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Approximation of Functions by Favard-Szász-Mirakyan Operators of Max-Product Type in Weighted Spaces

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Abstract. In this paper we study the uniform approximation of functions by Favard-Szász-Mirakyan operators of max-product type in some exponential weighted spaces. We estimate the rate of approximation in terms of a suitable modulus of continuity.

1. Introduction

The study of approximation of functions in weighted spaces has been intensified in the last period of time. Approximation results and estimates of the rate of convergence using positive linear operators were given. For example, for the positive linear operators defined by

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad x\in[0,\infty),\ n\geq 1 \tag{1}$$

introduced and studied independently by G. Mirakyan (also spelled Mirakjan) [1], J. Favard [2] and O. Szász [3], we can obtain pointwise convergence for functions of order $e^{\alpha x\ln x}$ and uniform convergence for functions of order $e^{\alpha x^\beta}$, with $\beta\in(0,1/2]$ (see [4]). In [5, 6], the weighted approximation of functions with the maximal weight $w(x) = e^{\alpha\sqrt{x}}$ is considered. There are many studies for weighted approximation of functions by modified or generalized Szász-Mirakyan operators. For example in [7–9] approximation with polynomial weights is considered, in [10–12] approximation in exponential weighted spaces is studied, and in [13, 14] general weights are used for approximation.

However, recently (see [15]), some nonlinear operators of max-product type were studied and the conclusion is that they have the same order of approximation as in the case of positive linear operators and even better for some subclasses of functions. For the operators defined by (1), the corresponding nonlinear operators of max-product type are

$$F_n(f,x) = \frac{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}}, \quad x\in[0,\infty), \tag{2}$$

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where $\bigvee$ denotes the supremum. These operators were studied in [16, 17] for the class of continuous and bounded functions defined on $[0,\infty)$. For a truncated version of these operators see the article [18]. For unbounded functions such a study has not yet been done.

In this paper we show that the operators F_n can be used for uniform approximation with the weight $w(x) = e^{\alpha\varphi(x)}$, where $\varphi(x) = \sqrt{x}$ and we estimate the rate of convergence of the these operators to the identity operator.

To present the results which we have obtained, we introduce some general notations. It seems that every sequence of positive linear operators can be used for uniform approximation of functions for a maximal class of weights $w(x) = e^{\alpha\varphi(x)}$, which is related to the given operators (see [6, 19]) through a function $\varphi : I \to J$, defined on a noncompact interval $I \subset \mathbb{R}$. This function φ is continuous and strictly increasing. The interval $J \subset \mathbb{R}$ is just $\varphi(I)$. We denote for $\alpha \geq 0$ the space of continuous functions

$$C_{\varphi,\alpha} = \left\{ f \in C(I), \text{ there exists } M > 0 \text{ such that } \frac{|f(x)|}{e^{\alpha\varphi(x)}} \leq M, \text{ for every } x \in I \right\}.$$

This space can be endowed with the norm

$$\|f\|_{\varphi,\alpha} = \sup_{x\in I} e^{-\alpha\varphi(x)}|f(x)|.$$

In the following section we introduce a new weighted modulus of continuity. This modulus is suitable for the uniform approximation of unbounded functions using operators of max-product type. But this new modulus can also be used for the approximation of functions using positive linear operators. In the last section, in Theorem 3.9 and 3.13, we give the main approximation results for the operators F_n and S_n, estimating the rate of convergence in terms of this new weighted modulus.

2. A new weighted modulus of continuity

Let $\delta \geq 0$ be a real number strictly less than the length of the interval $\varphi(I)$. For $f \in C_{\varphi,\alpha}$ we introduce the following modulus of continuity

$$\omega_{\varphi,\alpha}(f,\delta) = \sup_{\substack{x,t\in I \\ |\varphi(t)-\varphi(x)|\leq\delta}} \frac{|f(t) - f(x)|}{\max\left(e^{\alpha\varphi(t)}, e^{\alpha\varphi(x)}\right)},$$

where the supremum is taken for all $x \in I$ and $t \in I$ such that

$$\varphi(t) \in (\varphi(x) - \delta, \varphi(x) + \delta) \cap \varphi(I).$$

For $\alpha = 0$ we get

$$\omega_{\varphi,0}(f,\delta) = \sup_{\substack{x,t\in I \\ |\varphi(t)-\varphi(x)|\leq\delta}} |f(t) - f(x)| = \omega\left(f \circ \varphi^{-1}, \delta\right)$$

a modulus introduced in [20] (see also [6]). For $\alpha = 0$ and $\varphi(x) = x$ we get the usual modulus of continuity $\omega(f,\delta)$.

