Filomat 32:8 (2018), 2701–2706
https://doi.org/10.2298/FIL1808701G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Amenability-like Properties of $C(X, A)$

S. Ghoraishi[a], A. Mahmoodi[a], A. R. Medghalchi[b]

[a]*Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.*
[b]*Faculty of Mathematical Sciences and Computer, Kharazmi University, 50 Taleghani Avenue, Tehran, Iran.*

Abstract. Let A be a Banach algebra and X be a compact Hausdorff space. For given homomorphisms $\sigma \in Hom(A)$ and $\tau \in Hom(C(X, A))$, we introduce homomorphisms $\tilde{\sigma} \in Hom(C(X, A))$ and $\tilde{\tau}_x \in Hom(A)$, where $x \in X$. We then study both $\tilde{\sigma}$-(weak) amenability of $C(X, A)$, and $\tilde{\tau}_x$-(weak) amenability of A.

1. Introduction

Let X be a compact Hausdorff space and let A be a Banach algebra. It is known that $C(X, A)$, the set of all A-valued continuous functions on X, is a Banach algebra with pointwise algebraic operations and the uniform norm $\|f\|_\infty := \sup_{x\in X} \|f(x)\|$, $f \in C(X, A)$ [5]. In the nice papers [3, 11], Ghamarshoushtari and Zhang studied amenability and weak amenability of $C(X, A)$. They showed that $C(X, A)$ is amenable if and only if A is amenable. Further, if A is commutative, they proved that $C(X, A)$ is weakly amenable if and only if A is weakly amenable.

Let A be a Banach algebra. We denote by $Hom(A)$ the space of all continuous homomorphisms from A into A. Let A be a Banach algebra, E be a Banach A-bimodule and let $\sigma \in Hom(A)$. From [6], we recall that a bounded linear map $D : A \longrightarrow E$ is a *σ-derivation* if $D(ab) = D(a) \cdot \sigma(b) + \sigma(a) \cdot D(b)$, for $a, b \in A$. A σ-derivation $D : A \longrightarrow E$ is *σ-inner derivation* if there exists $x \in E$ such that $D(a) = \sigma(a) \cdot x - x \cdot \sigma(a)$, for all $a \in A$.

Let A be a Banach algebra and let $\sigma \in Hom(A)$. The notion of σ-amenable Banach algebras was introduced and studied by Mirzavaziri and Moslehian in [7] (see also [8]). We say A is *σ-amenable* if for every Banach A-bimodule E, every σ-derivation $D : A \longrightarrow E^*$ is σ-inner. Especially, a Banach algebra A is *σ-weakly amenable* if every σ-derivation $D : A \longrightarrow A^*$ is σ-inner [9].

For a Banach algebra A, it is known that the projective tensor product $A\hat{\otimes}A$ is a Banach A-bimodule in a natural way. A bounded net $(u_\alpha) \subset A\hat{\otimes}A$ is a *σ-bounded approximate diagonal* for A if

$$\sigma(a) \cdot u_\alpha - u_\alpha \cdot \sigma(a) \longrightarrow 0, \quad \text{and} \quad \pi(u_\alpha)\sigma(a) \longrightarrow \sigma(a) \quad (a \in A),$$

where $\sigma \in Hom(A)$, and $\pi : A\hat{\otimes}A \longrightarrow A$ is the product map defined by $\pi(a \otimes b) = ab$.

Before preceding further, we set up our notations. Let X be a compact Hausdorff space and let A be a Banach algebra. For each $x \in X$, we consider the continuous epimorphism $T_x : C(X, A) \longrightarrow A$ defined by $T_x(f) := f(x)$, for all $f \in C(X, A)$. For every $a \in A$, we define $1_a \in C(X, A)$ by the formula $1_a(x) := a$, for

