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The Classification of Closed Subspaces of Noncommutative L_2 Space Associated with a Factor of Type I

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Abstract. In this article, we discuss the relationship between the projections of a factor $\mathscr{M}$ of type I and the closed subspaces of the noncommutative L_2 space $L_2(\mathscr{M})$. Moreover, we consider the classification of these closed subspaces.

1. Introduction

The notation and terminology in this paper agrees, for the most part, with that in Jones[4] and Xu[13]. Here are a few specific items that are worthy of attention.

Let $\mathscr{H}$ be a Hilbert space with an inner product $\langle , \rangle$, denote by $B(\mathscr{H})$ the set of all bounded linear mappings from $\mathscr{H}$ to itself. If $\mathscr{M}$ is a strongly(weakly) closed $*$-subalgebra of $B(\mathscr{H})$ containing the unit I, $\mathscr{M}$ is called a von Neumann algebra. If $\mathscr{B}$ is a subset of $B(\mathscr{H})$, we define its commutant as $\mathscr{B}' = \{x \in B(\mathscr{H}) : xy = yx \text{ for all } y \in \mathscr{B}\}$, and the double commutant $\mathscr{B}'' = (\mathscr{B}')'$. Let $\mathscr{M}$ be a $*$-algebra on a Hilbert space $\mathscr{H}$ with $I \in \mathscr{M}$, then $\mathscr{M}$ is a von Neumann algebra if and only if $\mathscr{M} = \mathscr{M}''$.

We define the spectrum of x to be the set $\sigma(x) = \{\lambda \in \mathbb{C} | \lambda I - x \text{ is not invertible}\}$. An element $x \in \mathscr{M}$ is positive (denoted by $x \geqslant \theta$ where θ is the zero element in $\mathscr{M}$) if $x = x^*$ and $\sigma(x) \subset \mathbb{R}^+$, set $\mathscr{M}_+ = \{x \in \mathscr{M} | x \geqslant \theta\}$. If an element $p \in \mathscr{M}$ satisfies $p = p^* = p^2$, p is called a projection. We denote by $\mathscr{P}(\mathscr{M})$ the set of projections in $\mathscr{M}$. Two projections e and f in a von Neumann algebra $\mathscr{M}$ are said to be equivalent relative to $\mathscr{M}$, denoted as $e \sim_{\mathscr{M}} f$(written $e \sim f$ for convenience), if there is a partial isometry $u \in \mathscr{M}$ such that $u^*u = e$ and $uu^* = f$. We say $e \leq f$ if $e(\mathscr{H}) \subset f(\mathscr{H})$ and $e \precsim f$ if there is a projection $f_1 \in \mathscr{M}$ with $f_1 \leqslant f$ and $e \sim f_1$. A projection $e \in \mathscr{M}$ is finite if the only projection f in $\mathscr{M}$ such that $f \leq e$ and $f \sim e$ is $f = e$ and infinite if there is an $f \sim e$ with $f \lneqq e$.

A factor on the Hilbert space $\mathscr{H}$ is a von Neumann algebra $\mathscr{M}$ on $\mathscr{H}$ such that $\mathscr{M} \cap \mathscr{M}' = \mathbb{C}I$. Murray and von Neumann showed in [9] that if $\mathscr{M}$ is a factor there is a unique "dimension function" $d : \mathscr{P}(\mathscr{M}) \to [0, \infty]$ subject to

1. $d(\theta) = 0$;
2. $d(\sum\limits_{i=1}^{\infty} e_i) = \sum\limits_{i=1}^{\infty} d(e_i)$ if $e_i \perp e_j$ for $i \neq j$,

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3. $d(e) = d(f)$ if $e \sim f$.

It follows that $d(e) = d(f) \Rightarrow e \sim f$. A factor $\mathscr{M}$ is said to be of type I if the range of d is $\{1, 2, \cdots, n\}$ with $n = \infty$ possible – of type I_n if $n < \infty$ – of type I_∞ if $n = \infty$. It is fairly easy to prove that if $\mathscr{M}$ is of type I it is like $B(\mathscr{H}) \otimes id$ on $\mathscr{H} \otimes \mathscr{K}$.

Definition 1.1. *Let $\mathscr{M}$ be a von Neumann algebra. A trace on $\mathscr{M}$ is a mapping $\tau : \mathscr{M}_+ \to [0, \infty]$ satisfying:*

1. *for $x, y \in \mathscr{M}_+$, $\lambda \in \mathbb{R}_+$, $\tau(x + \lambda y) = \tau(x) + \lambda\tau(y)$;*
2. *for $x \in \mathscr{M}$, $\tau(x^*x) = \tau(xx^*)$.*

In addition, a trace τ is said to be normal if $\sup\limits_{\lambda} \tau(x_\lambda) = \tau(\sup\limits_{\lambda} x_\lambda)$ for any bounded monotonic increasing net $\{x_\lambda\}$ in $\mathscr{M}_+$; finite if $\tau(1) < \infty$; semi-finite if for any $x \in \mathscr{M}_+$, there is a $y \in \mathscr{M}_+$ such that $y \leqslant x$ and $\tau(y) < \infty$; faithful if for $x \in \mathscr{M}_+$, $\tau(x) = 0 \Rightarrow x = \theta$.