In the following, we give the main properties of this new modulus of continuity.

Lemma 2.1. *For $\delta \in [0, length(\varphi(I))$ and $f \in C_{\varphi,\alpha}$ the quantity $\omega_{\varphi,\alpha}(f,\delta)$ is finite.*

Proof. For $f \in C_{\varphi,\alpha}$ we have $|f(x)| \leq \|f\|_{\varphi,\alpha} e^{\alpha\varphi(x)}$ and

$$|f(t) - f(x)| \leq |f(t)| + |f(x)| \leq \|f\|_{\varphi,\alpha}\left(e^{\alpha\varphi(t)} + e^{\alpha\varphi(x)}\right).$$

Because $e^{\alpha\varphi(t)} + e^{\alpha\varphi(x)} \leq 2\max\left(e^{\alpha\varphi(t)}, e^{\alpha\varphi(x)}\right)$ we obtain $\omega_{\varphi,\alpha}(f,\delta) \leq 2\|f\|_{\varphi,\alpha}$. □

Lemma 2.2. *If $f \in C_{\varphi,\alpha}$ is a function such that $e^{-\alpha x} \cdot f(\varphi^{-1}(x))$ is uniformly continuous on $\varphi(I)$ then*

$$\lim_{\delta \searrow 0} \omega_{\varphi,\alpha}(f,\delta) = 0.$$

Proof. Let $w(x) = e^{\alpha\varphi(x)}$. If $\varphi(t) \geq \varphi(x)$ we consider the inequality

$$|f(t) - f(x)| \leq |w(t) - w(x)| \cdot \frac{|f(x)|}{w(x)} + w(t) \cdot \left| \frac{f(t)}{w(t)} - \frac{f(x)}{w(x)} \right|$$

and if $\varphi(t) < \varphi(x)$ we consider

$$|f(t) - f(x)| \leq |w(t) - w(x)| \cdot \frac{|f(t)|}{w(t)} + w(x) \cdot \left| \frac{f(t)}{w(t)} - \frac{f(x)}{w(x)} \right|.$$

In both cases we obtain

$$\omega_{\varphi,\alpha}(f,\delta) \leq \left(1 - e^{-\alpha\delta}\right) \left\|f\right\|_{\varphi,\alpha} + \omega\left(\frac{f}{w} \circ \varphi^{-1}, \delta\right). \tag{3}$$

Because $\left(\frac{f}{w} \circ \varphi^{-1}\right)(x) = e^{-\alpha x} \cdot f(\varphi^{-1}(x))$ is supposed to be uniformly continuous, we have, from the well-known property of the usual modulus of continuity, that $\lim_{\delta \searrow 0} \omega\left(\frac{f}{w} \circ \varphi^{-1}, \delta\right) = 0$. This fact and (3) prove that $\lim_{\delta \searrow 0} \omega_{\varphi,\alpha}(f,\delta) = 0$. □

Lemma 2.3. *We have*

$$\omega_{\varphi,\alpha}(f, n\delta) \leq n \cdot \omega_{\varphi,\alpha}(f,\delta), \quad \text{for every } n \geq 0, n \in \mathbb{Z}.$$

Proof. For $n = 0$ we have equality. Let us consider $t, x \in I$ with the property that $0 \leq \varphi(t) - \varphi(x) \leq n\delta$, $n \geq 1$. Because φ is continuous and strictly increasing there exist the points $x = x_0 < x_1 < \cdots < x_n = t$ such that

$$\varphi(x_k) - \varphi(x_{k-1}) = \frac{\varphi(t) - \varphi(x)}{n} \leq \delta.$$

Thus,

$$\begin{aligned} \left|f(t) - f(x)\right| &\leq \sum_{k=1}^{n} \left|f(x_k) - f(x_{k-1})\right| = \sum_{k=1}^{n} e^{-\alpha\varphi(x_k)} \left|f(x_k) - f(x_{k-1})\right| \cdot e^{\alpha\varphi(x_k)} \\ &\leq \omega_{\varphi,\alpha}(f,\delta) \sum_{k=1}^{n} e^{\alpha\varphi(x_k)} \leq \omega_{\varphi,\alpha}(f,\delta) \cdot n e^{\alpha\varphi(t)}. \end{aligned}$$