2010 *Mathematics Subject Classification*. 46M10; 46M18, 46H20.
Keywords. σ-amenability, σ-weak amenability, σ-bounded approximate diagonal.
Received: 07 June 2017; Accepted: 29 August 2017
Communicated by Mohammad Sal Moslehian
Email addresses: esm.ghoraishimadineh.sci@iauctb.ac.ir (S. Ghoraishi), a_mahmoodi@iauctb.ac.ir (A. Mahmoodi), a_medghalchi@khu.ac.ir (A. R. Medghalchi)

all $x \in X$. We notice that, every homomorphism $\sigma \in Hom(A)$ induces a homomorphism $\tilde{\sigma} \in Hom(C(X,A))$ defined by $\tilde{\sigma}(f) := \sigma f$, $f \in C(X,A)$. Conversely, for every homomorphism $\tau \in Hom(C(X,A))$ and $x \in X$, we introduce the map $\tilde{\tau}_x : A \longrightarrow A$ through $\tilde{\tau}_x(a) := T_x(\tau(1_a))$, $a \in A$. Since $1_{ab} = 1_a 1_b$, $\tilde{\tau}_x \in Hom(A)$. It is readily seen that $(T_x\tau)(1_a) = (\tilde{\tau}_x T_x)(1_a)$, for each $a \in A$. Next, for $f \in C(X)$ and $a \in A$ we define $fa \in C(X,A)$ via $fa(x) = f(x)a$ for each $x \in X$. Throughout the paper, we keep the above definitions and notations.

In this paper, motivated by [3, 6], we deal with amenability-like properties of the Banach algebra $C(X,A)$. Suppose that σ and τ are homomorphisms on A and $C(X,A)$, respectively. For a given σ-bounded approximate diagonal for A, we construct a $\tilde{\sigma}$-bounded approximate diagonal for $C(X,A)$ (Theorem 2.1). We show that under some certain conditions, τ-(weak) amenability of $C(X,A)$ implies $\tilde{\tau}_x$-(weak) amenability of A (Theorems 2.4 and 3.4). For a commutative Banach algebra A, we prove that σ-weak amenability of A yields $\tilde{\sigma}$-weak amenability of $C(X,A)$ in the presence of a bounded approximate identity for A (Theorem 3.2). Finally, we show that Theorem 3.2 is still true without the existence of a bounded approximate identity for A (Theorem 3.9).

2. σ-amenability

Suppose that A is a Banach algebra and X is a compact Hausdorff space. For $u = \sum_i u_i \otimes v_i \in C(X)\hat{\otimes}C(X)$ and $\alpha = \sum_j \alpha_j \otimes \beta_j \in A\hat{\otimes}A$, we consider

$$\Gamma(u,\alpha) = \sum_{i,j} u_i\alpha_j \otimes v_i\beta j \in C(X,A)\hat{\otimes}C(X,A)$$

so that $\|\Gamma(u,\alpha)\| \leq \|u\|\,\|\alpha\|$.

Theorem 2.1. *Let X be a compact Hausdorff space, A be a Banach algebra and let $\sigma \in Hom(A)$. If A has a σ-bounded approximate diagonal, then $C(X,A)$ has a $\tilde{\sigma}$-bounded approximate diagonal.*

Proof. We follow the standard argument in [3]. Let $(\alpha_v) \in A\hat{\otimes}A$ be a σ-bounded approximate diagonal for A such that $\|\alpha_v\|_p \leq M$ for all v. We claim that for any $\varepsilon > 0$ and any finite set $F \subset C(X,A)$, there is $U = U_{(F,\varepsilon)} \in C(X,A)\hat{\otimes}C(X,A)$ with $\|U\| \leq 2Mc$ such that

$$\|\tilde{\sigma}(a)\,.\,U - U\,.\,\tilde{\sigma}(a)\| < \varepsilon \quad \text{and} \quad \|\pi(U)\tilde{\sigma}(a) - \tilde{\sigma}(a)\| < \varepsilon \quad (a \in F)$$

where $c > 0$ is the constant asserted in [2, Corollary 1.2]. Given $\varepsilon > 0$ and a finite set $F \subset C(X,A)$. We first assume that each $a \in F$ is of the form of a finite sum $a = \sum_k f_k a_k$, with $f_k \in C(X)$ and $a_k \in A$. It is easy to check that $\tilde{\sigma}(a) = \sum_k f_k\sigma(a_k)$. We denote by F_A ($\subset A$) the finite set of all elements a_k associated to a for all $a \in F$, and by F_C ($\subset C(X)$) the finite set of all functions f_k associated to a for all $a \in F$. Let $N > 0$ be an integer that is greater than the number of the terms of $a = \sum_k f_k a_k$ for all $a \in F$, and set $L := \max\{\|\sigma(b)\|\,,\ \|f\| \ :\ b \in F_A, f \in F_C\}$. By the assumption, there is $\alpha \in (\alpha_v)$ such that

$$\|\sigma(b)\,.\,\alpha - \alpha\,.\,\sigma(b)\| < \frac{\varepsilon}{4cNL}, \quad \|\pi(\alpha)\,.\,\sigma(b) - \sigma(b)\| < \frac{\varepsilon}{NL} \quad (b \in F_A).$$