In the next section, unless stated in particular, $\mathscr{M}$ will always denote a von Neumann algebra on $\mathscr{H}$. If there is a normal faithful semi-finite trace τ on $\mathscr{M}$, we call $(\mathscr{M}, \tau)$ a noncommutative measure space.

For $x \in B(\mathscr{H})$, let $|x| = (x^*x)^{\frac{1}{2}}$, there is a unique partial isometry u from $(\ker x)^\perp$ onto $\overline{R(x)}$ such that $x = u|x|$. In addition, $u^*u = P_{(\ker x)^\perp}$ and $uu^* = P_{\overline{R(x)}}$. Let $r(x) = u^*u(l(x) = uu^*)$, then $r(x)(l(x))$ is called the right (left) support of x. If $x = x^*$, then $r(x) = l(x)$, this common projection is called the support of x and denoted by $s(x)$.

Definition 1.2. *Let $S_+(\mathscr{M}) = \{x \in \mathscr{M}_+ : \tau(s(x)) < \infty\}$ and $S(\mathscr{M})$ be the linear span of $S_+(\mathscr{M})$. Usually, we use S_+ and S to represent $S_+(\mathscr{M})$ and $S(\mathscr{M})$ respectively.*

An operator $x \in \mathscr{M}$ belongs to S if and only if there is an $e \in \mathscr{P}(\mathscr{M})$ satisfying $\tau(e) < \infty$ such that $exe = x$. $x \in S$ implies $|x|^2 \in S_+$, and so $\tau(|x|^2) < \infty$. Moreover, S is a strongly dense ideal of $\mathscr{M}$, and $x \in S$ implies $x^* \in S$.

Now let

$$\|x\|_2 = [\tau(|x|^2)]^{\frac{1}{2}}, \ x \in S.$$

Then $\|\cdot\|_2$ is a norm on S. We denote the completion of $(S, \|\cdot\|_2)$ by $L_2(\mathscr{M}, \tau)$ (shorthand for $L_2(\mathscr{M})$), it is a Hilbert space, and we call it noncommutative L_2 space.

In this paper, we classify the closed spaces of a noncommutative L_2 space associated with a factor of type I. If $e \in \mathscr{P}(\mathscr{M})$, $e\mathscr{M}e$ is a von Neumann subalgebra of $\mathscr{M}$, then $L_2(e\mathscr{M}e)$ is a closed subspace of $L_2(\mathscr{M})$. However, for any closed subspace of $L_2(\mathscr{M})$, is there a projection $e \in \mathscr{P}(\mathscr{M})$ such that this subspace can be expressed by $L_2(e\mathscr{M}e)$?

2. Main result

In this section, we study the relationship between the projections of a factor $\mathscr{M}$ of type I and the closed subspaces of the noncommutative L_2 space $L_2(\mathscr{M})$.

Lemma 2.1. *Let $(\mathscr{M}, \tau)$ be a noncommutative measure space, $e, f \in \mathscr{P}(\mathscr{M})$. If $e \sim f$, then $e\mathscr{M}e$ is $*$-isomorphism to $f\mathscr{M}f$.*

Proof. Since $e \sim f$, there is a partial isometry $u \in \mathscr{M}$ such that $u^*u = e$ and $uu^* = f$. Let

$$\begin{aligned} \varphi : & e\mathscr{M}e \to f\mathscr{M}f \\ & exe \mapsto fuxu^*f. \end{aligned}$$

We claim that φ is a $*$-isomorphism and left its proof to readers. □

Lemma 2.2. *If $(\mathscr{M}, \tau)$ and $(\mathscr{N}, \upsilon)$ are noncommutative measure spaces and $\pi : \mathscr{M} \to \mathscr{N}$ is an isomorphism such that $\upsilon \circ \pi = \tau$, then π maps $S(\mathscr{M})$ onto $S(\mathscr{N})$.*

Proof. For $x \in S(\mathscr{M})$, there is an $e \in \mathscr{P}(\mathscr{M})$ satisfying $\tau(e) < \infty$ such that $exe = x$. Since π is an isomorphism,

$$\pi(e)\pi(x)\pi(e) = \pi(exe) = \pi(x).$$

Moreover, $\pi(e)$ is a projection in $\mathscr{N}$ and

$$\upsilon(\pi(e)) = \tau(e) < \infty.$$

Therefore, $x \in S(\mathscr{N})$, and π maps $S(\mathscr{M})$ to $S(\mathscr{N})$.

For any $y \in S(\mathscr{N}) \subseteq \mathscr{N}$, there exists an $f \in \mathscr{P}(\mathscr{N})$ satisfying $\upsilon(f) < \infty$ such that $fyf = y$ and there is an $x \in \mathscr{M}$ such that $\pi(x) = y$. Then

$$\pi(\pi^{-1}(f)x\pi^{-1}(f)) = f\pi(x)f = fyf = y = \pi(x).$$

Since π is an injection, $\pi^{-1}(f)x\pi^{-1}(f) = x$. Besides, $\pi^{-1}(f)$ is a projection and $\tau(\pi^{-1}(f)) = \upsilon(f) < \infty$. Consequently, $x \in S(\mathscr{M})$, so π maps $S(\mathscr{M})$ onto $S(\mathscr{N})$. □

Proposition 2.3. *Let $(\mathscr{M}, \tau)$ be a noncommutative measure space, $e, f \in \mathscr{P}(\mathscr{M})$. If $e \sim f$, then $L_2(e\mathscr{M}e)$ is unitary isomorphic to $L_2(f\mathscr{M}f)$.*