□

Remark 2.4. *We have $\omega_{\varphi,\alpha}(f, \lambda\delta) \leq (1+\lambda) \cdot \omega_{\varphi,\alpha}(f,\delta)$, for every real numbers $\lambda, \delta \geq 0$. This is true because of the previous lemma:*

$$\omega_{\varphi,\alpha}(f,\lambda\delta) \leq \omega_{\varphi,\alpha}(f,(1+\lfloor\lambda\rfloor)\delta) \leq (1+\lfloor\lambda\rfloor)\omega_{\varphi,\alpha}(f,\delta) \leq (1+\lambda) \cdot \omega_{\varphi,\alpha}(f,\delta).$$

Lemma 2.5. *For every $t, x \in I$ and $\delta > 0$ we have*

$$\left|f(t) - f(x)\right| \leq \max\left(e^{\alpha\varphi(t)}, e^{\alpha\varphi(x)}\right) \left(1 + \frac{\left|\varphi(t) - \varphi(x)\right|}{\delta}\right) \omega_{\varphi,\alpha}(f,\delta). \tag{4}$$

Proof. From the definition of the modulus we get

$$\left|f(t) - f(x)\right| \leq \max\left(e^{\alpha\varphi(t)}, e^{\alpha\varphi(x)}\right) \omega_{\varphi,\alpha}(f, \left|\varphi(t) - \varphi(x)\right|.$$

By Remark 2.4 we obtain

$$\omega_{\varphi,\alpha}(f, \left|\varphi(t) - \varphi(x)\right| = \omega_{\varphi,\alpha}\left(f, \frac{\left|\varphi(t) - \varphi(x)\right|}{\delta}\delta\right) \leq \left(1 + \frac{\left|\varphi(t) - \varphi(x)\right|}{\delta}\right) \omega_{\varphi,\alpha}(f,\delta).$$

□

3. Weighted approximation by Favard-Szász-Mirakjan operators

In order to obtain some estimates of the rate of approximation of functions by operators (2) and (1) let us prove first some auxiliary results.

Lemma 3.1. *For $\alpha \geq 0$ and $n \in \mathbb{N}$ consider the intervals*

$$I_0 = \left[0, e^{-\frac{\alpha}{\sqrt{n}}}\right) \text{ and } I_k = \left[k e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{k}-\sqrt{k-1}\right)}, (k+1) e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{k+1}-\sqrt{k}\right)}\right),\ k \geq 1.$$

The intervals are nonempty, disjoint and their union is the positive half line.

Proof. We have

$$\ell_k = (k+1) e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{k+1}-\sqrt{k}\right)} - k e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{k}-\sqrt{k-1}\right)} = e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{k+1}-\sqrt{k}\right)}\left[1 + k\left(1 - e^{-\frac{\alpha}{\sqrt{n}}\left(2\sqrt{k}-\sqrt{k+1}-\sqrt{k-1}\right)}\right)\right].$$

Because

$$2\sqrt{k} - \sqrt{k+1} - \sqrt{k-1} = \frac{2}{(\sqrt{k-1}+\sqrt{k})(\sqrt{k-1}+\sqrt{k+1})(\sqrt{k}+\sqrt{k+1})} > 0,$$

we obtain $\ell_k > 0$. □

Lemma 3.2. *If $nx \in I_j$ then $\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{\alpha\sqrt{\frac{k}{n}}} = \frac{(nx)^j}{j!} e^{\alpha\sqrt{\frac{j}{n}}}$.*

Proof. The proof is similar with the proof of Lemma 3.3 from [17]. Let us denote $a_k = \frac{(nx)^k}{k!} e^{\alpha\sqrt{\frac{k}{n}}}$. We have

$$0 \leq a_{k+1} \leq a_k, \text{ if and only if } nx \in \left[0, (k+1) e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{k+1}-\sqrt{k}\right)}\right).$$

