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A Note on Multiplication and Composition Operators in Orlicz Spaces

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Abstract. In this note, we give some results on the ascent and descent of multiplication and composition operators on Orlicz spaces.

1. Preliminaries and Introduction

Let X be a linear space and $M : X \longrightarrow X$ be a linear operator with domain $\mathcal{D}(M)$ and range $\mathcal{R}(M)$ in X. The null space of the iterates of M, M^n, is denoted by $\mathcal{N}(M^n)$, and we know that the null spaces of M^n's form an increasing chain of subspaces $(0) = \mathcal{N}(M^0) \subset \mathcal{N}(M) \subset \mathcal{N}(M^2) \subset \ldots$. Also the ranges of iterates of M form a nested chain of subspaces $X = \mathcal{R}(M^0) \supset \mathcal{R}(M) \supset \mathcal{R}(M^2) \supset \ldots$. Note that if $\mathcal{N}(M^k)$ coincides with $\mathcal{N}(M^{k+1})$ for some k, it coincides with all $\mathcal{N}(M^n)$ for $n > k$. The smallest non-negative integer k such that $\mathcal{N}(M^k) = \mathcal{N}(M^{k+1})$ is called the *ascent* of M and denoted by $\alpha(M)$. If there is no such k, then we set $\alpha(M) = \infty$. Also if $\mathcal{R}(M^k) = \mathcal{R}(M^{k+1})$, for some non-negative integer k, then $\mathcal{R}(M^n) = \mathcal{R}(M^k)$ for all $n > k$. The smallest non-negative integer k such that $\mathcal{R}(M^k) = \mathcal{R}(M^{k+1})$ is called *descent* of M and denoted by $\delta(M)$. We set $\delta(M) = \infty$ when there is no such k. When ascent and descent of an operator are finite, then they are equal and the linear space X can be decomposed into the direct sum of the null and range spaces of a suitable iterates of M. The ascent and descent of an operator can be used to characterized when an operator can be broken into a nilpotent piece and an invertible one; see, for example, [1, 12]. For some results on ascent and descent of an operator in general setting see, for example, [11, 13].

Recently, R. Kumar in [8] has studied ascent and descent of weighted composition operators on L^p-spaces; see also [2]. In this paper, we, among other things, give some necessary and sufficient conditions for a product of multiplication and composition operators on Orlicz spaces to have finite ascent and decent. In particular, our results generalize and improve known results on the classical Lebesgue spaces.

First, for the convenience of the reader, we gather some necessary facts on Orlicz spaces. For more details on Orlicz spaces see [7, 9].

A function $\Phi : [0, \infty) \to [0, \infty]$ is called a *Young function* if Φ is convex and $\Phi(0) = 0$; we also assume that Φ is neither identically zero nor identically infinite on $(0, \infty)$. With each Young function Φ one can associate another convex function $\Psi : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\} \qquad (y \in \mathbb{R}).$$

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Then Ψ is called the complementary Young function of Φ.

A Young function Φ is said to satisfy the ∇' condition, if there exists $c > 0$ such that

$$\Phi(xy) \geq c\,\Phi(x)\Phi(y) \qquad (x, y \geq 0).$$

Throughout the paper, let (X, Σ, μ) be a σ-finite complete measure space and $L^0(X)$ be the linear space of all equivalence classes of all Σ-measurable real-valued functions on X, that is, we identify any two functions that are equal μ-almost everywhere on X. The support of a measurable function f is defined as $S(f) = \{x \in X : f(x) \neq 0\}$. Let Φ be a Young function, then the set of Σ-measurable functions defined as

$$L^\Phi(X) = \left\{ f \in L^0(X) : \exists k > 0, \int_X \Phi(k|f|)d\mu < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_\Phi = \inf\left\{ k > 0 : \int_X \Phi(|f|/k)d\mu \leq 1 \right\}.$$

The pair $(L^\Phi(X), \|\cdot\|_\Phi)$ is called an *Orlicz space*. If $\Phi(x) = x^p/p$ for $x \geq 0$, where $1 < p < \infty$, then Φ is a Young function and $\Psi(x) = x^{p'}/p'$, where $1 < p' < \infty$ and $1/p + 1/p' = 1$. In this case we recover the classical Lebesgue spaces: $L^\Phi(X) = L^p(X)$.

