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Hedging Error Estimate of the American Put Option Problem in Jump-Diffusion Processes

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Abstract. We consider discrete time hedging error of the American put option in case of brusque fluctuations in the price of assets. Since continuous time hedging is not possible in practice so we consider discrete time hedging process. We show that if the proportions of jump sizes in the asset price are identically distributed independent random variables having finite moments then the value process of the discrete time hedging uniformly approximates the value process of the corresponding continuous-time hedging in the sense of L_1 and L_2-norms under the real world probability measure.

1. Introduction

Merton [20] considered incomplete markets in order to introduce jump-diffusion process in the theory of option valuation and formulated pricing formulas for European kind options assuming that the jump risk is un-priced (in that setup the sample path is not continuous). Merton work [20] was later generalized by several authors, for examples Naik and Lee [21] analyzed the option price where the underlying asset is the market portfolio with discontinuous returns, Huyôn Pham [15] studied the pricing of the American put option by applying the free-boundary approach while Van Moerbeke [27] converted the American option pricing and analysed into free boundary problem in the case of diffusion model. Bayraktar and Xing [2] studied and approximated the price of the American put for jump diffusions by a sequence of functions, which converges exponentially to the price function uniformly. Hussain et al. [12] investigated regularity properties of value process of the American option problem in the jump-diffusion case.

It is shown in Shreve [25] that construction of perfect hedging of the American style options strongly depends on the first order derivative of the corresponding value function. As there does not exist perfect valuation of the American style options, so there is no perfect hedging strategy. Moreover, perfect hedging requires continuous time trading on financial markets which is not possible in practice. One can trade discrete times only. In this regards, Hussain and Shashiashvili [13] constructed discrete time hedging strategy of the American put option problem. In this setting the underlying stock neither pays dividends

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nor having brusque variations in its price. Hussain and Rehman [14] extended the latter work to American style option to the dividends paying stock.

Financial markets are full of uncertainties which make the stock price discontinuous like release of unexpected economic figure, political changes or natural disasters etc may lead to brusque changes in asset prices. In this case the stock market is incomplete and therefore, there is no perfect hedging strategy to hedge the underlying option. However, many authors studied some admissible delta hedging strategies (see for example [19]). To hedge the jump risk, He et al. [11] explored two different hedging strategies: a semi-static approach which uses a portfolio of the underlying and traded short maturity options to hedge a long maturity option, and a dynamic technique which involves frequent trading of options and the underlying. Kennedy et al. [18] study that, in the case of incomplete market, hedging a contingent claim written on the asset is not a trivial matter, and other instruments besides the underlying must be used to hedge in order to provide adequate protection against jump risk. They devised a dynamic hedging strategy that uses a hedge portfolio consisting of the underlying asset and liquidly traded options, where transaction costs are assumed present due to a relative bid-ask spread. In our knowledge, probably no body has investigated hedging error of the American options in discrete time setting under this phenomenon.

In this article , we consider American put option on a stock the price of which has brusque variations i.e., American put option in jump-diffusion process and investigate the corresponding discrete time hedging error. Our main result shows that the value process in discrete time setting converges uniformly to the continuous-time setting in the sense of L_1 and L_2-norms under the real world probability measure.

For further details, we refer the readers to Salman [23].

2. Formulation of Main Problem and some Preliminary Results

We consider the American put option in jump diffusion processes in a financial market which is incomplete so we have brusque variations in the assets price and focus on the delta hedging which in this framework is not perfect however used by practitioners. We divide the time interval $[0, T]$, where T is the time to maturity of the corresponding option, into n equal parts and denote by $\delta = \frac{T}{n}$. We consider the value process of the discrete time delta hedging strategy minimising the risk at maturity in jump-diffusion process and, using purely probabilistic approach, we estimate the discrete-time hedging error and find that it is proportional to the square root of $\delta \log \frac{T}{\delta}$.

In an incomplete financial market, we consider the probability space $(\Omega, \mathcal{F}, P)$ and define the Wiener Process $B = (B_t)$, Poisson Process $N = (N_t)$ with parameter λ and$(U_i)_{i\geq 1}$ a sequence of iid random variables taking values in the open interval $(-1, \infty)$ with common density $f(z)$, average expected value β with the assumption that the mathematical expectations $E(U_i^m) < \infty$, for all $i, m \in \mathbb{N}$. We take time T to be finite and $\sigma-$fields of (B_t), (N_t), (U_i) to be independent.

We make the $P-$completion of our natural filtration of (B_t), (N_t) and $(U_i)I_{i\leq N_t}$, $i \geq 1$ by $(\mathcal{F}_t)$. On the so formed filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$,let us consider a financial market having assets: $(M_t, Q_t, 0 \leq t \leq T)$ a bank account and the value of a stock respectively with jumps in the proportions $U_1, U_2, ...,$ at the random times. We also suppose that τ_i,s represent the jump timings of (N_t).

The assets M_t and Q_t evolve according to following ordinary and stochastic differential equations

$$dM_t = rM_t dt,\ M_0 = 1,\ \ 0 \leq t \leq T, \tag{1}$$

$$dQ_t = Q_{t-}\left(bdt + \sigma dB_t + d\left(\sum_{i=1}^{N_t} U_i\right)\right), Q_0 > 0, 0 \leq t \leq T, \tag{2}$$

where the drift term b, the interest rate r and the stock volatility σ are positive constants and Q_{t-} is the left-hand limit of Q_t at time t.