By taking $k = 0, 1, \ldots$ we get

$$a_1 \leq a_0, \quad \text{if and only if } nx \in \left[0, e^{-\frac{\alpha}{\sqrt{n}}}\right)$$
$$a_2 \leq a_1, \quad \text{if and only if } nx \in \left[0, 2e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{2}-1\right)}\right)$$
$$a_3 \leq a_2, \quad \text{if and only if } nx \in \left[0, 3e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{3}-\sqrt{2}\right)}\right)$$

and so on. From all these inequalities, we obtain

if $nx \in I_0$ then $a_k \leq a_0$, for all $k = 0, 1, \ldots$
if $nx \in I_1$ then $a_k \leq a_1$, for all $k = 0, 1, \ldots$
if $nx \in I_2$ then $a_k \leq a_2$, for all $k = 0, 1, \ldots$

and so on. In general, if $nx \in I_j$, then $a_k \leq a_j$, for all $k = 0, 1, \ldots$, which proves the lemma. □

Lemma 3.3. *For every $x \geq 0$ we have $F_n(e^{\alpha\sqrt{t}}, x) \leq e^{\frac{\alpha^2}{n}} \cdot e^{\alpha\sqrt{x}}$.*

Proof. Suppose $nx \in I_j$. By Lemma 3.2 we have

$$F_n(e^{\alpha\sqrt{t}}, x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{\alpha\sqrt{\frac{k}{n}}}}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} = \frac{\frac{(nx)^j}{j!} e^{\alpha\sqrt{\frac{j}{n}}}}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}.$$

Let $m = \lfloor nx \rfloor$. From Lemma 3.3 in [17] or from Lemma 3.2 for $\alpha = 0$ we have

$$\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} = \frac{(nx)^m}{m!}.$$

Because $m \le nx < (j+1)e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{j+1}-\sqrt{j}\right)} < j+1$ we have $m \le j$. Using the inequality $1 - e^{-x} \le x$ we also have

$$j - nx \le j - je^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{j}-\sqrt{j-1}\right)} = j\left(1 - e^{-\frac{\alpha}{\sqrt{n}}\left(\sqrt{j}-\sqrt{j-1}\right)}\right) \le \frac{j\alpha}{\sqrt{n}}\left(\sqrt{j} - \sqrt{j-1}\right) = \frac{j\alpha}{\sqrt{n}\left(\sqrt{j} + \sqrt{j-1}\right)} \le \frac{\alpha\sqrt{j}}{\sqrt{n}}.$$

Because $nx < \lfloor nx \rfloor + 1 = m + 1$ we finally obtain

$$e^{-\alpha\sqrt{x}} \cdot F_n(e^{\alpha\sqrt{t}}, x) = e^{-\alpha\sqrt{x}} \cdot \frac{\frac{(nx)^j}{j!} e^{\alpha\sqrt{\frac{j}{n}}}}{\frac{(nx)^m}{m!}} = \frac{m!(nx)^j}{j!(nx)^m} e^{\frac{\alpha}{\sqrt{n}}\left(\sqrt{j}-\sqrt{nx}\right)} \le \left(\frac{nx}{m+1}\right)^{j-m} e^{\frac{\alpha(j-nx)}{\sqrt{n}(\sqrt{j}+\sqrt{nx})}} \le e^{\frac{\alpha^2\sqrt{j}}{n(\sqrt{j}+\sqrt{nx})}} \le e^{\frac{\alpha^2}{n}}.$$

□

Remark 3.4. *We have* $F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right), x\right) \le e^{\frac{\alpha^2}{n}} \cdot e^{\alpha\sqrt{x}}$, *for every* $x \ge 0$. *Indeed,*

$$F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right), x\right) = \max\left(F_n\left(e^{\alpha\sqrt{t}}, x\right), F_n\left(e^{\alpha\sqrt{x}}, x\right)\right) \le \max\left(e^{\frac{\alpha^2}{n}} e^{\alpha\sqrt{x}}, e^{\alpha\sqrt{x}}\right) = e^{\frac{\alpha^2}{n}} \cdot e^{\alpha\sqrt{x}}.$$

Remark 3.5. *For* $\varphi(x) = \sqrt{x}$, *for every function* f *belonging to* $C_{\varphi,\alpha}$ *the functions* $F_n f$ *also belong to* $C_{\varphi,\alpha}$. *Indeed,*

$$\left|F_n(f, x)\right| \le F_n(|f|, x) \le F_n\left(\|f\|_{\varphi,\alpha} e^{\alpha\sqrt{t}}, x\right) = \|f\|_{\varphi,\alpha} F_n(e^{\alpha\sqrt{t}}, x) \le \|f\|_{\varphi,\alpha} e^{\frac{\alpha^2}{n}} \cdot e^{\alpha\sqrt{x}}.$$