For a measurable function $u \in L^0(X)$, the rule taking u to $u \cdot f$, is a linear transformation on $L^0(X)$ and we denote this transformation by M_u. In the case that M_u is continuous, it is called a *multiplication operator* induced by u.

Let $T : X \to X$ be a measurable transformation, that is, $T^{-1}(A) \in \Sigma$ for any $A \in \Sigma$. If $\mu(T^{-1}(A)) = 0$ for all $A \in \Sigma$ with $\mu(A) = 0$, then T is said to be non-singular. This condition means that the measure $\mu \circ T^{-1}$, defined by $\mu \circ T^{-1}(A) = \mu(T^{-1}(A))$ for $A \in \Sigma$, is absolutely continuous with respect to the μ (it is usually denoted $\mu \circ T^{-1} \ll \mu$). The Radon-Nikodym theorem ensures the existence of a non-negative locally integrable function h_0 on X such that

$$\mu \circ T^{-1}(A) = \int_A h_0\, d\mu \qquad (A \in \Sigma).$$

Any non-singular measurable transformation T induces a linear operator (*composition operator*) C_T from $L^0(X)$ into itself defined by

$$C_T(f)(x) = f(T(x)) \qquad (x \in X,\ f \in L^0(X)).$$

Here the non-singularity of T guarantees that the operator C_T is well-defined as a mapping from $L^0(X)$ into itself. If M_u and C_T are well-defined, then we denote their composition as $M_{T,u} = C_T M_u$. Composition operators have been studied extensively in many function spaces and recently the investiagtion of such operators on Orlicz spaces has attracted some attention.

2. Main Results

Throughout the paper we assume that Φ is a Young function, $T : X \longrightarrow X$ is a non-singular measurable transformation and $u : X \longrightarrow \mathbb{R}$ is a Σ-measurable function such that $S(u) = X$. Also, for a Young function Φ, we assume that $M_{T,u}$ is a bounded operator on the Orlicz space $L^\Phi(X)$.

Here we give a sequence of measures $(\mu^n)_n$ defined inductively by

$$\mu^n(A) = \int_{T^{-1}(A)} h_{n-1}(\Phi \circ |u|)\, d\mu \qquad (A \in \Sigma)$$

where for natural number n, h_n is the Radon-Nikodym derivative $d\mu^n/d\mu$ and $h_0 = d(\mu \circ T^{-1})/d\mu$. Moreover, we set

$$S_n = S(h_n), \qquad L^\Phi(X_n) = \{f \in L^\Phi(X) : S(f) \subseteq S_n^c,\ \mu\text{-a.e.}\}$$

where $S_n^c = X - S_n$ for $n \in \mathbb{N}$.

First we prove a technical lemma that we use in the next assertions.

Lemma 2.1. *If $A \in \Sigma$ and $A \subseteq S_n^c$ for some $n \in \mathbb{N}$, then $T^{-k}(A) \subseteq S_{n-k}^c$ for $k \leq n$.*

Proof. we have

$$\mu^n(A) = \int_{T^{-1}(A)} h_{n-1}(\Phi \circ |u|)\, d\mu = \int_A h_n\, d\mu = 0.$$

Therefore $h_{n-1} \mid_{T^{-1}(A)} = 0$, and hence $T^{-1}(A) \subseteq S_{n-1}^c$. Applying this to the set $T^{-1}(A) \subseteq S_{n-1}^c$ and measure μ^{n-1}, we find that $T^{-2}(A) \subseteq S_{n-2}^c$. By continuing this, we get $T^{-k}(A) \subseteq S_{n-k}^c$ for $k \leq n$. □

A non-singular measurable transformation $T : X \longrightarrow X$ is called essentially surjective if $T(X) \in \Sigma$ and $\mu[X \setminus T(X)] = 0$.

Remark 2.2. In [8] it was claimed that a non-singular measurable transformation T is essentially surjective if and only if $M_{T,u}$ is injective. As noted in [5] this is not true. Consider the following example taken from [10]: $X = [0, 1]$, and let m denote the Lebesgue measure defined on σ-algebra of Lebesgue measurable subsets of X. Let $T(x) = (x + h(x))/2$, where h is the singular Cantor-Lebesgue function. Then T is an onto homeomorphism on X. Note that $d(\mu \circ T^{-1})/d\mu = 1/2$ almost everywhere. Letting $u = 1$ and noting that $M_{T,u}$ is a well-defined bounded weighted composition operator on $L^p([0, 1])$. If K denotes the Cantor set, then $m(T(K)) = 1/2$ while $m(K) = 0$. It follows that $M_{T,u}(\chi_{T(K)}) = 0$; i.e., $M_{T,u}$ is not injective.