On the other hand, by the same argument as in the proof of [2, Theorem 2.1], we obtain an element $u \in C(X) \otimes C(X)$ with $\pi(u) = 1$ and $\|u\| \leq 2c$ for which

$$\|f\,.\,u - u\,.\,f\| < \frac{\varepsilon}{2\|\alpha\|LN} \quad (f \in F_C).$$

Putting $U := \Gamma(u, \alpha)$, for $a = \sum_k f_k a_k$ we see that

$$\begin{aligned}\|\tilde{\sigma}(a) \,.\, U - U \,.\, \tilde{\sigma}(a)\| &= \|\sum_k (\Gamma(f_k u, \sigma(a_k)\alpha) - \Gamma(u f_k, \alpha\sigma(a_k))\| \\ &= \|\sum_k \Gamma(f_k u, \sigma(a_k)\alpha - \alpha\sigma(a_k)) + \Gamma(f_k u - u f_k, \alpha\sigma(a_k))\| \\ &\leq \sum_k (L\|u\| \, \|\sigma(a_k)\alpha - \alpha\sigma(a_k)\| + L\|\alpha\| \, \|f_k u - u f_k\|) \\ &\leq NL(2c\frac{\varepsilon}{4cNL} + \|\alpha\|\frac{\varepsilon}{2\|\alpha\|_P LN}) = \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|\pi(U)\tilde{\sigma}(a) - \tilde{\sigma}(a)\| = \|\pi(u)\pi(\alpha)\tilde{\sigma}(a) - \tilde{\sigma}(a)\| &= \|\sum_k f_k(\pi(\alpha)\sigma(a_k) - \sigma(a_k))\| \\ \leq L\sum_k \|\pi(\alpha)\sigma(a_k) - \sigma(a_k)\| < NL\frac{\varepsilon}{NL} = \varepsilon.\end{aligned}$$

Now, we assume that $F \subset C(X, A)$ is an arbitrary finite set. From the proof of [2, Theorem 2.1], we know that for each $a \in F$ there exists an element $a_\varepsilon = \sum_k f_k a_k$ where the right side of a_ε is a finite sum, $f_k \in C(X)$ and $a_k \in A$ such that $\|\tilde{\sigma}(a) - \tilde{\sigma}(a_\varepsilon)\| < \min\{\frac{\varepsilon}{4}, \frac{\varepsilon}{8Mc}\}$. Applying the above argument for the finite set $F_\varepsilon := \{a_\varepsilon : a \in F\}$, we get $U \in C(X, A)\hat{\otimes}C(X, A)$ such that $\|U\| \leq 2cM$ and

$$\|\tilde{\sigma}(a_\varepsilon) \,.\, U - U \,.\, \tilde{\sigma}(a_\varepsilon)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\pi(U)\tilde{\sigma}(a_\varepsilon) - \tilde{\sigma}(a_\varepsilon)\| < \frac{\varepsilon}{2} \quad (a_\varepsilon \in F_\varepsilon).$$

Therefore

$$\|\tilde{\sigma}(a) \,.\, U - U \,.\, \tilde{\sigma}(a)\| < \varepsilon \quad \text{and} \quad \|\pi(U)\tilde{\sigma}(a) - \tilde{\sigma}(a)\| < \varepsilon \quad (a \in F)$$

so that the claim is proved. Finally, the net $(U_{F,\varepsilon})$ with the natural partial order $(F_1, \varepsilon_1) < (F_2, \varepsilon_2)$ if and only if $F_1 \subset F_2$ and $\varepsilon_1 \geq \varepsilon_2$ is the desired $\tilde{\sigma}$-approximate diagonal for $C(X, A)$. □

Remark 2.2. *To our knowledge, we do not know whether or not the existence of σ-bounded approximate diagonal is equivalent to σ-amenability. Hence, we can not prove or disprove if σ-amenability of A implies $\tilde{\sigma}$-amenability of $C(X, A)$.*