Proof. Let φ be the isomorphism from $e\mathscr{M}e$ to $f\mathscr{M}f$, then φ maps $S(e\mathscr{M}e)$ onto $S(f\mathscr{M}f)$. For any $x \in \mathscr{M}$,

$$\begin{aligned}&\|\varphi(exe)\|_2^2 = \|fuxu^*f\|_2^2 = \tau(fux^*u^*f \cdot fuxu^*f)\\ =&\tau(uex^*exeu^*) = \tau(ex^*exeu^*u) = \tau(ex^*exe) = \|exe\|_2^2.\end{aligned}$$

Since $e\mathscr{M}e$ is $\|\cdot\|_2$−norm dense in $L_2(e\mathscr{M}e)$, $L_2(e\mathscr{M}e)$ is unitary isomorphic to $L_2(f\mathscr{M}f)$. □

Proposition 2.3 shows that $e \sim f \Rightarrow L_2(e\mathscr{M}e) \cong L_2(f\mathscr{M}f)$. Thus, it is natural to ask whether the inverse proposition is true.

As we have known, for $e \in \mathscr{P}(\mathscr{M})$, $L_2(e\mathscr{M}e) \subseteq L_2(\mathscr{M})$. However, the closed subspaces of $L_2(\mathscr{M})$ can not always expressed as $L_2(e\mathscr{M}e)$ for any $e \in \mathscr{P}(\mathscr{M})$. Indeed, let $\mathscr{M} = M_n(\mathbb{C})$ and τ be a normalized trace on $\mathscr{M}$, that is $\tau(I) = 1$. Since all the norms are equivalent on a finite dimensional normed linear space, $L_2(\mathscr{M}) = M_n(\mathbb{C})$. There are only $n + 1$ projections in $\mathscr{M}$ up to projection equivalent, but $L_2(\mathscr{M})$ has at least $n^2 + 1$ closed subspaces up to isomorphism. Thus, there must be some closed subspaces of $L_2(\mathscr{M})$ which can not be expressed as $L_2(e\mathscr{M}e)$ for any $e \in \mathscr{P}(\mathscr{M})$. Then we will ask that under which conditions the closed subspace of $L_2(\mathscr{M})$ can be expressed as $L_2(e\mathscr{M}e)$ for some $e \in \mathscr{P}(\mathscr{M})$.

In this paper, we answer these questions under the case of factor of type I_n and factor of type I_∞. To answer the two questions, we need the following definition.

Definition 2.4. *Let $(\mathscr{M}, \tau)$ be a noncommutative measure space and E be the orthogonal projection from $L_2(\mathscr{M})$ onto its closed subspace $\mathcal{L}$. E is called a projection with bimodule property if $E(y_1xy_2) = y_1E(x)y_2$ and $E(x^*) = E(x)^*$ for any $x \in L_2(\mathscr{M})$, $y_1, y_2 \in \mathcal{L}$.*

First of all, we discuss the above two questions in the case of type I_n-factor.

Lemma 2.5. [11] *If $\mathcal{A}$ is a finite dimensional C^*-algebra, then $\mathcal{A}$ can be decomposed into the direct sum $\mathcal{A} = \sum_{k=1}^{n} \bigoplus \mathcal{A}_k$, where each $\mathcal{A}_k$ is isomorphic to the algebra of $n_k \times n_k$ matrices.*

Theorem 2.6. *Let $\mathscr{M} = M_n(\mathbb{C})$ and τ be a normalized trace on $\mathscr{M}$. If $\mathcal{L}$ is a closed subspace of $L_2(\mathscr{M})$, then $E : L_2(\mathscr{M}) \to \mathcal{L}$ is a projection with bimodule property if and only if $\mathcal{L} = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \oplus O_{l\times l}(1 \leq k \leq n, l \geq 0$ and $\sum_{i=1}^{k} n_i + l = n)$. Furthermore, the decomposition is unique in the sense that if $\mathcal{L}_1 = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$ and $\mathcal{L}_2 = M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_t}(\mathbb{C})$, then $\mathcal{L}_1 \cong \mathcal{L}_2 \Leftrightarrow s = t$ and there is a $\sigma \in S_t$ such that $n_i = m_{\sigma(i)}(1 \leq i \leq s)$, where S_t is the permutation group on $\{1, 2, \cdots, t\}$.*

Proof. If $\mathcal{L} = \begin{bmatrix} B_{11} & & & \\ & \ddots & & \\ & & B_{kk} & \\ & & & 0 \end{bmatrix}$ where $B_{ii} \in M_{n_i}(\mathbb{C})$, 0 is a null matrix of order l, $1 \le k \le n, l \ge 0$ and $\sum_{i=1}^{k} n_i + l = n$. Then for $A \in M_n(\mathbb{C})$, A can be written as the form $[A_{ij}]$ where $A_{ij} \in M_{n_i \times n_j}(\mathbb{C})(1 \le i, j \le k+1,\ n_{k+1} = l)$ such that

$$E(A) = \begin{bmatrix} A_{11} & & & \\ & \ddots & & \\ & & A_{kk} & \\ & & & 0 \end{bmatrix}.$$