In the next theorem we obtain an upper bound for the null space of $M_{T,u}^n$ on the Orlicz space $L^\Phi(X)$.

Lemma 2.3. *Let $M_{T,u}$ be a bounded operator on $L^\Phi(X)$ with $\Phi \in \triangledown'$. If T is an essentially surjective non-singular measurable transformation, then for all $n \in \mathbb{N}$, we have $\mathcal{N}(M_{T,u}^n) \subseteq L^\Phi(S_{n-1}^c)$.*

Proof. If $f \in \mathcal{N}(M_{T,u}^n)$, then, for all $x \in X$,

$$M_{T,u}^n f(x) = \prod_{i=1}^n u(T^i(x))\, f(T^n(x)) = 0.$$

Therefore, for all $\alpha > 0$

$$\begin{aligned}
0 &= \int_X \Phi\Big(\alpha\Big|\prod_{i=1}^n (u \circ T^i(x))\, f \circ T^n(x)\Big|\Big) d\mu \\
&= \int_X h_0(x)\Phi\Big(\alpha\Big|u(x)\prod_{i=1}^{n-1} (u \circ T^i(x))\, f \circ T^{n-1}(x)\Big|\Big) d\mu \\
&\geq c \int_X h_0(x)\Phi(|u(x)|).\Phi\Big(\alpha\Big|\prod_{i=1}^{n-1} (u \circ T^i(x)).f \circ T^{n-1}(x)\Big|\Big) d\mu \\
&\geq c \int_X h_0(T^{-1}(x))\Phi(|u(T^{-1}(x))|).\Phi\Big(\alpha\Big|u(x)\prod_{i=1}^{n-2} (u \circ T^i(x)).f \circ T^{n-2}(x)\Big|\Big) d\mu \circ T^{-1} \\
&\geq c \int_X \Phi\Big(\alpha\Big|u(x)\prod_{i=1}^{n-2} (u \circ T^i(x)).f \circ T^{n-2}(x)\Big|\Big) d\mu^1 \\
&= c \int_X h_1(x).\Phi\Big(\alpha\Big|u(x)\prod_{i=1}^{n-2} (u \circ T^i(x)).f \circ T^{n-2}(x)\Big|\Big) d\mu,
\end{aligned}$$

where $c > 0$ is a constant satisfies in the definition of condition $\Phi \in \triangledown'$. Repeating the argument, we see that

$$c^n \int_X h_{n-1}(x).\Phi\Big(\alpha\Big|u(x).f(x)\Big|\Big) d\mu \leq 0,$$

and $h_{n-1}\mid_{s(f)}=0$, this means that, $f \in L^{\Phi}(S^c_{n-1})$. □

Here we give a lower bound for the null space of $M^n_{T,u}$.

Lemma 2.4. *Let $M_{T,u}$ be a bounded operator on $L^{\Phi}(X)$. For all $n \in \mathbb{N}$, we have $L^{\Phi}(S^c_{n-1}) \subseteq \mathcal{N}(M^n_{T,u})$.*

Proof. Suppose $f \in L^{\Phi}(S^c_{n-1})$. So $S(f) \subseteq S^c_{n-1}$ and, by Lemma 2.1, $T^{-(n-1)}(S(f)) \subseteq S^c_0$. Therefore for all $\alpha > 0$, we have

$$\int_X \Phi(\alpha|\prod_{i=1}^{n}(u \circ T^i(x)).f \circ T^n(x)|)d\mu =$$
$$\int_X h_0(x)\Phi(\alpha|u(x).\prod_{i=1}^{n-1}(u \circ T^i(x)).f \circ T^{n-1}(x)|)d\mu = 0;$$

i.e., $f \in \mathcal{N}(M^n_{T,u})$. □

Now, under a weak condition on the Young function, a combination of lemmas 2.3 and 2.4 gives us the null space of $M^n_{T,u}$.