From (2), unique solution of the stock price Q_t is given by

$$Q_t = Q_0\left(\prod_{i=1}^{N_t}(1 + U_i)\right)e^{(b-\frac{\sigma^2}{2})t+\sigma B_t}. \tag{3}$$

Under the assumption that $E|U_1| < \infty$, where E stands for the mathematical expectation, the discounted stock price $\tilde{Q}_t = e^{-rt}Q_t$ is a martingale (see, Lamberton and Lapeyre [19]) if

$$b = r - \lambda E(U_1). \tag{4}$$

In the corresponding framework, let $v(t,x), 0 \le t \le T, x \ge 0$ denotes arbitrary price function (as the market is incomplete, so there is no unique price of the put American option under the physical probability measure) of the American put option then, from Lamberton and Lapeyre [19], an admissible strategy minimising the risk at maturity the writer of the option should hold

$$\triangle(t) = \frac{1}{\sigma^2 + \lambda E(U_1^2)}\left[\sigma^2 \frac{\partial v}{\partial x}(t,x) + \lambda \int \frac{z}{x}\Big(v(t, x(1+z)) - v(t,x)\Big)\, df(z)\right] \tag{5}$$

number of stocks at time t, where $f(z)$ is the law of random variables U_i.

The above admissible delta hedging strategy is replicable (that is the payoff at maturity time will be equal to the final value of the delta hedging with P a.s. (see Lamberton and Lapyere [19] Chapter 7)) however, it requires continuous time trading on the financial market. No one can trade continuously in time.

Practically, the discrete time strategy defined through $\triangle(t)$ is the following:

$$\triangle_\delta(t) = \triangle(t_{k-1}), t_{k-1} \le t < t_k, k = 1, 2, ...n, \tag{6}$$

where $\delta = \frac{T}{n}, t_k = k \cdot \delta$, the subscript δ indicates the error of approximation.

The aim here is to derive an estimate for the error due to the fact that the portfolio is re-adjusted at discrete non-random dates (say $k\delta$, $k = 0, 1, ..., n$ with $\delta = \frac{k}{n}$). Let us denote by $\Pi_\delta(t), 0 \le t \le T$, the value of the portfolio corresponds to the discrete time strategy $\triangle_\delta(t)$ at time t defined in (6). Assume the hedger of the option starts with initial amount $v(0, Q_0) = \pi(0)$ and adjusts his portfolio at each time $t_k, k = 0, 1, ..., n-1$. He holds $\triangle_\delta(t_k)$ number of shares from stock in time interval $[t_k, t_{k+1})$ and invest the remainder $\pi_\delta(t_k) - \triangle_\delta(t_k)Q_{t_k}$ in money market account at time t_k. The value process $\Pi_\delta(t)$ can be expressed as

$$\pi_\delta(t) = e^{rt}\left[\pi(0) + \int_0^t \triangle_\delta(u) d\tilde{Q}_u\right], 0 \le t \le T. \tag{7}$$

Similar representation is valid for self-financing continuous time portfolio value process $\pi(t)$

$$\pi(t) = e^{rt}\left[\pi(0) + \int_0^t \triangle(u) d\tilde{Q}_u\right]. \tag{8}$$

Discrete time error of American option for admissible strategy (5) is then given as

$$E \sup_{0\le t\le T} |\pi(t) - \pi_\delta(t)| = E \sup_{0\le t\le T} e^{rt}\left|\int_0^t (\triangle(u) - \triangle_\delta(u))\, d\tilde{Q}_u\right|, \tag{9}$$

where E represents the mathematical expectation with respect to the real world probability measure P.

To estimate the above discrete-time error, we have to pass on to the new probability measure $\widetilde{P}$. Let $\tilde{f}(y)$ be a density function such that $\tilde{f}(y) = 0$ whenever $f(y) = 0$ and define the process

$$Z_t = e^{-\theta B_t - \frac{1}{2}\theta^2 t} e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\tilde{\lambda}\tilde{f}(U_i)}{\lambda f(U_i)}, 0 \le t \le T, \tag{10}$$

where $\theta = \frac{b - r + \tilde{\lambda}\tilde{\beta}}{\sigma}$ is the market price of risk, $\tilde{\lambda}$ is the intensity of N_t, $\tilde{\beta}$ is the average value and $\tilde{f}(y)$ is the common density of $U_1, U_2, ...$ with $\widetilde{P}$, where

$$\widetilde{P}(C) = \int_C Z_T dP, \text{ for all } C \in \mathcal{F}_T.$$

We will denote by $\widetilde{E}$ the mathematical expectation with respect to new probability measure $\widetilde{P}$ and come to main result:

Theorem 2.1. *Assume $f(y)$ is the common density under the real world probability measure P of the sequence of random variables $U_1, U_2, ...$ and $\tilde{f}(y)$ under the new measure $\widetilde{P}$. The process*

$$Y_t = e^{2(\tilde{\lambda}-\lambda)t} \prod_{i=1}^{N_t} \left(\frac{\lambda f(U_i)}{\tilde{\lambda} \tilde{f}(U_i)} \right)^2, 0 \le t \le T \tag{11}$$

is a martingale under $\widetilde{P}$ if

$$\mathcal{L} \equiv \int_{-\infty}^{\infty} \frac{f^2(u)}{\tilde{f}(u)} du = \frac{\tilde{\lambda}^2}{\lambda^2}(2\lambda - 3\tilde{\lambda}), \tag{12}$$

and in particular, $\widetilde{E} Y_t = 1$, for all $t \ge 0$.