Proposition 2.3. *Suppose that $\sigma \in Hom(A)$ and $\tau \in Hom(B)$, where A and B are Banach algebras. Suppose that $\phi : A \longrightarrow B$ is a continuous homomorphism with a dense range and $\tau\phi = \phi\sigma$. If A is σ-amenable, then B is τ-amenable.*

Proof. We may either prove it or else look at [8, Proposition 3.3]. □

Theorem 2.4. *Let X be a compact Hausdorff space, let A be a Banach algebra and let $\tau \in Hom(C(X, A))$ such that $T_{x_0}\tau = \tilde{\tau}_{x_0}T_{x_0}$, for some $x_0 \in X$. If $C(X, A)$ is τ-amenable, then A is $\tilde{\tau}_{x_0}$-amenable.*

Proof. We have already seen that the map T_{x_0} is surjective, so this is immediate by Proposition 2.3. □

We note that homomorphisms $\tau \in Hom(C(X, A))$ satisfying the condition of Theorem 2.4 exist in abundance. Take an arbitrary homomorphism $\eta \in Hom(C(X, A))$ and define the map $\tau : C(X, A) \longrightarrow C(X, A)$ by

$$\tau(f)(x) := \tau(1_{f(x)})(x) := \eta(1_{f(x)})(x) \quad (f \in C(X, A), x \in X)\,.$$

Then, we may check that $\tau \in Hom(C(X, A))$ and $T_x\tau = \tilde{\tau}_x T_x$ for all $x \in X$.

3. σ-weak amenability

For a Banach algebra A and a homomorphism σ on A, we write $Z^1_\sigma(A, E)$ for the set of all σ-derivations from A into a Banach A-bimodule E.

Let A be a σ-weakly amenable Banach algebra, and $\sigma \in Hom(A)$ such that $\sigma(A)$ is a dense subset of A. A more or less verbatim of the classic argument, shows that $\overline{A^2} = A$ (see [6, Theorem 6] for details).

Proposition 3.1. *Let A be a commutative Banach algebra and let $\sigma \in Hom(A)$ with a dense range. If A is σ-weakly amenable, then $Z^1_\sigma(A, E) = 0$ for each Banach A-module E.*

Proof. Assume towards a contradiction that there is a nonzero element $D \in Z^1_\sigma(A, E)$. Since $\overline{A^2} = A$, there exists $a_0 \in A$ with $D(a_0^2) \neq 0$. Hence $\sigma(a_0) \cdot Da_0 \neq 0$ and so there exists $\lambda \in E^*$ with $\langle \sigma(a_0) \cdot Da_0, \lambda \rangle = 1$. Consider $R_\lambda \in {}_AB(E, A^*)$ such that $\langle a, R_\lambda x\rangle = \langle a \cdot x, \lambda\rangle$, for $a \in A$ and $x \in E$ [2, Proposition 2.6.6]. It is not hard to see that $R_\lambda \circ D \in Z^1_\sigma(A, A^*)$. Then we have

$$\langle \sigma(a_0), (R_\lambda \circ D)(a_0)\rangle = \langle \sigma(a_0) \cdot Da_0, \lambda\rangle = 1$$

so that $R_\lambda \circ D \neq 0$, a contradiction of the fact that A is σ-weakly amenable. □

Proposition 3.2. *Let X be a compact Hausdorff space and let A be a commutative Banach algebra with a bounded net (e_v) for which $(\sigma(e_v))$ is a bounded approximate identity for A, where σ belongs to $Hom(A)$ with a dense range. If A is σ-weakly amenable, then $C(X, A)$ is $\tilde{\sigma}$-weakly amenable.*