For any $A = [A_{ij}] \in M_n(\mathbb{C})$, $B \in \mathcal{L}$,

$$E(A^*) = E(A)^*,$$

$$\begin{aligned} E(AB) &= E\left(\begin{bmatrix} A_{11} & \cdots & A_{1k} & A_{1k+1} \\ \vdots & & \vdots & \vdots \\ A_{k1} & \cdots & A_{kk} & A_{kk+1} \\ A_{k+11} & \cdots & A_{k+1k} & A_{k+1k+1} \end{bmatrix} \begin{bmatrix} B_{11} & & & \\ & \ddots & & \\ & & B_{kk} & \\ & & & 0 \end{bmatrix}\right) \\ &= E\left(\begin{bmatrix} A_{11}B_{11} & A_{12}B_{22} & \cdots & A_{1k}B_{kk} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{k1}B_{11} & A_{k2}B_{22} & \cdots & A_{kk}B_{kk} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right) \\ &= E(A)B. \end{aligned}$$

Similarly, $E(BA) = BE(A)$. Thus, E is a projection with bimodule property.

Conversely, if $E : L_2(\mathscr{M}) \to \mathcal{L}$ is a projection with bimodule property, then $(\mathcal{L}, \|\cdot\|)$ is a C*-algebra. Indeed, for any $x, y \in \mathcal{L}$, $E(x) = x, E(y) = y$, then

$$xy = E(x)y = E(xy) \in \mathcal{L},$$

$$x^* = E(x)^* = E(x^*) \in \mathcal{L},$$

$$\|xy\| \le \|x\| \cdot \|y\|,$$

$$\|x^*x\| = \|x\|^2.$$

Thus $(\mathcal{L}, \|\cdot\|)$ is a C*-algebra with $\dim \mathcal{L} < \infty$. Since all the norms are equivalent on a finite dimensional space, $\mathcal{L}$ can be written as $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \oplus O_{l\times l}$ for some $1 \le k \le n$ and $\sum_{i=1}^{k} n_i + l = n$.

Without loss of generality, we can assume $n_1 \le n_2 \le \cdots \le n_s$ and $m_1 \le m_2 \le \cdots \le m_t$.

$'' \Leftarrow ''$ If $s = t$ and $m_i = n_i (1 \le i \le s)$, then $\mathcal{L}_1 = \mathcal{L}_2$.

$'' \Rightarrow ''$ If $\mathcal{L}_1 \cong \mathcal{L}_2$, we can show $s = t$ and $m_i = n_i (1 \le i \le s)$ by induction. □

Definition 2.7. *If $\mathcal{L} = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$, where $n_1 \le n_2 \le \cdots \le n_s$ and $\sum_{i=1}^{k} n_i \le n$, we call $\mathcal{L}$ a type $(n_1, n_2, \cdots, n_s)$ subspace of $M_n(\mathbb{C})$.*

The answers of the two questions in the case of type I_n-factor is in the following.

Corollary 2.8. *Suppose $\mathscr{M}$ is a factor of type I_m, that is $\mathscr{M} \cong M_m(\mathbb{C}) \otimes id_{\mathscr{H}}$ where $\mathscr{H}$ is a finite dimensional Hilbert space.*

1. *For* $e, f \in \mathscr{P}(\mathscr{M})$, $L_2(e\mathscr{M}e) \cong L_2(f\mathscr{M}f) \Leftrightarrow e \sim f$;
2. *For a closed subspace* $\mathcal{L}$ *of* $L_2(\mathscr{M})$, *there is an* $e \in \mathscr{P}(\mathscr{M})$*(that is* $e = e_0 \otimes id_{\mathscr{H}}$ *where* $e_0 \in \mathscr{P}(M_m(\mathbb{C}))$*) such that* $\mathcal{L} \cong L_2(e\mathscr{M}e) \Leftrightarrow \mathcal{L} = \mathcal{L}_0 \otimes I_{\mathscr{H}}$ *where* $\mathcal{L}_0$ *is a type* $(m_1)(m_1 \leq m)$ *subspace of* $M_m(\mathbb{C})$.

In particular, if $\mathscr{M} = M_n(\mathbb{C})$, *then for* $e, f \in \mathscr{P}(\mathscr{M})$, $L_2(e\mathscr{M}e) \cong L_2(f\mathscr{M}f) \Leftrightarrow e \sim f$; *for a closed subspace* $\mathcal{L}$ *of* $L_2(\mathscr{M})$, *there is an* $e \in \mathscr{P}(\mathscr{M})$ *such that* $\mathcal{L} \cong L_2(e\mathscr{M}e) \Leftrightarrow \mathcal{L}$ *is a type* $(n_1)(n_1 \leq n)$ *subspace of* $L_2(\mathscr{M})$.

We now describe an example to indicate how to calculate the number of pairwise inequivalent subspaces of type $(n_1, n_2, \cdots, n_s)$.