Proof. In the proof, we follow the book by Shreve [25]. Let us define the pure jump process J_t, the compensated Poisson processes H_t and R_t as

$$J_t = \prod_{i=1}^{N_t} \left(\frac{\lambda f(U_i)}{\tilde{\lambda} \tilde{f}(U_i)} \right)^2, \quad H_t = \sum_{i=1}^{N_t} \left(\frac{\lambda f(U_i)}{\tilde{\lambda} \tilde{f}(U_i)} \right)^2, \quad R_t = \sum_{i=1}^{N_t} U_i, \quad 0 \le t \le T.$$

As $\triangle R_t \equiv R_t - R_{t-} = U_{N_t}, 0 \le t \le T$, at the jump times of R, we have

$$\triangle J_t = \left[\left(\frac{\lambda f(\triangle R_t)}{\tilde{\lambda} \tilde{f}(\triangle R_t)} \right)^2 - 1 \right] J_{t-}, \quad \triangle H_t = \left(\frac{\lambda f(\triangle R_t)}{\tilde{\lambda} \tilde{f}(\triangle R_t)} \right)^2, \quad 0 \le t \le T. \tag{13}$$

Let us calculate

$$\begin{aligned} \widetilde{E} \left(\frac{\lambda f(U_i)}{\tilde{\lambda} \tilde{f}(U_i)} \right)^2 &= \left(\frac{\lambda}{\tilde{\lambda}} \right)^2 \int_{-\infty}^{\infty} \frac{f^2(u)}{\tilde{f}^2(u)} \tilde{f}(u) du \\ &= \left(\frac{\lambda}{\tilde{\lambda}} \right)^2 \mathcal{L}. \end{aligned} \tag{14}$$

It is clear (see, for example, Shreve [25] Theorem 11.3.1) that the compensated Poisson process

$$H_t - \frac{\lambda^2}{\tilde{\lambda}^2} \mathcal{L} t \tag{15}$$

is a martingale with $\widetilde{P}$.

Note that if there is no jump at time t, then $N_t = N_{t-}$ which further gives

$$\triangle N_t = 0, \ \triangle J_t = 0 \text{ and } \ \triangle H_t = 0,$$

and in the case there is a jump at t then $\triangle N_t = 1$ and therefore, we can write

$$\triangle J_t = J_t - J_{t-} \triangle N_t.$$

Thus from (13), we can write

$$\triangle J_t = J_{t-} \triangle H_t - J_{t-} \triangle N_t, \quad 0 \le t \le T.$$

Since J_t, H_t and N_t are all pure jump process, therefore the latter expression can be written as

$$dJ_t = J_{t-} dH_t - J_{t-} dN_t, \quad 0 \le t \le T. \tag{16}$$

Using Itô's product rule for the jump process $\prod_{i=1}^{N_t}\left(\frac{\lambda f(U_i)}{\tilde{\lambda}\tilde{f}(U_i)}\right)^2$ and continuous process $e^{2(\tilde{\lambda}-\lambda)t}$, we express

$$\begin{aligned} Y_t &= Y_0 + 2(\tilde{\lambda}-\lambda)\int_0^t J_{s-}e^{2(\tilde{\lambda}-\lambda)s}ds + \int_0^t e^{2(\tilde{\lambda}-\lambda)s}dJ_s \\ &= 1 - \int_0^t e^{2(\tilde{\lambda}-\lambda)s}J_{s-}d(N_s - \tilde{\lambda}s) + \int_0^t e^{2(\tilde{\lambda}-\lambda)s}J_{s-}d(H_s - 2\lambda s + 3\tilde{\lambda}s), \end{aligned} \tag{17}$$

where we have used equation (16).

From expressions (15) and (17), it is clear that Y_t is a martingale if

$$H_t - \frac{\lambda^2}{\tilde{\lambda}^2}\mathcal{L}t = H_t - 2\lambda t + 3\tilde{\lambda}t,\ 0 \le t \le T,$$

from where we get the required value of $\mathcal{L}$.

Moreover, because $Z(0) = 1$ and $Z(t)$ is a martingale, so we have $\widetilde{E}Z(t) = 1$ for all $t \ge 0$ under the condition (12). □

Let us define a new Brownian motion $W_t = B_t + \theta t, 0 \le t \le T$. By Girsanov theorem W_t is a Wiener Process with $\widetilde{P}$ with the same $\mathcal{F}_t$. Equation (2) is transformed as

$$dQ_t = Q_{t-}\left[(rdt + \sigma dW_t) + d\left(R_t - \tilde{\lambda}\tilde{\beta}t\right)\right], \tag{18}$$

and the discounted stock price $\widetilde{Q}_t = e^{-rt}Q_t$ satisfies

$$d\widetilde{Q}_t = \widetilde{Q}_{t-}\left(\sigma dW_t + d\left(R_t - \tilde{\lambda}\tilde{\beta}t\right)\right), \widetilde{Q}_0 = Q_0. \tag{19}$$

Moreover, both the expectations are related as

$$EC = \widetilde{E}\left(Z_T^{-1}C\right),$$

for any arbitrary square integrable $\mathcal{F}_T$−measurable random variable C.