Corollary 2.5. *Let $M_{T,u}$ be a bounded operator on $L^{\Phi}(X)$ and $\Phi \in \nabla'$. If T is an essentially surjective non-singular measurable transformation, then $L^{\Phi}(S^c_{n-1}) = \mathcal{N}(M^n_{T,u})$.*

Here we can give a necessary and sufficient condition for $M_{T,u}$ to have finite ascent.

Theorem 2.6. *Let $M_{T,u}$ be a bounded operator on $L^{\Phi}(X)$, $\Phi \in \nabla'$, and T be an essentially surjective non-singular measurable transformation. For some $n \in \mathbb{N}$, $\alpha(M_{T,u}) = n$ if and only if $\mu^{k-1} \ll \mu^k \ll \mu^{k-1}$ for every $k \geq n$.*

Proof. Suppose $\mu^{k-1} \ll \mu^k \ll \mu^{k-1}$ for all $k \geq n$. First, we show that $S^c_{k-1} = S^c_k$ for all $k \geq n$. Since $\mu^k \ll \mu^{k-1}$ and $\mu^{k-1}(S^c_{k-1}) = 0$,

$$\mu^k(S^c_{k-1}) = \int_{S^c_{k-1}} h_k d\mu = 0.$$

It means that $h_k\mid_{S^c_{k-1}} = 0$ and also $S^c_{k-1} \subseteq S^c_k$. Now since $\mu^{k-1} \ll \mu^k$, with the same method, we have $S^c_k \subseteq S^c_{k-1}$. Thus $S^c_{k-1} = S^c_k$. In other words for every $k \geq n$;

$$\mathcal{N}(M^k_{T,u}) = L^{\Phi}(S^c_{k-1}) = L^{\Phi}(S^c_k) = \mathcal{N}(M^{k+1}_{T,u}).$$

That means $\alpha(M) = n$.

The converse is obvious. □

Remark 2.7. Our Theorem 2.6 in the special case of Lebesgue spaces, generalizing Theorem 2.4 in [8], gives necessary and sufficient condition for a weighted composition operator to have finite ascent.

Proposition 2.8. *If n is the smallest natural number such that $S_{n-1} \subset T^{n+1}(S_{n-1})$, then $\alpha(M_{T,u}) = n$.*

Proof. We know that for every natural number n, $L^{\Phi}(S) = L^{\Phi}(S^c_{n-1}) + L^{\Phi}(S_{n-1})$. Thus if $f \in \mathcal{N}(M^{n+1}_{T,u})$, then there exist $g \in L^{\Phi}(S_{n-1})$ and $h \in L^{\Phi}(S^c_{n-1}) \subseteq \mathcal{N}(M^n_{u,T})$, with $f = g + h$. From that we see that $M^{n+1}_{T,u}f = M^{n+1}_{T,u}g = \prod_{i=1}^{n+1}(u \circ T^i)(g \circ T^{n+1}) = 0$ almost everywhere. Since $u \circ T^i \neq 0$ almost everywhere, we conclude that for every measurable subset A, we have $(g \circ T^{n+1}).\chi_A = 0$, almost everywhere. This means that if we take $A = S_{n-1}$, because of $S_{n-1} \subset T^{n+1}(S_{n-1})$ we get $g.\chi_A = 0$ a.e. Thus $g = 0$ a.e. and $f = h$. It shows that $\mathcal{N}(M^{n+1}_{T,u}) \subset \mathcal{N}(M^n_{T,u})$, and consequently, $\alpha(M_{T,u}) = n$. □

In the next result we provide a necessary condition for $M_{T,u}$ to have infinite ascent.