Lemma 3.6. *For every* $x \ge 0$ *and* $n \in \mathbb{N}$ *the following inequality holds true*

$$\frac{\bigvee_{k \le nx} \frac{(nx)^k}{k!}\left(\sqrt{nx} - \sqrt{k}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} \le 1.$$

Proof. We have already remarked that $\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} = \frac{(nx)^m}{m!}$, with $m = \lfloor nx \rfloor$. If $m = 0$, the inequality to be proved is $\sqrt{nx} \le 1$, which is true because $nx \in [0, 1)$. Consider the case $m = 1$. The inequality to be proved is true, because

$$\frac{\max\left(\sqrt{nx}, nx(\sqrt{nx} - 1)\right)}{nx} = \max\left(\frac{1}{\sqrt{nx}}, \sqrt{nx} - 1\right) \le \max(1, \sqrt{2} - 1) \le 1.$$

In what follows we consider $m \ge 2$. Let us denote $b_k = \frac{(nx)^k}{k!}\left(\sqrt{nx} - \sqrt{k}\right)$. First, we observe that

$$\frac{b_0}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} = \frac{m!\sqrt{nx}}{(nx)^m} = \frac{2}{nx} \cdots \frac{m}{nx} \cdot \frac{1}{\sqrt{nx}} \le 1.$$

It remains to evaluate the maximum of b_k, for $k \ge 1$. We have

$$b_k = \frac{(nx)^k}{k!}\left(\sqrt{nx} - \sqrt{k}\right) = \frac{(nx)^k}{k!} \cdot \frac{nx - k}{\sqrt{nx} + \sqrt{k}} \le \frac{(nx)^k}{k!} \frac{nx - k}{\sqrt{nx} + 1}.$$

Let us denote $c_k = \frac{(nx)^k}{k!}\frac{nx-k}{\sqrt{nx}+1}$. We have

$$\frac{c_k}{c_{k-1}} = \frac{nx}{k} \cdot \frac{nx-k}{nx-k+1} \le 1$$

if and only if $nx(nx-k) \le k(nx-k)+k$ which is equivalent to $(nx-k)^2 \le k$. So, for every integer $k \le nx$ with the property $nx \le k + \sqrt{k}$ we have $b_k \le b_{k-1}$.

In particular, by taking $k = 2, 3, \dots$ we have

$$\begin{aligned} c_2 \le c_1 &\text{ if and only if } nx \in [2, 2+\sqrt{2}) \\ c_3 \le c_2 &\text{ if and only if } nx \in [2, 3+\sqrt{3}) \\ c_4 \le c_3 &\text{ if and only if } nx \in [2, 6) \end{aligned}$$

and so on. Let us denote $J_k = \left[k+\sqrt{k}, k+1+\sqrt{k+1}\right)$, for $k = 1, 2, \dots$. We deduce that if $nx \in J_j$ then $c_k \le c_j$, for every $k \ge 1$. We obtain

$$\frac{\bigvee\limits_{1\le k\le nx} b_k}{\frac{(nx)^m}{m!}} \le \frac{\bigvee\limits_{1\le k\le nx} c_k}{\frac{(nx)^m}{m!}} = \frac{\frac{(nx)^j}{j!}}{\frac{(nx)^m}{m!}} \cdot \frac{nx-j}{\sqrt{nx}+1} \le \frac{nx-j}{\sqrt{nx}+1} \le \frac{\sqrt{j+1}+1}{\sqrt{j+\sqrt{j}}+1} \le 1.$$

□

Lemma 3.7. *For every $x \ge 0$ and for every $\alpha \ge 0$ and $n \in \mathbb{N}$, such that $n \ge \alpha^2$ the following inequality holds true*

$$\frac{\bigvee_{k>nx} \frac{(nx)^k}{k!} e^{\alpha\sqrt{\frac{k}{n}}}\left(\sqrt{k}-\sqrt{nx}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} \le e^{\frac{2\alpha}{\sqrt{n}}} \cdot e^{\alpha\sqrt{x}}.$$