Making use of (10) and (11), the latter relation gives

$$EC \le \left(\widetilde{E}Y_T\right)^{\frac{1}{2}}\left(\widetilde{E}C^2\right)^{\frac{1}{2}}. \tag{20}$$

To estimate explicitly $\widetilde{E}Y_T$, we use the fact that the expressions (15) and $N_t - \tilde{\lambda}t$ are martingales, from (17) we can write

$$\widetilde{E}Y_T = 1 + \left[\left(\frac{\lambda}{\tilde{\lambda}}\right)^2\mathcal{L} - 2\lambda + 3\tilde{\lambda}\right]\widetilde{E}\int_0^T e^{2(\tilde{\lambda}-\lambda)s}J_{s-}ds.$$

From Fubini's theorem, Gronwall's inequality and the process J having finitely many jumps, we obtain

$$\begin{aligned} \widetilde{E}Y_T &= 1 + \left[\left(\frac{\lambda}{\tilde{\lambda}}\right)^2\mathcal{L} - 2\lambda + 3\tilde{\lambda}\right]\int_0^T \widetilde{E}e^{2(\tilde{\lambda}-\lambda)s}J_{s-}ds \\ &\le exp\left\{\left[\left(\frac{\lambda}{\tilde{\lambda}}\right)^2\mathcal{L} - 2\lambda + 3\tilde{\lambda}\right]T\right\}. \end{aligned}$$

By using the latter estimate and the bound (20), we get the following estimate for (9)

$$\begin{aligned}
E \sup_{0\le t\le T} |\pi(t) - \pi_\delta(t)| &\le e^{\left(\left(\frac{\lambda}{\tilde\lambda}\right)^2 \mathcal{L} - 2\lambda + 3\tilde\lambda + 2r\right)\frac{T}{2}} \left[\widetilde{E} \sup_{0\le t\le T} \left(\int_0^t (\triangle(u) - \triangle_\delta(u))\, d\tilde{Q}_u\right)^2\right]^{\frac{1}{2}} \\
&\le 2e^{\left(\left(\frac{\lambda}{\tilde\lambda}\right)^2 \mathcal{L} - 2\lambda + 3\tilde\lambda + 2r\right)\frac{T}{2}} \left[\widetilde{E} \sup_{0\le t\le T} \left\{\left(\int_0^t (\triangle(u) - \triangle_\delta(u))\, \tilde{Q}_u \sigma dW_u\right)^2\right.\right. \\
&\quad \left.\left. + \left(\int_0^t (\triangle(u) - \triangle_\delta(u))\, \tilde{Q}_{u-} d\left(R_u - \tilde\lambda\tilde\beta u\right)\right)^2\right\}\right]^{\frac{1}{2}}.
\end{aligned}$$

Using Doob's maximal inequality, we can write

$$\begin{aligned}
E \sup_{0\le t\le T} |\pi(t) - \pi_\delta(t)| &\le 8e^{\left(\left(\frac{\lambda}{\tilde\lambda}\right)^2 \mathcal{L} - 2\lambda + 3\tilde\lambda + 2r\right)\frac{T}{2}} \left[\widetilde{E} \int_0^T (\triangle(u) - \triangle_\delta(u))^2\, \tilde{Q}_u^2 \sigma^2 du\right. \\
&\quad \left. + \widetilde{E} \int_0^T (\triangle(u) - \triangle_\delta(u))^2\, \tilde{Q}_{u-}^2 \left[d(R_u - \tilde\lambda\tilde\beta u)\right]^2\right]^{\frac{1}{2}} \\
&\le 8e^{\left(\left(\frac{\lambda}{\tilde\lambda}\right)^2 \mathcal{L} - 2\lambda + 3\tilde\lambda + 2r\right)\frac{T}{2}} \left[\sigma^2 \widetilde{E} \int_0^T (\triangle(u) - \triangle_\delta(u))^2\, Q_u^2 du\right. \\
&\quad \left. + \widetilde{E} \int_0^T (\triangle(u) - \triangle_\delta(u))^2\, Q_{u-}^2 d\left(\sum_{i=1}^{N_u} U_i^2\right)\right]^{\frac{1}{2}} \\
&= 8e^{\left(\left(\frac{\lambda}{\tilde\lambda}\right)^2 \mathcal{L} - 2\lambda + 3\tilde\lambda + 2r\right)\frac{T}{2}} \sqrt{\sigma^2 + \tilde\lambda \widetilde{E}(U_1^2)} \left[\widetilde{E} \int_0^T (\triangle(u) - \triangle_\delta(u))^2\, Q_u^2 du\right]^{\frac{1}{2}}, \qquad (21)
\end{aligned}$$

where we have used the assumption that the number of jumps is finite.

Assume $v(t,x)$ be value function of American put option at time t and $\mathcal{T}_{t,T}$ denotes all $(\mathcal{F}_u)_{0\le u\le T}$-measurable stopping times τ from t to T. Then in particular, we have

$$v(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \widetilde{E}\left[e^{-r(\tau-t)} g\left(Q_\tau(t,x)\right)\right], 0 \le t \le T, x \ge 0, \qquad (22)$$

where $g(x), x \ge 0$, is non-increasing bounded below convex payoff function with assumption $g(0) = g(0+)$ and the stock price $Q_u(t,x)$, evolves as

$$dQ_u(t,x) = Q_{u-}(t,x)\left(rdu + \sigma dW_u + d\left(R_u - \tilde\lambda\tilde\beta u\right)\right), t \le u \le T,$$

with $Q_t(t,x) = x$.