Example 2.9. *Let* $\mathcal{L}$ *be a type* $(n_1, n_2, \cdots, n_s)$ *nonzero subspace of* $M_n(\mathbb{C})$. *We can show the following conclusion by induction.*

1. *If* $s = 1$, $\mathcal{L}$ *is isomorphic to* $L_2(e\mathscr{M}e)$ *for some* $e \in \mathscr{P}(\mathscr{M})$, *the number of such subspaces is* n *in a sense of isometric* $*$*-isomorphism.*
2. *If* $s = 2$, $\mathcal{L}$ *is isomorphic to* $L_2(e\mathscr{M}e) \oplus L_2(f\mathscr{M}f)$ *for some* $e, f \in \mathscr{P}(\mathscr{M})$, *the number of such subspaces up to isometric* $*$*-isomorphism is* $\begin{cases} k^2, & \text{if } n = 2k; \\ k(k+1), & \text{if } n = 2k+1. \end{cases}$
3. *If* $s = 3$, *for a fixed* m, *the number of type* (m, n_2, n_3) *subspaces up to isometric* $*$*-isomorphism is* $\begin{cases} k^2, & \text{if } n = 2k + 3m - 2; \\ k(k+1), & \text{if } n = 2k + 3m - 1. \end{cases}$
4. *For any* $s > 3$, *and fixed* $n_1, n_2, \cdots, n_{s-2}$, *the number of type* $(n_1, n_2, \cdots, n_{s-2}, n_{s-1}, n_s)$ *subspaces up to isometric* $*$*-isomorphism is* $\begin{cases} k^2, & \text{if } n = 2k + 3n_{s-2} + \sum_{i=1}^{s-3} n_i - 2; \\ k(k+1), & \text{if } n = 2k + 3n_{s-2} + \sum_{i=1}^{s-3} n_i - 1. \end{cases}$

Proof. 1. If $\mathcal{L} = M_{n_1}(\mathbb{C})$, let $e = e_{11} + \cdots + e_{n_1 n_1}$ where e_{ii} is a matrix unit. Then $\mathcal{L} \cong L_2(e\mathscr{M}e)$ and n_1 may be $1, 2, \cdots, n$. Therefore, the number of type (n_1) subspaces is n.

If $\mathcal{L} = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C})$, let $e = e_{11} + \cdots + e_{n_1 n_1}$, $f = e_{(n_1+1)(n_1+1)} + \cdots + e_{(n_1+n_2)(n_1+n_2)}$ where e_{ii} is a matrix unit. Then $\mathcal{L} \cong L_2(e\mathscr{M}e) \oplus L_2(f\mathscr{M}f)$ and (n_1, n_2) may be

$$\begin{array}{lllllll} (1,1), & (1,2), & \cdots, & (1,n-5), & (1,n-4), & (1,n-3), & (1,n-2), \quad (1,n-1), \\ (2,2), & (2,3), & \cdots, & (2,n-4), & (2,n-3), & (2,n-2), & \\ (3,3), & (3,4), & \cdots, & (3,n-3), & & & \\ \vdots & & & & & & \end{array}$$

(k,k) (if $n = 2k$) or $(k,k), (k,k+1)$ (if $n = 2k+1$).
Therefore, the number of type (n_1, n_2) subspaces is

$$\begin{cases} (n-1) + (n-3) + \cdots + 1 = k^2, & \text{if } n = 2k; \\ (n-1) + (n-3) + \cdots + 2 = k(k+1), & \text{if } n = 2k+1. \end{cases}$$

2. If $s = 3$, $n_1 = m$, $n = 2k + 3m - 2$, $\mathcal{L}$ may be type

$$\begin{array}{cccc} (m,m,m), & (m,m,m+1), & \cdots, & (m,m,n-2m), \\ (m,m+1,m+1), & (m,m+1,m+2), & \cdots, & (m,m+1,n-2m-1), \\ \vdots & & & \\ (m,m+k-1,m+k-1) & & & \end{array}$$

If $s = 3$, $n_1 = m$, $n = 2k + 3m - 1$, $\mathcal{L}$ may be type

$$\begin{array}{cccc} (m,m,m), & (m,m,m+1), & \cdots, & (m,m,n-2m), \\ (m,m+1,m+1), & (m,m+1,m+2), & \cdots, & (m,m+1,n-2m-1), \\ \vdots & & & \\ (m,m+k-1,m+k-1), & (m,m+k-1,m+k) & & \end{array}$$

Therefore, the number of type (m, n_2, n_3) subspaces is

$$\begin{cases} (n-3m+1)+(n-3m-1)+\cdots+1 = k^2, & \text{if } n = 2k+3\times m-2; \\ (n-3m+1)+(n-3m-1)+\cdots+2 = k(k+1), & \text{if } n = 2k+3\times m-1. \end{cases}$$

3. If $s > 3, n = 2k + 3n_{s-2} + \sum_{i=1}^{s-3} n_i - 2$ for fixed $n_1, n_2, \cdots, n_{s-2}$, $\mathcal{L}$ may be type

$$\begin{array}{ccc} (n_1,\cdots,n_{s-2},n_{s-2},n_{s-2}), & \cdots, & (n_1,\cdots,n_{s-2},n_{s-2},n-\sum_{i=1}^{s-2} n_i - n_{s-2}), \\ (n_1,\cdots,n_{s-2},n_{s-2},n_{s-2}+1), & \cdots, & (n_1,\cdots,n_{s-2},n_{s-2}+1,n-\sum_{i=1}^{s-2} n_i - n_{s-2}-1), \\ \vdots & & \end{array}$$