Proposition 2.9. *Suppose for all $A \in \Sigma$ with $\mu(A) > 0$, we have $T(A) \in \Sigma$. If $\alpha(M_{T,u}) = \infty$, then there exists a sequence $\{A_k\}$ of measurable sets with positive measure such that $A_k \subseteq T^k(B_k)$ for some $B_k \in \Sigma$ and $A_k \neq T^{k+1}(C_k)$ for all $C_k \in \Sigma$.*

Proof. Suppose $\alpha(M_{T,u}) = \infty$, then $\mathcal{N}(M^k_{T,u}) \subsetneq \mathcal{N}(M^{k+1}_{T,u})$ for each $k \in \mathbb{N}$. Thus there exists a non-zero function $f \in L^{\Phi}(X)$ such that $f \in \mathcal{N}(M^{k+1}_{T,u})$ but $f \notin \mathcal{N}(M^k_{T,u})$. Since $f \in \mathcal{N}(M^{k+1}_{T,u})$, then for all $C \in \Sigma$, we have

$$M^{k+1}_{T,u} f.\chi_C = 0.$$

Therefore $f \circ T^{k+1}.\chi_C = 0$, for all $C \in \Sigma$. Moreover, since $f \notin \mathcal{N}(M^k)$, then there exists a set B_k such that

$$M^k f.\chi_{B_k} \neq 0.$$

Hence $f \circ T^k.\chi_{B_k} \neq 0$. Since $T(B_k)$ is a measurable set with positive measure and μ is σ-finite, then there exists a measurable set A_k with positive measure such that $A_k \subseteq T^k(B_k)$. We can see that $f.\chi_{A_k} \neq 0$. Suppose on the contrary that there exists a measurable set C_k with positive measure such that $A_k = T^{k+1}(C_k)$, and we have

$$M^{k+1}_{T,u} f.\chi_{C_k} = 0$$

This means that $f \circ T^{k+1}.\chi_{C_k} = f.\chi_{A_k} = 0$, which is a contradiction.

Therefore we get a sequence $\{A_k\}$ of sets with positive measure such that $A_k \subseteq T^k(B_k)$ for some $B_k \in \Sigma$ and $A_k \neq T^{k+1}(C_k)$ for all $C_k \in \Sigma$. □

Theorem 2.10. *If $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of measurable sets with positive measure such that for all $k \in \mathbb{N}$, $\mu(T^{-(k+1)}(A_k)) = 0$ but $\mu(T^{-k}(A_k)) \neq 0$, then $\alpha(M_{T,u}) = \infty$.*

Proof. Assume that $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of non-zero measurable sets such that for all $k \in \mathbb{N}$, $\mu(T^{-(k+1)}(A_k)) = 0$ and $\mu(T^{-k}(A_k)) \neq 0$. If we take $f_k = \chi_{A_k}$, it is obvious that $f_k \in L^{\Phi}(X)$ and $f_k \notin \mathcal{N}(M^k_{T,u})$ for $k \in \mathbb{N}$, since $f_k \circ T^k.\chi_{B_k} \neq 0$ for a non-zero measurable B_k such that $B_k \subseteq T^{-K}(A_k)$. But $f_k \circ T^{k+1} = \chi_{T^{-(k+1)}(A_k)} = 0$ a.e. Therefore for all $C \in \Sigma$ with $\mu(C) \neq 0$, $f_k \circ T^{k+1}.\chi_C = 0$. This means that $f_k \in \mathcal{N}(M^{k+1}_{T,u})$. So, we conclude that $\mathcal{N}(M^k_{T,u}) \subsetneq \mathcal{N}(M^{k+1}_{T,u})$ for all $k \in \mathbb{N}$. Thus $\alpha(M_{T,u}) = \infty$. □

An atom in a measure space (X, Σ, μ) is a set $A \in \Sigma$ with $\mu(A) > 0$ such that for every $B \in \Sigma$, if $B \subseteq A$, then we have either $\mu(B) = 0$ or $\mu(A) = \mu(B)$. Let A be an atom, since μ is σ-finite, it follows that $\mu(A) < \infty$. It is a well-known fact that every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as $X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint Σ-atoms and B, being disjoint from each A_n, is non-atomic [14]. If $B = \emptyset$, then we say that X is purely atomic measure space. For an arbitrary purely atomic measure space, we can assume, without loss of generality, that the measure space is of the form $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is counting measure, so that atoms can be considered as singletons $\{n\}$ with $n \in \mathbb{N}$.

Proposition 2.11. *Suppose $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be a measure space with the counting measure μ and $u : \mathbb{N} \longrightarrow \mathbb{R}$ be a function such that $S(u) = \mathbb{N}$. For a non-singular transformation $T : \mathbb{N} \longrightarrow \mathbb{N}$ and the operator $M_{T,u}$ on $L^{\Phi}(\mathbb{N})$, we have $\alpha(M_{T,u}) = \infty$ if and only if there exists a sequence $\{a_k\} \subset \mathbb{N}$ with $a_k \in T^k(\mathbb{N})$ and $a_k \notin T^{k+1}(\mathbb{N})$ for all $k \geq 1$.*

Proof. The proof is similar to the proof of Theorem 3.1. in [2], and we omit it. □

Our final results deal with infinite descent case.