Assume the mathematical expectation of U_1, U_1^2, $|\ln(1+U_1)|$ and $\ln^2(1+U_1)$ is finite, we come to the following result:

Proposition 2.2. *Assume $\triangle(t), 0 \le t \le T$, represents the continuous time delta strategy minimizing the risk at maturity, defined as (5), of the American option and $\triangle_\delta(t)$ the corresponding discrete time trading strategy given in (6). Then the following estimate is valid*

$$\widetilde{E} \int_0^T (\triangle(u) - \triangle_\delta(u))^2\, Q_u^2 du \le a \, \ln\frac{T}{\delta} \cdot \delta, \qquad (23)$$

where a depends on Q_0, $g(0)$, r, σ, λ, β, T, $\widetilde{E}(U_1)$, $\widetilde{E}(U_1^2)$, $\widetilde{E}(|\ln(1+U_1)|)$, $\widetilde{E}(\ln^2(1+U_1))$, and $\mathcal{N}(2\sigma^2 T)$, where $\mathcal{N}(\cdot)$ stands for standard normal distribution.

Proof. Using expressions (5) and (6), we can express

$$
\begin{aligned}
&\widetilde{E}\int_0^T (\triangle(t)-\triangle_\delta(t))^2 Q_t^2 dt \\
= &\ \frac{1}{(\sigma^2+\lambda E(U_1)^2)^2}\widetilde{E}\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\Bigg[\sigma Q_t\left(\frac{\partial v(t,Q_t)}{\partial x}-\frac{\partial v(t_{k-1},Q_{t_{k-1}})}{\partial x}\right) \\
- &\ \lambda\int z\left(v(t,Q_t)-v(t,Q_t(1+z))-\frac{Q_t}{Q_{t_{k-1}}}\left(v(t_{k-1},Q_{t_{k-1}})-v(t_{k-1},Q_{t_{k-1}}(1+z))\right)\right)df(z)\Bigg]^2 dt \\
\leq &\ \frac{2}{(\sigma^2+\lambda E(U_1)^2)^2}\widetilde{E}\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\Bigg[\sigma^2 Q_t^2\left(\frac{\partial v(t,Q_t)}{\partial x}-\frac{\partial v(t_{k-1},Q_{t_{k-1}})}{\partial x}\right)^2 \\
+ &\ 2\lambda^2\int z^2\left(\left(v(t,Q_t)-\frac{Q_t}{Q_{t_{k-1}}}v(t_{k-1},Q_{t_{k-1}})\right)^2+\left(v(t,Q_t(1+z))-\frac{Q_t}{Q_{t_{k-1}}}v(t_{k-1},Q_{t_{k-1}}(1+z))\right)^2\right)df(z)\Bigg]dt.
\end{aligned}
\tag{24}
$$

We denote by $\Gamma(t,y)=y\frac{\partial v(t,y)}{\partial y}, 0\leq t<T, y>0$ and express

$$
\widetilde{E}\sum_{k=1}^n\int_{t_{k-1}}^{t_k}Q_t^2\left(\frac{\partial v(t,Q_t)}{\partial x}-\frac{\partial v(t_{k-1},Q_{t_{k-1}})}{\partial x}\right)^2 dt=\widetilde{E}\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\left(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_{t_{k-1}})\frac{Q_t}{Q_{t_{k-1}}}\right)^2 dt. \tag{25}
$$

Let us write

$$
\begin{aligned}
\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_{t_{k-1}})\frac{Q_t}{Q_{t_{k-1}}} &= \Gamma(t,Q_t)-\Gamma(t_{k-1},Q_t)+\Gamma(t_{k-1},Q_t)-\Gamma(t_{k-1},Q_{t_{k-1}}) \\
&\quad +\Gamma(t_{k-1},Q_{t_{k-1}})\frac{Q_{t_{k-1}}-Q_t}{Q_{t_{k-1}}},
\end{aligned}
\tag{26}
$$

therefore, we can write

$$
\begin{aligned}
&\widetilde{E}\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\left(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_{t_{k-1}})\frac{Q_t}{Q_{t_{k-1}}}\right)^2 dt \\
&\leq 4\widetilde{E}\sum_{k=1}^n\Bigg[\int_{t_{k-1}}^{t_k}(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_t))^2 dt+\int_{t_{k-1}}^{t_k}(\Gamma(t_{k-1},Q_t)-\Gamma(t_{k-1},Q_{t_{k-1}}))^2 dt \\
&\qquad +\int_{t_{k-1}}^{t_k}\Gamma^2(t_{k-1},Q_{t_{k-1}})\left(\frac{Q_{t_{k-1}}-Q_t}{Q_{t_{k-1}}}\right)^2 dt\Bigg].
\end{aligned}
\tag{27}
$$