Proof. We have $\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} = \frac{(nx)^m}{m!}$, with $m = \lfloor nx \rfloor$. For $m = 0$ we have

$$\frac{\bigvee_{k>nx} \frac{(nx)^k}{k!} e^{\alpha\sqrt{\frac{k}{n}}}\left(\sqrt{k}-\sqrt{nx}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} < \bigvee_{k\ge 1} \frac{e^{\frac{\alpha\sqrt{k}}{\sqrt{n}}} \cdot \sqrt{k}}{k!} \le e^{\frac{2\alpha}{\sqrt{n}}},$$

because for $k \ge 2$ the sequence $d_k = \frac{e^{\frac{\alpha(\sqrt{k}-2)}{\sqrt{n}}} \cdot \sqrt{k}}{k!}$ is decreasing and we have $d_k \le \max(d_2, d_1) < 1$.

Consider the case $m \ge 1$. Using the inequality

$$\sqrt{k}-\sqrt{nx} = \frac{k-nx}{\sqrt{k}+\sqrt{nx}} \le \frac{k-nx}{2\sqrt{nx}}$$

and denoting $d_k = \frac{(nx)^k}{k!} e^{\frac{\alpha}{\sqrt{n}} \cdot \frac{k-nx}{2\sqrt{nx}}} \cdot \frac{k-nx}{2\sqrt{nx}}$, it remains to prove that

$$\frac{\bigvee_{k>nx} d_k}{\frac{(nx)^m}{m!}} \le e^{\frac{2\alpha}{\sqrt{n}}}.$$

The inequality $\frac{d_{k+1}}{d_k} \ge 1$ is true if and only if $k > nx$ is an integer such that $nxe^{\frac{\alpha}{2n\sqrt{x}}}(k+1-nx) \ge (k+1)(k-nx)$, which is equivalent to

$$k^2 + k\left[1 - nx\left(1 + e^{\frac{\alpha}{2n\sqrt{x}}}\right)\right] + n^2x^2 e^{\frac{\alpha}{2n\sqrt{x}}} - nx\left(1 + e^{\frac{\alpha}{2n\sqrt{x}}}\right) \le 0.$$

So, $d_{k+1} \geq d_k$ if and only if $k \in [k_1, k_2]$, where

$$k_1 = \frac{-1 + nx\left(1 + e^{\frac{\alpha}{2n\sqrt{x}}}\right) - \sqrt{\left[1 + nx\left(e^{\frac{\alpha}{2n\sqrt{x}}} - 1\right)\right]^2 + 4nx}}{2}$$

$$k_2 = \frac{-1 + nx\left(1 + e^{\frac{\alpha}{2n\sqrt{x}}}\right) + \sqrt{\left[1 + nx\left(e^{\frac{\alpha}{2n\sqrt{x}}} - 1\right)\right]^2 + 4nx}}{2}$$

Because $k_1 < nx < k_2$ we deduce that $\bigvee_{k>nx} d_k = d_j$, where $j = \lfloor k_2 \rfloor + 1$. Because the function $f : [1, \infty) \to \mathbb{R}$ defined by

$$f(t) = \frac{1 + t\left(e^{\frac{\alpha}{2\sqrt{nt}}} - 1\right)}{2\sqrt{t}}$$

is decreasing, we have that $f(nx) \leq f(m) \leq f(1) = \frac{1}{2}e^{\frac{\alpha}{2\sqrt{n}}}$. Using this inequality and the fact that $g(t) = \frac{t+\sqrt{t^2+1}}{2} \leq \frac{1}{2} + t$, we obtain

$$\frac{j - nx}{2\sqrt{nx}} = \frac{\lfloor k_2 \rfloor + 1 - nx}{2\sqrt{nx}} \leq \frac{k_2 + 1 - nx}{2\sqrt{nx}} = g\left(f(nx)\right) \leq \frac{1}{2} + \frac{1}{2}e^{\frac{\alpha}{2\sqrt{n}}}.$$

We finally obtain

$$\frac{\bigvee_{k>nx} d_k}{\frac{(nx)^m}{m!}} \leq e^{\frac{\alpha}{\sqrt{n}}\frac{j-nx}{2\sqrt{nx}}} \cdot \frac{j - nx}{2\sqrt{nx}} \leq e^{\frac{\alpha}{2\sqrt{n}}(1+e^{\frac{\alpha}{2\sqrt{n}}})} \cdot \frac{1 + e^{\frac{\alpha}{2\sqrt{n}}}}{2} \leq e^{\frac{2\alpha}{\sqrt{n}}}.$$