Proof. Using the map $a \longmapsto 1_a$, we may consider A as a closed subalgebra of the commutative Banach algebra $C(X, A)$. Suppose that $D : C(X, A) \longrightarrow C(X, A)^*$ is a $\tilde{\sigma}$-derivation. Notice that $C(X, A)$ is naturally a commutative A-bimodule with actions $a \,.\, f(x) = f \,.\, a(x) := af(x)$, $(a \in A, f \in C(X, A), x \in X)$. We also note that $\tilde{\sigma} = \sigma$ on A. Therefore $D\,|_A \colon A \longrightarrow C(X, A)^*$ is a σ-derivation and then, by Proposition 3.1, $D\,|_A = 0$. On the other hand, $(\sigma(e_v))$ is also a bounded approximate identity for $C(X, A)$ and $wk^*\text{-}\lim D(\sigma(e_v)) = 0$. An argument similar to that in the proof of [2, Proposition 4.1] shows that $wk^*\text{-}\lim D(f\sigma(e_v))$ exists for each $f \in C(X)$. So we may define $\tilde{D} : C(X) \longrightarrow C(X, A)^*$ via $\tilde{D}(f) = wk^* - \lim_v D(f\sigma(e_v))$. Clearly $C(X, A)$ is a commutative $C(X)$-bimodule. Then

$$\begin{aligned} D(fg\sigma(e_v)) &= wk^* - \lim D(fg\sigma(e_\mu)\sigma(e_v))) = wk^* - \lim D(f\sigma(e_\mu)g\sigma(e_v))) \\ &= wk^* - \lim D(f\sigma(e_\mu)) \,.\, \tilde{\sigma}(g\sigma(e_v)) + \tilde{\sigma}(f\sigma(e_\mu)) \,.\, D(g\sigma(e_v)) \\ &= \tilde{D}(f) \,.\, \tilde{\sigma}(g\sigma(e_v)) + f \,.\, D(g\sigma(e_v)) \end{aligned}$$

for $f, g \in C(X)$. Taking wk^*-limit in v, we get

$$\tilde{D}(fg) = \tilde{D}(f) \,.\, g + f \,.\, \tilde{D}(g) \quad (f, g \in C(X)).$$

Hence $\tilde{D}$ is a derivation on amenable $C(X)$. Therefore $\tilde{D} = 0$, and then

$$D(f\sigma(a)) = \tilde{D}(f) \cdot \sigma(a) + f \cdot \tilde{D}(\sigma(a)) = 0 \quad (a \in A, f \in C(X)).$$

Whence $D = 0$ on the linear span of $\{f\sigma(a) : f \in C(X), a \in A\}$ which is dense in the linear span of $\{fa : f \in C(X), a \in A\}$, by the density of range of σ. The latter is itself dense in $C(X, A)$ by [2], so that $D = 0$ on the whole $C(X, A)$. □

In Proposition 3.2, as an special case, we may suppose that (e_v) is itself a bounded approximate identity for A. Indeed if (e_v) is a bounded approximate identity for A, then the density of the range of σ shows that $(\sigma(e_v))$ is still a bounded approximate identity for A.

The following was proved in [9, Proposition 18], and so we omit its proof.

Proposition 3.3. *Suppose that $\sigma \in Hom(A)$ and $\tau \in Hom(B)$, where A and B are Banach algebras such that A is commutative and σ has a dense range. Suppose that $\phi : A \longrightarrow B$ is a continuous epimorphism for which $\tau\phi = \phi\sigma$. If A is σ-weakly amenable, then B is τ-weakly amenable.*

The following is the converse of Proposition 3.2.

Theorem 3.4. *Let X be a compact Hausdorff space, let A be a commutative Banach algebra and let $\tau \in Hom(C(X, A))$ with a dense range such that $T_{x_0}\tau = \tilde{\tau}_{x_0}T_{x_0}$, for some $x_0 \in X$. If $C(X, A)$ is τ-weakly amenable, then A is $\tilde{\tau}_{x_0}$-weakly amenable.*

Proof. We use Proposition 3.3. □

We extend [1, Theorem 1.8.4] as follows, where the proof reads somehow the same lines.

Proposition 3.5. *Let I be an ideal in a commutative algebra A, E be an A-module and $D : I \longrightarrow E$ be a σ-derivation. Then the map*

$$\tilde{D} : I \times A \longrightarrow E, \quad \tilde{D}(a, b) := D(ab) - \sigma(b) \cdot D(a)$$

is bilinear such that

(i) $\tilde{D}(a, b) = \sigma(a) \cdot D(b) \quad (a, b \in I)$;

(ii) for each $a \in I^2$, the map $b \longmapsto \tilde{D}(a, b)$, $A \longrightarrow E$ is a σ-derivation.