We estimate all the integrals on the right side of previous inequality by using Propositions 3.1 and 3.4 from Hussain et al. [12] which state that the mapping $\varsigma(t,y)=yv(t,y)$ is Lipschitz continuous in y, that is,

$$
|\varsigma(t,x)-\varsigma(t,y)|\leq 2g(0)|x-y|,\ 0\leq t\leq T, 0<x\leq y<\infty, \tag{28}
$$

and locally Lipschitz continuous in t, that is,

$$
|\varsigma(t,y)-\varsigma(s,y)|\leq\frac{Cy}{\sqrt{T-t}}|t-s|,\ 0\leq s\leq t\leq T, y>0, \tag{29}
$$

and $\Gamma(t,y)=y\frac{\partial v(t,y)}{\partial y}$ satisfies

$$
|\Gamma(t,y)-\Gamma(s,y)|\leq\frac{G+yH}{\sqrt{T-t}}|t-s|^{\frac{1}{2}}, 0\leq t<T, y>0, \tag{30}
$$

where the positive constants C, G and H are functions of r, σ, $g(0)$, λ, $E|U_1|$, $E|\ln(1+U_1)|$ and T and the fact that $v(t,x) \le g(0), 0 \le t \le T, x > 0$, we write

$$\begin{aligned}
\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_t))^2\,dt &\le \widetilde{E}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{(G+Q_tH)^2}{T-t}(t-t_{k-1})dt+\widetilde{E}\int_{t_{n-1}}^{T}(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_t))^2\,dt\\
&\le 2\delta\widetilde{E}\int_0^{t_{n-1}}\frac{G^2+Q_t^2H^2}{T-t}dt+36g^2(0)\cdot\delta,
\end{aligned}$$

where $\delta = \frac{T}{n}, n = 1, 2, ...$

Using the unique solution Q_t of (18), we find

$$\widetilde{E}Q_t^2 = Q_0^2 e^{\left(2r-2\tilde{\lambda}\tilde{\beta}+\sigma^2+\tilde{\lambda}\left(\widetilde{E}(1+U_1)^2-1\right)\right)t}.$$

Therefore

$$\begin{aligned}
&\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_t))^2\,dt\\
\le\ & 2\left\{G^2+H^2Q_0^2\exp\left[\left(2r+2\tilde{\lambda}\tilde{\beta}+\sigma^2+\tilde{\lambda}(\widetilde{E}(1+U_1)^2+1)\right)T\right]\right\}\cdot\delta\ln\frac{T}{\delta}+36g^2(0)\cdot\delta. \qquad (31)
\end{aligned}$$

To estimate the second expectation on the right hand side of (27), we introduce the change of variable $z = \ln x$ and define $u(t,z)$ such that

$$u(t,z) = v(t,e^z),\ \ 0 \le t \le T, -\infty < z < \infty. \qquad (32)$$

Replace the new function $u(t,z)$ in (bounds (22) in [12]) the system of inequalities

$$\begin{cases}
-rv(t,x)+\frac{\partial v(t,x)}{\partial t}-\lambda xEU_1e^{-rt}\frac{\partial v(t,x)}{\partial x}+\frac{x^2\sigma^2}{2}e^{-2rt}\frac{\partial^2 v(t,x)}{\partial x^2} & \\
-\lambda\int(v(t,x(1+z))-v(t,x))df(z)\le 0, & \text{a.e. in } [0,T)\times\mathbb{R}^+;\\
\frac{\partial^2 v(t,x)}{\partial x^2}\ge 0, & x>0,
\end{cases}$$

where f is the common law of the random variables U_i's, we obtain

$$\begin{cases}
-ru(t,z)+\frac{\partial u(t,z)}{\partial t}-\left(\lambda EU_1+\frac{\sigma^2}{2}e^{-rt}\right)e^{-rt}\frac{\partial u(t,z)}{\partial z} & \\
+\frac{\sigma^2}{2}e^{-2rt}\frac{\partial^2 u(t,z)}{\partial z^2}-\lambda\int(u(t,z+\ln(1+z))-u(t,z))df(z)\le 0, & \\
\frac{\partial^2 u(t,z)}{\partial z^2}\ge\frac{\partial u(t,z)}{\partial z}, & \text{a.e. in } [0,T)\times\mathbb{R}.
\end{cases}$$

This system gives

$$\begin{aligned}
\left|\frac{\partial^2 u(t,z)}{\partial z^2}\right| &\le \left|\frac{\partial u(t,z)}{\partial z}\right|+\frac{2}{\sigma^2}e^{2rT}\left[ru(t,z)+\left|\frac{\partial u(t,z)}{\partial t}\right|+\left(\lambda EU_1+\frac{\sigma^2}{2}\right)\left|\frac{\partial u(t,z)}{\partial z}\right|\right.\\
&\quad\left.+\lambda\int|u(t,z+\ln(1+z))-u(t,z)|\,df(z)\right].
\end{aligned}$$

Using Proposition 2.3 from [12], which states that the mapping $u(t,z)$ satisfies

$$\begin{aligned}
|u(t,y)-u(t,z)| &\le g(0)|y-z|, 0\le t\le T,\ y,z\in\mathbb{R},\\
|u(t,z)-u(s,z)| &\le \frac{A}{\sqrt{T-t}}|t-s|, 0\le s\le t<T,\ z\in\mathbb{R},
\end{aligned} \qquad (33)$$

where A depends on Q_0, r, σ, $g(0)$, λ, $E(U_1)$, and T, we obtain

$$\left|\frac{\partial^2 u(t,z)}{\partial z^2}\right| \leq \frac{F}{\sqrt{T-t}}, \tag{34}$$

where F depends on r, σ, $g(0)$, λ, $E(U_1)$, $E|\ln(1+U_1)|$ and T.