$(n_1, \cdots, n_{s-2}, n_{s-2} + k - 1, n_{s-2} + k - 1)$.
If $s > 3, n = 2k + 3n_{s-2} + \sum_{i=1}^{s-3} n_i - 1$ for fixed $n_1, n_2, \cdots, n_{s-2}$, $\mathcal{L}$ may be type

$$\begin{array}{ccc} (n_1,\cdots,n_{s-2},n_{s-2},n_{s-2}), & \cdots, & (n_1,\cdots,n_{s-2},n_{s-2},n-\sum_{i=1}^{s-2} n_i - n_{s-2}), \\ (n_1,\cdots,n_{s-2},n_{s-2},n_{s-2}+1), & \cdots, & (n_1,\cdots,n_{s-2},n_{s-2}+1,n-\sum_{i=1}^{s-2} n_i - n_{s-2}-1), \\ \vdots & & \end{array}$$

$(n_1, \cdots, n_{s-2}, n_{s-2} + k - 1, n_{s-2} + k - 1), (n_1, \cdots, n_{s-2}, n_{s-2} + k - 1, n_{s-2} + k)$.
Therefore, the number of type (m, n_2, n_3) subspaces is

$$\begin{cases} (n-\sum_{i=1}^{s-2} n_i - n_{s-2} - n_{s-2} + 1)+\cdots+1 = k^2, & \text{if } n = 2k+3n_{s-2}+\sum_{i=1}^{s-3} n_i - 2; \\ (n-\sum_{i=1}^{s-2} n_i - n_{s-2} - n_{s-2} + 1)+\cdots+2 = k(k+1), & \text{if } n = 2k+3n_{s-2}+\sum_{i=1}^{s-3} n_i - 1. \end{cases}$$

□

Next, we discuss the case of type I_∞-factor.

Suppose that $\mathscr{M} = B(\mathscr{H})$ where $\mathscr{H}$ is a separable infinite dimensional Hilbert space and $\{\xi_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$. We define the trace τ on $\mathscr{M}$ to be $\tau(x) = \sum_{i=1}^{\infty} \langle x\xi_i, \xi_i \rangle$. Then $S(\mathscr{M}) = F(\mathscr{H})$, which is the class of all finite rank operators and $L_2(\mathscr{M}) = L^2(\mathscr{H})$, which is the class of all Hilbert-Schmidt operators on $\mathscr{H}$. Moreover, for $e \in \mathscr{P}(\mathscr{M})$, $L_2(e\mathscr{M}e) = L^2(e\mathscr{H})$.

In the following, we list several basic properties of $L^2(\mathscr{H})$ that we shall use, often without comment, in the sequel.

1. $\|x\| \le \|x\|_2, \forall x \in L^2(\mathscr{H})$;
2. $L^2(\mathscr{H})$ is a self-adjoint ideal of $B(\mathscr{H})$ and a normed $*$-algebra;
3. $\|\xi \otimes \eta\|_2 = \|\xi\|\|\eta\| = \|\xi \otimes \eta\|$, $\forall \xi, \eta \in \mathscr{H}$;
4. $F(\mathscr{H})$ is dense in $L^2(\mathscr{H})$ in the norm $\|\cdot\|_2$ and $F(\mathscr{H})$ is linearly spanned by the rank-one projections;

Lemma 2.10. *Let $\mathscr{M} = B(\mathscr{H})$ with $\mathscr{H}$ a separable infinite dimensional Hilbert space, $e, f \in \mathscr{P}(\mathscr{H})$. If $L^2(e\mathscr{H}) \cong L^2(f\mathscr{H})$, then $e \sim f$.*

Proof. If $L^2(e\mathscr{H}) \cong L^2(f\mathscr{H})$, since $\mathscr{H}$ is separable, $\dim(L^2(e\mathscr{H})) = \dim(L^2(f\mathscr{H})) = n$ with $n = \infty$ possible. Let

$$d : \mathscr{P}(\mathscr{M}) \to [0, \infty]$$

be the dimension function. Then $d(e) = \dim(e\mathscr{H})$.

1. If $n = \infty$, then e, f are infinite projections. Therefore, $d(e) = d(f) = \infty$, that is $e \sim f$.
2. If $n < \infty$, then $\dim(e\mathscr{H}) = \dim(f\mathscr{H}) < \infty$. Therefore $d(e) = d(f) = \dim(e\mathscr{H})$, thus $e \sim f$.

□

Theorem 2.11. *Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space, $e_i, f_i \in \mathscr{P}(\mathscr{H})$. If $\mathcal{L}_1 = \sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$ with $e_i \perp e_j$ for $i \neq j$, $\mathcal{L}_2 = \sum_{j=1}^{m} \oplus L^2(f_j\mathscr{H})$ with $f_i \perp f_j$ for $i \neq j$. Then $\mathcal{L}_1 \cong \mathcal{L}_2 \Leftrightarrow m = n$ and there is a $\sigma \in S_n$ such that $e_i \sim f_{\sigma(i)} (1 \leq i \leq n)$ with $m, n = \infty$ possible.*

Proof. We may suppose $n = \infty$, $e_i \precsim e_{i+1}$, $f_j \precsim f_{i+1}$.
"$\Leftarrow$" If $m = n = \infty$ and $e_i \sim f_i$, then $L^2(e_i\mathscr{H}) \cong L^2(f_i\mathscr{H})$. Let φ_i be the isomorphic mapping from $L^2(e_i\mathscr{H})$ onto $L^2(f_i\mathscr{H})$ and $\varphi = \sum_{i=1}^{\infty} \oplus\varphi_i$, then φ is an isomorphism from $\mathcal{L}_1$ onto $\mathcal{L}_2$.
"$\Rightarrow$" If φ is the isomorphic mapping from $\mathcal{L}_1$ onto $\mathcal{L}_2$. Let $\varphi_i = \varphi|_{L^2(e_i\mathscr{H})}$, then φ_i is an isomorphic mapping from $L^2(e_i\mathscr{H})$ onto $L^2(f_i\mathscr{H})$. Therefore, $e_i \sim f_i$. □