Proof. We only prove the clause (ii). For $a_1, a_2 \in I$, and $b_1, b_2 \in A$, we have

$$\begin{aligned}
\tilde{D}(a_1a_2, b_1b_2) &= D(a_1a_2b_1b_2) - \sigma(b_1b_2) \cdot D(a_1a_2)\\
&= \sigma(a_1b_1) \cdot D(a_2b_2) + \sigma(a_2b_2) \cdot D(a_1b_1) - \sigma(b_1b_2) \cdot D(a_1a_2)\\
&= \sigma(b_1) \cdot [\sigma(a_1) \cdot D(a_2b_2) - \sigma(b_2)\sigma(a_1) \cdot D(a_2)]\\
&+ \sigma(b_2) \cdot [\sigma(b_1a_1) \cdot D(a_2) + \sigma(a_2) \cdot D(a_1b_1) - \sigma(b_1) \cdot D(a_1a_2)]\\
&= \sigma(b_1) \cdot [D(a_1a_2b_2) - \sigma(a_2b_2) \cdot D(a_1) - \sigma(b_2a_1) \cdot D(a_2)]\\
&+ \sigma(b_2) \cdot [D(a_1a_2b_1) - \sigma(b_1) \cdot D(a_1a_2)]\\
&= \sigma(b_1) \cdot [D(a_1a_2b_2) - \sigma(b_2) \cdot D(a_1a_2)]\\
&+ \sigma(b_2) \cdot [D(a_1a_2b_1) - \sigma(b_1) \cdot D(a_1a_2)]\\
&= \sigma(b_1) \cdot \tilde{D}(a_1a_2, b_2) + \sigma(b_2) \cdot \tilde{D}(a_1a_2, b_1)
\end{aligned}$$

as required. □

The following is an analogue of [1, Lemma 2.8.68].

Lemma 3.6. *Let A be a σ-weakly amenable commutative Banach algebra, I be a closed ideal in A, and E be a Banach A-module. Take $\sigma \in Hom(A)$ with a dense range such that $\sigma(I) \subseteq I$. Then $D\mid_{I^4} = 0$ for each $D \in Z^1_\sigma(I, E)$.*

Proof. We first observe that $F := {}_A\mathcal{B}(I, E)$ is a Banach A-module for the action $(a \cdot T)(b) = T(ab)$, $(a \in A, b \in I)$. Then we notice that the map $j : E \longrightarrow F$ with $j(x)(a) = a \cdot x$, $(a \in I, x \in E)$ belongs to ${}_A\mathcal{B}(E, F)$. Whence $j \circ D \in Z^1_\sigma(I, E)$. Clearly the map

$$\tilde{D} : I \times A \to F, \quad (a, b) \longmapsto (j \circ D)(a, b) - \sigma(b) \cdot (j \circ D)(a)$$

is bilinear. Therefore $\tilde{D}(a, b) = \sigma(a) \cdot (j \circ D)(b)$, by Proposition 3.5 (i), for $a, b \in I$. Take $a \in I^2$. By Proposition 3.5 (ii), the map $A \longrightarrow F$, $b \longmapsto \tilde{D}(a, b)$, is a σ-derivation. It follows from Proposition 3.1, that this map is zero and so $\tilde{D}(I^2 \times A) = 0$. Hence, for $a \in I^2$ and $b, c \in I$, we have

$$\sigma(a)c \cdot D(b) = (j \circ D)(b)(\sigma(a)c) = (\sigma(a) \cdot (j \circ D)(b))(c) = \tilde{D}(a, b)(c) = 0.$$

In particular, choosing $c \in \sigma(I)$, we may see that $\sigma(I^3) \cdot D(I) = 0$, and whence $D \mid I^4 = 0$. □

Proposition 3.7. *Let A be a σ-weakly amenable commutative Banach algebra, I be a closed ideal in A, and $\sigma \in Hom(A)$ with a dense range such that $\overline{\sigma(I)} = I$. Then I is σ-weakly amenable if and only if $\overline{I^2} = I$.*

Proof. If I is σ-weakly amenable, then $\overline{I^2} = I$ by [9, Theorem 6].