By using the continuity of $\frac{\partial u(t,z)}{\partial z}$ (see, Pham [15] and Zhang [29]) and the bound (34), we get

$$\left|\Gamma(t,x) - \Gamma(t,y)\right| \leq \frac{F}{\sqrt{T-t}}|\ln x - \ln y|,\ 0 \leq t < T,\ x,y \in (0,\infty). \tag{35}$$

Thus we can write

$$\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} (\Gamma(t_{k-1},Q_t) - \Gamma(t_{k-1},Q_{t_{k-1}}))^2\,dt \leq \widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{F^2}{T-t_{k-1}}(\ln Q_t - \ln Q_{t_{k-1}})^2\,dt. \tag{36}$$

Solution of (18) further gives

$$\begin{aligned}\widetilde{E}(\ln Q_t - \ln Q_{t_{k-1}})^2 &\leq 4\widetilde{E}\left(\sum_{i=N_{t_{k-1}}+1}^{N_t}\ln(1+U_i)\right)^2 + 4\left(\left(r-\tilde{\lambda}\tilde{\beta}-\frac{\sigma^2}{2}\right)(t-t_{k-1})\right)^2 + 2\widetilde{E}(\sigma(W_t - W_{t_{k-1}}))^2\\ &\leq 2\left(4\tilde{\lambda}\widetilde{E}\ln^2(1+U_1) + 2\left(r-\tilde{\lambda}\tilde{\beta}-\frac{\sigma^2}{2}\right)^2 T + \sigma^2\right)(t-t_{k-1}),\end{aligned}$$

thus (36) becomes

$$\begin{aligned}&\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(\Gamma(t_{k-1},Q_t) - \Gamma(t_{k-1},Q_{t_{k-1}}))^2\,dt\\ \leq\ & 2F^2\left(4\tilde{\lambda}\widetilde{E}\ln^2(1+U_1) + 2\left(r-\tilde{\lambda}\tilde{\beta}-\frac{\sigma^2}{2}\right)^2 T + \sigma^2\right)\left[\delta\int_0^{t_{n-1}}\frac{dt}{T-t} + \int_{t_{n-1}}^{T}\frac{t-t_{n-1}}{T-t_{n-1}}dt\right]\\ =\ & 2F^2\left(4\tilde{\lambda}\widetilde{E}\ln^2(1+U_1) + 2\left(r-\tilde{\lambda}\tilde{\beta}-\frac{\sigma^2}{2}\right)^2 T + \sigma^2\right)\left(\ln\frac{T}{\delta}+1\right)\delta.\end{aligned} \tag{37}$$

To estimate the third integral in (27), we use (28) and the bound $v(t,x) \leq g(0), 0 \leq t \leq T, x > 0$, and write

$$\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\Gamma^2(t_{k-1},Q_{t_{k-1}})\left(\frac{Q_{t_{k-1}}-Q_t}{Q_{t_{k-1}}}\right)^2 dt \leq g^2(0)\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\widetilde{E}\left(1-\frac{Q_t}{Q_{t_{k-1}}}\right)^2 dt.$$

Since, for $t_{k-1} \leq t \leq t_k$, we have

$$\begin{aligned}\widetilde{E}\left(1-\frac{Q_t}{Q_{t_{k-1}}}\right)^2 &= \widetilde{E}\Bigg[1 - 2\exp\Bigg(\sum_{i=N_{t_{k-1}}+1}^{N_t}\ln(1+U_i) + \left(r-\tilde{\lambda}\tilde{\beta}-\frac{\sigma^2}{2}\right)(t-t_{k-1}) + \sigma(W_t - W_{t_{k-1}})\Bigg)\\ &\quad + \exp\Bigg(2\sum_{i=N_{t_{k-1}}+1}^{N_t}\ln(1+U_i) + 2\left(r-\tilde{\lambda}\tilde{\beta}-\frac{\sigma^2}{2}\right)(t-t_{k-1}) + 2\sigma(W_t - W_{t_{k-1}})\Bigg)\Bigg]\\ &= 2\left[1 - \exp\left(\left(r-\tilde{\lambda}\tilde{\beta}+\tilde{\lambda}\widetilde{E}U_1\right)(t-t_{k-1})\right)\right]\\ &\quad + \left[\exp\left(\left(2r - 2\tilde{\lambda}\tilde{\beta} + \sigma^2 + \tilde{\lambda}\left(\widetilde{E}(1+U_1)^2 - 1\right)\right)(t-t_{k-1})\right) - 1\right].\end{aligned}$$

Mean value theorem then gives

$$\widetilde{E}\left(1-\frac{Q_t}{Q_{t_{k-1}}}\right)^2 \leq \Big[\left(2r+2\tilde{\lambda}\tilde{\beta}+\sigma^2+\tilde{\lambda}\widetilde{E}(1+U_1)^2\right)e^{\left(2r+2\tilde{\lambda}\tilde{\beta}+\sigma^2+\tilde{\lambda}\widetilde{E}(1+U_1)^2\right)T} +2\left(r+\tilde{\lambda}\tilde{\beta}+\tilde{\lambda}\widetilde{E}|U_1|\right)e^{\left(r+\tilde{\lambda}\tilde{\beta}+\tilde{\lambda}\widetilde{E}|U_1|\right)T}\Big](t-t_{k-1}),$$

for $t_{k-1} \leq t \leq t_k$.