□

Lemma 3.8. *For every $x, \alpha \geq 0$ and $n \geq \alpha^2$ we have*

$$F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right)\left|\sqrt{t} - \sqrt{x}\right|, x\right) \leq \frac{1}{\sqrt{n}}e^{\frac{2\alpha}{\sqrt{n}}} \cdot e^{\alpha\sqrt{x}}.$$

Proof. We have

$$F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right)\left|\sqrt{t} - \sqrt{x}\right|, x\right) = \max\left(A_n, B_n\right),$$

where

$$A_n = \frac{\bigvee_{k>nx} \frac{(nx)^k}{k!} e^{\alpha\sqrt{\frac{k}{n}}}\left(\sqrt{\frac{k}{n}} - \sqrt{x}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}},$$

$$B_n = \frac{\bigvee_{k\leq nx} \frac{(nx)^k}{k!} e^{\alpha\sqrt{x}}\left(\sqrt{x} - \sqrt{\frac{k}{n}}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}.$$

By Lemma 3.7 we have $A_n \leq \frac{1}{\sqrt{n}}e^{\frac{2\alpha}{\sqrt{n}}} \cdot e^{\alpha\sqrt{x}}$ and by Lemma 3.6, $B_n \leq \frac{1}{\sqrt{n}}e^{\alpha\sqrt{x}}$. □

Theorem 3.9. *For $\varphi(x) = \sqrt{x}$, for every $f \in C_{\varphi,\alpha}$ the estimation of the error of uniform approximation by F_n is given by*

$$\left\|F_n f - f\right\|_{\varphi,\alpha} \leq \left(e^{\frac{\alpha^2}{n}} + e^{\frac{2\alpha}{\sqrt{n}}}\right) \cdot \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

for every $n \in \mathbb{N}$, $n \geq \alpha^2$.

Proof. Because $F_n(1,x) = 1$, using [17, Lemma 2.1] and (4) we obtain

$$\left|F_n(f,x) - f(x)\right| \le F_n(\left|f(t) - f(x)\right|, x) \le F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right)\left(1 + \frac{\left|\sqrt{t} - \sqrt{x}\right|}{\delta_n}\right), x\right) \cdot \omega_{\varphi,\alpha}(f,\delta_n)$$
$$\le \left(C_n(x) + \frac{D_n(x)}{\delta_n}\right)\omega_{\varphi,\alpha}(f,\delta_n),$$

where

$$C_n(x) = F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right), x\right)$$
$$D_n(x) = F_n\left(\max\left(e^{\alpha\sqrt{t}}, e^{\alpha\sqrt{x}}\right)\left|\sqrt{t} - \sqrt{x}\right|, x\right).$$

Using Remark 3.4 and Lemma 3.8 and choosing $\delta_n = \frac{1}{\sqrt{n}}$ we get

$$e^{-\alpha\sqrt{x}}\left|F_n(f,x) - f(x)\right| \le \left(e^{\frac{\alpha^2}{n}} + e^{\frac{2\alpha}{\sqrt{n}}}\right)\omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

which proves the theorem. □

For the particular case $\alpha = 0$ we obtain

Corollary 3.10. *The Favard-Szász-Mirakjan operators of max-product type F_n have the property that $\left\|F_n f - f\right\| \to 0$ if $f(t^2)$ is uniformly continuous on $[0,\infty)$. Moreover,*

$$\left|F_n(f,x) - f(x)\right| \le 2 \cdot \omega\left(f(t^2), \frac{1}{\sqrt{n}}\right), \quad \textit{for every } n \in \mathbb{N} \textit{ and } x \in [0,\infty).$$

Remark 3.11. *Corollary 3.10 extends the result of [17, Theorem 4.1] to some unbounded functions. For example, the function $f(x) = \sqrt{x}$ is unbounded and has the property that $f(t^2)$ is uniformly continuous on $[0,\infty)$, so the square root function can be uniformly approximated on $[0,\infty)$. Theorem 3.9 extends the result to unbounded functions of order $O(e^{\alpha\sqrt{x}})$.*