Conversely if $\overline{I^2} = I$, then $\overline{I^4} = I$. Suppose that $D \in Z^1_\sigma(I, I^*)$. Then by Lemma 3.6, $D\,|_{I^4} = 0$ so that $D = 0$. Hence I is σ-weakly amenable. □

Lemma 3.8. *Let X be a compact Hausdorff space, let A be a Banach algebra and let $\sigma \in Hom(A)$. If σ has a dense range, then $\tilde{\sigma}$ has a dense range as well.*

Proof. Take $f \in C(X)$ and $a \in A$ and put $h := fa \in C(X, A)$. By the assumption, there exists a sequence $(a_n) \subseteq A$ such that $\lim_n \sigma(a_n) = a$. Define $h_n := fa_n$, $(n = 1, 2, ...)$. Then it is easy to verify that $\lim_n \tilde{\sigma}(h_n) = h$. This completes the proof, since the set of all linear combinations of elements of $C(X, A)$ of the form fa $(f \in C(X), a \in A)$ is dense in $C(X, A)$ [2]. □

We recall that a homomorphism $\sigma \in Hom(A)$ is extended to a homomorphism $\sigma^\sharp \in Hom(A^\sharp)$ through $\sigma^\sharp(a + \lambda e) = \sigma(a) + \lambda e$ $(a \in A, \lambda \in \mathbb{C})$, where e is the identity of $A^\sharp$, the unitization of A.

Now, we are ready to prove our last goal.

Theorem 3.9. *Let X be a compact Hausdorff space, let A be a commutative Banach algebra and let $\sigma \in Hom(A)$ with a dense range. If A is σ-weakly amenable, then $C(X, A)$ is $\tilde{\sigma}$-weakly amenable.*

Proof. Suppose that A is σ-weakly amenable. By [6, Theorem 12], $A^\sharp$ is $\sigma^\sharp$-weakly amenable. Applying Proposition 3.2, we see that $C(X, A^\sharp)$ is $\widetilde{\sigma^\sharp}$-weakly amenable. Our assumptions together with [6, Theorem 6], imply that A^2 is dense in A. We learn from the proof of [8, Theorem 1] that $C(X, A)^2$ is dense in $C(X, A)$, and also $C(X, A)$ is a closed ideal of $C(X, A^\sharp)$. Next, it is easy to verify that $\widetilde{\sigma^\sharp}(f) = \tilde{\sigma}(f)$, for all $f \in C(X, A)$. Hence

$$\overline{\widetilde{\sigma^\sharp}(C(X, A))} = \overline{\tilde{\sigma}(C(X, A))} = C(X, A)$$

by Lemma 3.8. Now, an application of Proposition 3.7 yields that $C(X, A)$ is $\tilde{\sigma}$-weakly amenable. □

References

[1] A. Bodaghi, M. Eshaghi Gordji, A. R. Medghalchi, A generalization of the weak amenability of Banach algebras, *Banach J. Math. Anal.* **3**, no. 1, (2009), 131-142.

[2] H. G. Dales, Banach Algebras and Automatic continuity, *Clarendon Press*, Oxford, 2000.

[3] R. Ghamarshoushtari, Y. Zhang, Amenability properties of Banach algebra valued continuous functions, *J. Math. Anal. Appl.* **422** (2015), 1335-1341.

[4] B. E. Johnson, Cohomology in Banach Algebras. *Mem. Amer. Math. Soc.* **127**, 1972.

[5] L. Kaplansky, Topological rings, *Amer. J. Math.* **69** (1947), 153-183.

[6] M. Mirzavaziri, M. S. Moslehian, σ-Derivations in Banach algebras, *Bull. Iranian Math. Soc.* **32** (1) (2006), 65-78.

[7] M. Mirzavaziri, M. S. Moslehian, σ-amenability of Banach algebras, *Southeast Asian Bull. Math.* **33** (2009), 89-99.

[8] M. Moslehian, A. N. Motlagh, Some notes on (σ, τ)-amenability of Banach Algebras, *Stud. Univ. Babes-Bolyai Math.* **53** (3) (2008), 57-68.

[9] T. Yazdanpanah, I. Moazzami Zadeh, σ-weak Amenability of Banach Algebras. *Int. J. Nonlinear Anal. Appl.* **4**, no. 1, (2013), 66-73.

[10] T. Yazdanpanah and H. Najafi, σ-contractible and σ-biprojective Banach algebras, *Quaestiones Math.* **33** (2010), 485-495.

[11] Y. Zhang, Addendum to "Amenability properties of Banach algebra valued continuous functions", *J. Math. Anal. Appl.* **431** (2015), 702-703.