Proposition 2.12. *Let $u : X \longrightarrow \mathbb{R}$ be a measurable function that is bounded away from zero. If $\delta(M_{T,u}) = \infty$, then the map $\theta_k : \mathcal{R}(T^k) \longrightarrow \mathbb{C} \times \mathcal{R}(T^k)$ is not one-to-one for all $k \geq 1$ where $\theta_k(T^k(x)) = (\prod_{i=1}^{k+1} u(T^i(x)), T^{k+1}(x))$ for all $x \in X$.*

Proof. Assume that $\delta(M_{T,u}) = \infty$. We show that θ_k is not injective for all $k \geq 1$. On the contrary, if θ_k is injective for some k, then for $f \in \mathcal{R}(M^k_{T,u})$, there is $g \in L^\Phi(X)$ such that $M^k_{T,u}g = f$. Now define h as follows

$$h(x) = \begin{cases} g(x)/u(x) & \text{if } x \in \mathcal{R}(T^k), \theta(x) = (\prod_{i=1}^{k+1} u(T^i(x)), x) \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $h \in L^\Phi(X)$. Note that

$$\begin{aligned} M^{k+1}_{T,u}h(x) &= \prod_{i=1}^{k+1} u(T^i(x)).(h \circ T^{k+1})(x) \\ &= \prod_{i=1}^{k} u(T^i(x)).(g \circ T^{k})(x) \\ &= f(x) \end{aligned}$$

Therefore $f \in \mathcal{R}(M^{k+1}_{T,u})$. In other words, $\mathcal{R}(M^{k+1}_{T,u}) = \mathcal{R}(M^k_{T,u})$. This implies that $\delta(M_{T,u}) < \infty$ which is a contradiction. □

A measure space (X, Σ, μ) is called separable if every two distinct points of X can be separated by two measurable sets A and B with positive measures such that $A \cap B = \emptyset$.

Proposition 2.13. *Let (X, Σ, μ) be a separable measure space. Then $\delta(M_{T,u}) = \infty$, if the map $\theta_k : \mathcal{R}(T^k) \longrightarrow \mathbb{C}\times\mathcal{R}(T^k)$ is not one-to-one for all $k \geq 1$, where $\theta_k(T^k(x)) = (\prod_{i=1}^{k+1} u(T^i(x)), T^{k+1}(x))$ for all $x \in X$.*

Proof. Suppose θ_k is not one-to-one for all $k \geq 1$. Therefore there exist a, b such that $a_1 = T^k(a) \neq T^k(b) = b_1 \in \mathcal{R}(T^k)$ and $\prod_{i=1}^{k+1} u(T^i(a)) = \prod_{i=1}^{k+1} u(T^i(b)), T^{k+1}(a) = T^{k+1}(b)$. Since (X, Σ, μ) is a separable measure space, then we can choose two measurable sets A and B with positive measure such that $a_1 \in A$, $b_1 \in B$ and $A \cap B = \emptyset$. Since (X, Σ, μ) is σ-finite, we can choose two measurable sets $A' \subseteq A$ and $B' \subseteq B$ such that $\mu(A') < 0, \mu(B') < 0$. Now, we define two measurable functions $f(x) = \chi_{A'} - \chi_{B'}$ and $g(x) = M^k_{T,u}(f(x)) = \prod_{i=1}^{k} u(T^i(x)).(f \circ T^k)(x)$. Since $M^k_{T,u}$ is bounded on $L^\Phi(X)$ and $f \in L^\Phi(X)$, we see that $g \in L^\Phi(X)$.