Theorem 2.12. *Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space and $\mathcal{L}$ be a closed subspace of $L^2(\mathscr{H})$, then $E : L^2(\mathscr{H}) \to \mathcal{L}$ is an orthogonal projection with bimodule property if and only if there are $\{e_i\} \subset \mathscr{P}(\mathscr{M})$ satisfying $e_i \perp e_j$ for $i \neq j$ such that $\mathcal{L} = \sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$ with $n = \infty$ possible.*

Proof. The proof of sufficiency can be divided into three steps.

Case a. $\mathcal{L} = L^2(e\mathscr{H})$ for some $e \in \mathscr{P}(\mathscr{H})$ and E is the orthogonal projection from $L^2(\mathscr{H})$ onto $L^2(e\mathscr{H})$ such that $E(x) = exe$ for all $x \in L^2(\mathscr{H})$. Then for any $x \in L^2(\mathscr{H})$, $y = eye \in L^2(e\mathscr{H})$,

$$E(x^*) = ex^*e = (exe)^* = E(x)^*,$$

$$E(xy) = exye = exe \cdot eye = E(x)y,$$

$$E(yx) = eyxe = eye \cdot exe = yE(x).$$

Hence, E is a projection with bimodule property.

Case b. $\mathcal{L} = \sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$ for some $e_i \in \mathscr{P}(\mathscr{M})$ such that $e_i \perp e_j$ when $i \neq j$. Let E be the orthogonal projection from $L^2(\mathscr{H})$ onto $\sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$ such that for any $x \in L^2(\mathscr{H})$, $E(x) = \begin{bmatrix} e_1xe_1 & & \\ & \ddots & \\ & & e_nxe_n \end{bmatrix}$. The proof of bimodule property of E is similar to the proof of sufficiency of Theorem 2.7.

Case c. $\mathcal{L} = \sum_{i=1}^{\infty} \oplus L^2(e_i\mathscr{H})$ for some $e_i \in \mathscr{P}(\mathscr{M})$ such that $e_i \perp e_j$ when $i \neq j$. For $x \in L^2(\mathscr{H})$, let $E(x) = \sum_{i=1}^{\infty} \oplus e_ixe_i$, then E is the orthogonal projection from $L^2(\mathscr{H})$ onto $\mathcal{L}$. Indeed, for $x \in L^2(\mathscr{H})$, $e_ixe_i \in L^2(e_i\mathscr{H})$. Let F_i be an orthonormal basis in $e_i\mathscr{H}$, then

$$\|e_ixe_i\|_2^2 = \sum_{\xi_j^{(i)} \in F_i} \langle e_ixe_i\xi_j^{(i)}, \xi_j^{(i)} \rangle = \sum_{\xi_j^{(i)} \in F_i} \langle x\xi_j^{(i)}, \xi_j^{(i)} \rangle.$$

The set $\{\xi_j^{(i)} | \xi_j^{(i)} \in F_i, i = 1, 2, \cdots\}$ is an orthonormal set in $\mathscr{H}$, it is contained in an orthonormal basis, denote this orthonormal basis by F. Then, $\sum_{i=1}^{\infty} \|e_ixe_i\|_2^2 \le \sum_{\xi \in F} \langle x\xi, \xi\rangle = \|x\|_2^2 < \infty$. Therefore, $\sum_{i=1}^{\infty} \oplus e_ixe_i \in \mathcal{L}$. For any $y = \sum_{i=1}^{\infty} \oplus y_i \in \mathcal{L}$, where $y_i \in L^2(e_i\mathscr{H})$, then $y_i = e_iy_ie_i$. Since $e_i \perp e_j$ for $i \neq j$,

$$
\begin{aligned}
y &= \sum_{i=1}^{\infty} \oplus e_iy_ie_i = (\sum_{i=1}^{\infty} \oplus e_i) \cdot (\sum_{i=1}^{\infty} \oplus y_i) \cdot (\sum_{i=1}^{\infty} \oplus e_i) \\
&= (\sum_{i=1}^{\infty} \oplus e_i) y (\sum_{i=1}^{\infty} \oplus e_i) = \sum_{i=1}^{\infty} \oplus e_iye_i. \qquad (1)
\end{aligned}
$$

For any $x \in L^2(\mathscr{H})$, $y = \sum_{i=1}^{\infty} \oplus e_iye_i \in \mathcal{L}$,

$$E(x^*) = \sum_{i=1}^{\infty} \oplus e_ix^*e_i = \sum_{i=1}^{\infty} \oplus (e_ixe_i)^* = (\sum_{i=1}^{\infty} \oplus e_ixe_i)^* = E(x)^*.$$

$$E(xy) = \sum_{i=1}^{\infty} e_ixye_i = \sum_{i=1}^{\infty} e_ix(\sum_{j=1}^{\infty} e_jye_j)e_i = \sum_{i=1}^{\infty} e_ixe_iye_i,$$

$$E(x)y = (\sum_{i=1}^{\infty} e_ixe_i)(\sum_{j=1}^{\infty} e_jye_j) = \sum_{i=1}^{\infty} e_ixe_iye_i.$$

Therefore, $E(xy) = E(x)y$. Similarly, we can get $E(yx) = yE(x)$.