This bound leads us

$$\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\Gamma^2(t_{k-1},Q_{t_{k-1}})\left(\frac{Q_{t_{k-1}}-Q_t}{Q_{t_{k-1}}}\right)^2 dt \leq g^2(0)T\Big[\left(2r+2\tilde{\lambda}\tilde{\beta}+\sigma^2+\tilde{\lambda}\widetilde{E}(1+U_1)^2\right)e^{\left(2r+2\tilde{\lambda}\tilde{\beta}+\sigma^2+\tilde{\lambda}\widetilde{E}(1+U_1)^2\right)T} +2\left(r+\tilde{\lambda}\tilde{\beta}+\tilde{\lambda}\widetilde{E}|U_1|\right)e^{\left(r+\tilde{\lambda}\tilde{\beta}+\tilde{\lambda}\widetilde{E}|U_1|\right)T}\Big]\delta. \tag{38}$$

Combining (31), (37) and (38) in (27), we obtain

$$\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\Gamma(t,Q_t)-\Gamma(t_{k-1},Q_{t_{k-1}})\frac{Q_t}{Q_{t_{k-1}}}\right)^2 dt \leq \left(A_1\ln\frac{T}{\delta}+B_1\right)\delta, \tag{39}$$

where the positive constants A_1 and B_1 are functions of the parameters Q_0, r, β, σ, $g(0)$, λ, T, $\widetilde{E}(U_1)$, $\widetilde{E}(U_1^2)$, $\widetilde{E}(|\ln(1+U_1)|)$ and $\widetilde{E}(\ln^2(1+U_1))$.

Using the similar expression as (26), we write

$$\begin{aligned}
&\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int z^2\left(v(t,Q_t)-\frac{Q_t}{Q_{t_{k-1}}}v(t_{k-1},Q_{t_{k-1}})\right)^2 df(z)dt \\
\leq\ & 4\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int z^2\Big[(v(t,Q_t)-v(t_{k-1},Q_t))^2+(v(t_{k-1},Q_t)-v(t_{k-1},Q_{t_{k-1}}))^2 \\
&+v^2(t_{k-1},Q_{t_{k-1}})\frac{(Q_{t_{k-1}}-Q_t)^2}{Q^2_{t_{k-1}}}\Big]df(z)dt.
\end{aligned} \tag{40}$$

Using the estimates (28), (29) and the bound $v(t,x) \leq g(0), 0 \leq t \leq T, x > 0$, we obtain

$$\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int z^2\left(v(t,Q_t)-\frac{Q_t}{Q_{t_{k-1}}}v(t_{k-1},Q_{t_{k-1}})\right)^2 df(z)dt \leq \left(A_2\ln\frac{T}{\delta}+B_2\right)\delta, \tag{41}$$

where A_2 and B_2 depend on r, σ, $g(0)$, λ, T, $\widetilde{E}(U_1)$, $\widetilde{E}(|\ln(1+U_1)|)$, $\widetilde{E}(\ln^2(1+U_1))$ for all $n \in \mathbb{N}$ and $\mathcal{N}(2\sigma^2 T)$, where $\mathcal{N}(\cdot)$ stands for standard normal distribution.

Same estimate as (41) is valid for the series

$$\widetilde{E}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int z^2\left(v(t,Q_t(1+z))-\frac{Q_t}{Q_{t_{k-1}}}v(t_{k-1},Q_{t_{k-1}}(1+z))\right)^2 df(z)dt. \tag{42}$$

Further, for $k \geq 2$, we have the obvious inequality $1 \leq \ln\frac{T}{\delta}$, combining the estimates (39), (41) and (42) in (24), we complete the proof. □

3. Main Result

In this section, we state and prove our main result for the jump-diffusion processes.

Proposition 3.1. *If $\Pi(t)$ denotes the continuous-time portfolio for the delta hedging strategy (5) and $\Pi_\delta(t), 0 \leq t \leq T$, the portfolio value process of the corresponding discrete time hedging process defined in (6), then we have the following hedging error*

$$E \sup_{0 \leq t \leq T} |\pi(t) - \pi_\delta(t)|^2 \leq d \cdot \delta \, \ln \frac{T}{\delta},$$

where the non negative constant d depends on $\mathcal{L}$, r, σ, λ, $g(0)$, T, $\widetilde{E}(U_1)$, $\widetilde{E}(U_1^2)$, Q_0, $\widetilde{E}|\ln(1+U_1)|$, $\widetilde{E}\ln^2(1+U_1)$ and $\mathcal{N}(2\sigma^2 T)$, where $\mathcal{N}(\cdot)$ stands for standard normal distribution.

Proof. Proof follows from the application of Proposition 2.2 in the estimate (21). □

Conclusion

We consider American style option in jump-diffusion model with finite time expiry T, and divide the time interval $[0, T]$ into n equal parts and denote by $\delta = \frac{T}{n}$. We consider the value process of the discrete time delta hedging strategy minimising the risk at maturity. Using purely probabilistic approach, we estimate the discrete time hedging error and find that it is proportional to the square root of $\delta \log \frac{T}{\delta}$.

Financial Competing Interests

1. We have received neither reimbursement, fees, funding, salary nor we have stocks and shares anywhere that may have gain or loss financially from the publication of this article, either now or in future.
2. We have not any patents and not applying for patents for this article.

Non-Financial Competing Interests

We have no non-financial competing interests (political, personal, religious, ideo- logical, intellectual, commercial etc) which are related to this manuscript.

Contribution

All the authors have equally contributed. All the authors read and approved the final form of the manuscript.

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