, Theorem 23], for $|\lambda| < \nu$, we have $\hat{T} - \lambda\hat{I}$ is an s-regular linear relation and consequently $R(\hat{T} - \lambda\hat{I}) = \Big(R(T - \lambda I) + \overline{H_0(T)}\Big) / \overline{H_0(T)}$ is closed. According to Lemma 5.1, if $0 < |\lambda| < \alpha$ then $\overline{H_0(T)} \subseteq R(T - \lambda I)$ so if we take $\delta = \min(\alpha, \nu)$ then $R(T - S) = R(T - S) + \overline{H_0(T)}$ which is closed for $0 < |\lambda| < \delta$. Thus $T - \lambda I$ is s-regular. □

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