Remark 3.12. *Let us observe that for some classes of functions the order of approximation is better that $\frac{1}{\sqrt{n}}$. Using the proofs of Lemma 4.2, Lemma 4.3, Corollary 4.4 from [17] we deduce that the estimate*

$$\left|F_n(f,x) - f(x)\right| \le \frac{M}{n}, \quad n \ge 1,$$

is true for a positive, increasing, concave and Lipschitz function f, which is not necessarily bounded. For example, if f is an increasing concave polygonal line, the order of approximation by linear Favard-Szász-Mirakjan operators S_n is $\frac{1}{\sqrt{n}}$ (see [17, Remark 2, p. 66]) and by max-product Favard-Szász-Mirakjan operators F_n is $\frac{1}{n}$, which is essentially better.

Theorem 3.13. *For $\varphi(x) = \sqrt{x}$, for every $f \in C_{\varphi,\alpha}$ we have*

$$\left\|S_n f - f\right\|_{\varphi,\alpha} \le C_\alpha \cdot \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

for every $n \in \mathbb{N}$, where $C_\alpha > 0$ is a constant depending only on α.

Proof. From [6, Lemma 3.1] we have $S_n(e^{\alpha\sqrt{t}}, x) \leq M_\alpha \cdot e^{\alpha\sqrt{x}}$, where $M_\alpha > 0$ is a constant depending only on α. From (4) we obtain

$$\begin{aligned}\left|S_n(f,x) - f(x)\right| \leq S_n(\left|f(t) - f(x)\right|, x) &\leq S_n\left(\left(e^{\alpha\sqrt{t}} + e^{\alpha\sqrt{x}}\right)\left(1 + \frac{\left|\sqrt{t} - \sqrt{x}\right|}{\delta_n}\right), x\right) \cdot \omega_{\varphi,\alpha}(f, \delta_n) \\ &\leq \left(C_n(x) + \frac{D_n(x)}{\delta_n}\right)\omega_{\varphi,\alpha}(f, \delta_n),\end{aligned}$$

where

$$\begin{aligned}C_n(x) &= S_n\left(e^{\alpha\sqrt{t}} + e^{\alpha\sqrt{x}}, x\right) \leq (M_\alpha + 1) \cdot e^{\alpha\sqrt{x}} \\ D_n(x) &= S_n\left(\left(e^{\alpha\sqrt{t}} + e^{\alpha\sqrt{x}}\right)\left|\sqrt{t} - \sqrt{x}\right|, x\right) = S_n\left(e^{\alpha\sqrt{t}}\left|\sqrt{t} - \sqrt{x}\right|, x\right) + e^{\alpha\sqrt{x}} S_n\left(\left|\sqrt{t} - \sqrt{x}\right|, x\right).\end{aligned}$$

Using the Cauchy-Schwarz inequality for positive linear operators $|S_n(f \cdot g, x)| \leq \sqrt{S_n(f^2, x)} \cdot \sqrt{S_n(g^2, x)}$ and the estimation $S_n\left(\left|\sqrt{t} - \sqrt{x}\right|^2, x\right) \leq \frac{1}{n}$ (see the proof of Corollary 3.2 from [6]) we obtain

$$\begin{aligned}D_n(x) &\leq \sqrt{S_n\left(e^{2\alpha\sqrt{t}}, x\right)} \cdot \sqrt{S_n\left(\left|\sqrt{t} - \sqrt{x}\right|^2, x\right)} + e^{\alpha\sqrt{x}}\sqrt{S_n\left(\left|\sqrt{t} - \sqrt{x}\right|^2, x\right)} \\ &\leq \sqrt{M_{2\alpha} \cdot e^{2\alpha\sqrt{x}}} \cdot \frac{1}{\sqrt{n}} + e^{\alpha\sqrt{x}} \cdot \frac{1}{\sqrt{n}} = \left(\sqrt{M_{2\alpha}} + 1\right) \cdot e^{\alpha\sqrt{x}} \cdot \frac{1}{\sqrt{n}}.\end{aligned}$$

Choosing $\delta_n = \frac{1}{\sqrt{n}}$ and $C_\alpha = 2 + M_\alpha + \sqrt{M_{2\alpha}}$ we get

$$e^{-\alpha\sqrt{x}}\left|S_n(f,x) - f(x)\right| \leq C_\alpha \cdot \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

which proves the theorem. □

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