"Necessity" If E is an orthogonal projection from $L^2(\mathscr{H})$ onto its closed subspace $\mathcal{L}$, then $\|E\|_2 \le 1$. For $\xi, \eta \in \mathscr{H}$, $\|E(\xi \otimes \eta)\| \le \|E(\xi \otimes \eta)\|_2 \le \|E\|_2 \|\xi \otimes \eta\|_2 \le \|\xi \otimes \eta\|_2 = \|\xi \otimes \eta\|$, then $E|_{F(\mathscr{H})}$ is a projection with $\|E\| \le 1$. Since $F(\mathscr{H})$ is dense in $K(\mathscr{H})$ in the norm $\|\cdot\|$, $E|_{F(\mathscr{H})}$ has a unique norm topology extension (denoted $\widetilde{E}$) to $K(\mathscr{H})$ and $\|\widetilde{E}\| \le 1$. Therefore, $\widetilde{E}$ is a projection from $K(\mathscr{H})$ onto $\overline{\mathcal{L}}^{\|\cdot\|}$ with bimodule property. Hence, $\overline{\mathcal{L}}^{\|\cdot\|}$ is a C*-subalgebra of $K(\mathscr{H})$. Then $\overline{\mathcal{L}}^{\|\cdot\|} \cong \sum_{\varphi \in P(\overline{\mathcal{L}}^{\|\cdot\|})} \oplus B(\mathscr{H}_\varphi)$, where $P(\overline{\mathcal{L}}^{\|\cdot\|})$ is the set of extreme point of the state space on $\overline{\mathcal{L}}^{\|\cdot\|}$. Since $\mathscr{H}$ is separable, so is $K(\mathscr{H})$. Thus the number of φ in $P(\overline{\mathcal{L}}^{\|\cdot\|})$ is countable. Therefore, $\overline{\mathcal{L}}^{\|\cdot\|} \cong \sum_{i=1}^{n} \oplus B(\mathscr{H}_{\varphi_i})$ where $\varphi_i \in P(\overline{\mathcal{L}}^{\|\cdot\|})$ with $n = \infty$ possible. Let Φ be the isometric $*$-isomorphism from $\overline{\mathcal{L}}^{\|\cdot\|}$ onto $\sum_{i=1}^{n} \oplus B(\mathscr{H}_{\varphi_i})$ and I_i be the identity of $B(\mathscr{H}_{\varphi_i})$. Set $e_i = \Phi^{-1}(I_i)$, then e_i is a projection in $B(\mathscr{H})$. We claim $\mathcal{L} \cong \sum_{i=1}^{n} \oplus \mathcal{L}^2(e_i\mathscr{H})$.

Indeed, for any $x \in F(\mathscr{H})$, $E(x) = \sum_{i=1}^{n} \oplus e_ixe_i$. Since x is the linear combination of rank-one projections, we may suppose $x = \xi \otimes \xi$, where $\xi \in \mathscr{H}$ and $\|\xi\| = 1$. Then $e_ixe_i = e_i\xi \otimes e_i\xi$ is a rank-one projection or 0, that is $e_ixe_i \in F(e_i\mathscr{H})$. Therefore, $E(F(\mathscr{H})) \subset (\sum_{i=1}^{n} \oplus F(e_i\mathscr{H}), \|\cdot\|_2) \subset \sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$. Hence, $\mathcal{L} = \overline{E(F(\mathscr{H}))}^{\|\cdot\|_2} \subset \sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$.

Conversely, for any $x \in \sum_{i=1}^{n} \oplus L^2(e_i\mathscr{H})$. If $n = 1$, $x \in L^2(e_1\mathscr{H})$, there is a sequence $\{x_n\} \subset F(e_1\mathscr{H}) \subset F(\mathscr{H})$, such that $E(x_n) = x_n \xrightarrow{\|\cdot\|_2} x$. Since E is $\|\cdot\|_2$−continuous, $x \in \mathcal{L}$. Therefore, $L^2(e_1\mathscr{H}) \subset \mathcal{L}$. If $n < \infty$, $x = \sum_{i=1}^{k} \oplus x_i$ where $x_i \in L^2(e_i\mathscr{H})$ with $k < \infty$, then for any $1 \le i \le k$, there is a sequence $\{x_i^{(n)}\} \subset F(e_i\mathscr{H}) \subset F(\mathscr{H})$, such that

$E(x_i^{(n)}) = x_i^{(n)} \xrightarrow{\|\cdot\|_2} x_i$. Since $k < \infty$ and E is $\|\cdot\|_2$−continuous, $x \in \mathcal{L}$. Therefore, $\sum_{i=1}^{k} \oplus L^2(e_i\mathscr{H}) \subset \mathcal{L}$. If $n = \infty$, $\sum_{i=1}^{\infty} \oplus L^2(e_i\mathscr{H}) = \lim_{k\to\infty} \sum_{i=1}^{k} \oplus L^2(e_i\mathscr{H})$. By the continuity of E, $\sum_{i=1}^{\infty} \oplus L^2(e_i\mathscr{H}) \subset \mathcal{L}$. □

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