We claim that $g \notin \mathcal{R}(M^{k+1}_{T,u})$. Suppose, on the contrary, that there exists a non-zero function $h \in L^\Phi(X)$ with $M^{k+1}_{T,u}(h) = g$, then

$$\begin{aligned} \prod_{i=1}^{k} u(T^i(a)) &= g(a) \\ &= \prod_{i=1}^{k+1} u(T^i(a)).(h \circ T^{k+1})(a) \\ &= \prod_{i=1}^{k+1} u(T^i(b)).(h \circ T^{k+1})(b) \\ &= g(b) \\ &= -\prod_{i=1}^{k} u(T^i(b)) \end{aligned}$$

This is a contradiction and hence $g \notin \mathcal{R}(M^{k+1}_{T,u})$. This means that $\mathcal{R}(M^{k+1}_{T,u}) \subset \mathcal{R}(M^k_{T,u})$ for all $k \geq 1$ and $\delta(M_{T,u}) = \infty$ □

Finally we provide some examples to illustrate our main results.

Example 2.14. i) Let $X = [0,1]$, μ be the Lebesgue measure. If $T(x) = x/2$ and u is a non-zero arbitrary measurable function, then, by letting $A_n = (\frac{1}{2^n}, \frac{1}{2^{n-1}})$ in Theorem 2.10, we find that $\alpha(M_{T,u}) = \infty$, where $M_{T,u}$ is a bounded operator on $L^{\Phi}(X)$ for any Young function Φ. Also, if u is an injective measurable function on X, then by Proposition 2.13, $\delta(M_{T,u})$ is finite.

ii) Consider $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure and let $u : \mathbb{N} \longrightarrow \mathbb{R}$ be a function such that $S(u) = \mathbb{N}$. Define $T : \mathbb{N} \longrightarrow \mathbb{N}$ by $T(n) = n + 1$. Then $(k+1)_{k\in\mathbb{N}}$ is a sequence with the properties $k + 1 \in T^k(\mathbb{N})$ and $k + 1 \notin T^{k+1}(\mathbb{N})$ for each $k \in \mathbb{N}$. Therefor, by Theorem2.11, for all bounded operator $M_{T,u}$ on the Orlicz space $L^{\Phi}(\mathbb{N})$ with Young function Φ, $\alpha(M_{T,u}) = \infty$. If we take $T(n) = n$, then there is not any sequence $(a_k) \subset \mathbb{N}$ with $a_k \in T^k(\mathbb{N})$ and $a_k \notin T^{k+1}(\mathbb{N})$ for each $k \in \mathbb{N}$. Thus, by Theorem 2.11, $\alpha(M_{T,u})$ is finite.

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References

[1] Y. A. Abramovich, C. D. Aliprantis, An invitation to operator theory, Amer. Math. Soc., Providence, Rhode Island, 2002.
[2] H. Chandra, P. Kumar, Ascent and descent of composition operators on ℓ^p-spaces, Demonstratio Math., 43 (2010) 161–165.
[3] R. K. Giri, S. Pradhan, On the properties of ascent and descent of composition operators on Orlicz spaces, Math. Sci. Appl. E-Notes, 5 (2017) 70-76.
[4] R. K. Giri, D. Choudhuri, Composition operators on Orlicz-Sobolev spaces, J. Adv. Math. Stud., 10 (2017) 410-417.
[5] W. H. Hornor, Review on [8], Mathematical Reviews, MR 2403270 (2010b:47058).
[6] M. R. Jabbarzadeh, Weighted composition operators between L^p-spaces, Bull. Korean Math. Soc., 42 (2005) 369-378.
[7] M. A. Krasnosel'skii, Ya. B. Rutickii, Convex functions and Orlicz spaces, Noordhoff, Netherlands, 1961.
[8] R. Kumar, Ascent and descent of weighted composition operators of L^p-spaces, Mat. Vesnik, 60 (2008) 45–51.
[9] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Marcel Dekker, New York, 1991.
[10] R. K. Singh, J. S. Manhas, Composition operators on function spaces, North-Holland Mathematics Studies, 179, North-Holland Publishing Co., Amsterdam, 1993.
[11] A. E. Taylor, Theorems on ascent, descent, nullity and defect of linear operators, Math. Ann., 163 (1966) 18-49.
[12] A.E. Taylor, D. C. Lay, Introduction to functional analysis, 2nd ed., Wiley, New York, 1980.
[13] B. Yood, Ascent, descent and roots of Fredholm operators, Studia Math., 158 (2003) 219-226.
[14] A. C. Zaanen, Integration, 2nd ed., North-Holland, Amsterdam, 1967.