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Two Dimensional Operator Preinvex Functions and associated Hermite-Hadamard type Inequalities

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Abstract. We first introduce the notion of operator (h_1, h_2)-preinvex on the co-ordinates. After this some new two dimensional version of integral inequalities of Hermite-Hadamard type associated with this new class of operator (h_1, h_2)-preinvex are obtained. Some new and novel particular cases are also discussed.

1. Introduction

A set $\mathcal{K} \subset \mathbb{R}$ is said to be convex, if

$$ta + (1-t)b \in \mathcal{K}, \quad \forall a, b \in \mathcal{K}, t \in [0,1].$$

A function $f : \mathbb{K} \to \mathbb{R}$ is said to be convex, if

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b), \quad \forall a, b \in \mathcal{K}, t \in [0,1].$$

It has been observed that in recent decades the classical notion of convexity has been extended and generalized in different directions. For example see [1, 2, 4, 6, 7, 10–13, 15–19] and the references therein. The relation between theory of convexity and theory of inequalities attracted many researchers to study these theories in depth. Consequently many inequalities have been obtained via convex functions and via its variant forms. For more information readers are referred to [4].
Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, the following inequality

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\mathrm{d}x \le (b-a)\frac{f(a)+f(b)}{2},$$

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holds. This result is due to Hermite and Hadamard independently and is known as Hermite-Hadamard's inequality. This result provides us a necessary and sufficient condition for a function to be convex.
Inspired by the ongoing research, we derive some new extensions of two dimensional Hermite-Hadamard inequalities via operator (h_1, h_2)-preinvex functions on the co-ordinates. Some new special cases are also discussed. This is the main motivation of this paper. The ideas and techniques of this paper may stimulate future research in this direction.

2. Preliminaries

In this section, we discuss some previously known concepts. From [2], let us consider a bidimensional interval $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be convex function on Δ, if the following inequality

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w),$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the partial functions $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $x \in [a, b]$ and $y \in [c, d]$.

Definition 2.1. *Consider a rectangle $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates function on Δ, if*

$$\begin{aligned} &f(tx + (1-t)y, ru + (1-r)w) \\ &\leq trf(x, u) + t(1-r)f(x, w) + r(1-t)f(y, u) + (1-t)(1-r)f(y, w), \end{aligned} \tag{2.1}$$

whenever $x, y \in [a, b]$, $u, w \in [c, d]$ and $t, r \in [0, 1]$.

Definition 2.2. *[13] Let $(u, v) \in X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$. We say $X_1 \times X_2$ is invex at (u, v) with respect to η_1 and η_2 if for each $(x, y) \in X_1 \times X_2$ and $t_1, t_2 \in [0, 1]$*

$$(u + t_1\eta_1(x, u), v + t_2\eta_2(y, v)) \in X_1 \times X_2.$$

We also need the following assumtion regarding the function η which is due to Mohan and Neogy [14].
Condition C. Let $X \in \mathbb{R}$ be an open invex subset with respect to η. For any $x, y \in X$ and $t \in [0, 1]$

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y); \eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y).$$

Definition 2.3 ([13]). *Let h_1 and h_2 be non-negative functions on $[0, 1], h_1 \not\equiv 0, h_2 \not\equiv 0$. The non-negative function $f : X_1 \times X_2 \to \mathbb{R}$ is said to be (h_1, h_2)-preinvex on the co-ordinates with respect to η_1 and η_2 if*

$$\begin{aligned} &f(x + t_1\eta_1(b, x), f(x + t_2\eta_2(d, y)) \\ &\leq h_1(1-t_1)h_2(1-t_2)f(x, y) + h_1(1-t_1)h_2(t_2)f(x, d) + h_1(t_1)h_2(1-t_2)f(b, y) \\ &+ h_1(t_1)h_2(t_2)f(b, d). \end{aligned} \tag{2.2}$$

Remark 2.4. *Note that if $\eta_1(b, x) = b - x$, $\eta_2(d, y) = d - y$ and $h_1(t) = h_2(t) = t$, then the definition of a co-ordinates (h_1, h_2)-preinvex function reduces to the definition of a convex function on the co-ordinates proposed by Dragomir [2]. Moreover, if $h_1(t) = h_2(t) = t^s$, then definition reduces to the definition of an s-convex function on the co-ordinates proposed by Alomari and Darus [1].*

Definition 2.5 ([10]). *Consider $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$. A function $f : \Delta \to \mathbb{R}$ is said to be s-preinvex on the co-ordinates function on Δ, where $s \in (0, 1]$ if*

$$\begin{aligned} &f(x + t\eta_1(y, x), u + r\eta_2(w, u)) \\ &\leq (1-t)^s(1-r)^s f(x, u) + (1-t)^s r^s f(x, w) + t^s(1-r)^s f(y, u) + t^s r^s f(y, w), \end{aligned} \tag{2.3}$$

whenever $(x, u), (x, w), (y, u), (y, w) \in \Delta$ and $t, r \in [0, 1]$.

Note that, if $s = 1$ in above definition we have definition of preinvex on the co-ordinates functions.

Definition 2.6 ([9]). *Consider $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$. A function $f : \Delta \to \mathbb{R}$ is said to be preinvex on the co-ordinates function on Δ, if*

$$\begin{aligned} & f(x + t\eta_1(y, x), u + r\eta_2(w, u)) \\ & \leq (1-t)(1-r)f(x, u) + (1-t)rf(x, w) + t(1-r)f(y, u) + trf(y, w), \end{aligned} \tag{2.4}$$

whenever $(x, u), (x, w), (y, u), (y, w) \in \Delta$ *and* $t, r \in [0, 1]$.

To investigate the operator version of the Hermite-Hadamard inequality associated with operator (h_2, h_2)-preinvex functions we need the following preliminary definitions and results.
We now review the operator order in $B(H)$ which is the set of all bounded linear operators on a Hilbert space $(H; \langle ., . \rangle)$, and the continuous functional calculus for bounded self adjoint operator. For self adjoint operators $A, B \in B(H)$, we write

$$A \leq B \quad \text{if} \quad \langle Ax, x \rangle \leq \langle Bx, x \rangle,$$

for every vector $x \in H$, we call it operator order. The set of all self adjoint elements in $B(H)$ is denoted with $B(H)_{sa}$.
Let A be a bounded self adjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The Gelfand map establishes a $\star$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the $C^\star$-algebra $C^\star(A)$ generated by A and the identity operator 1_H on H as follows(see for instance [5]). For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

1. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
2. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^\star) = \Phi(f)^\star$;
3. $\| \Phi(f) \| = \| f \| := \sup_{t \in Sp(A)} |f(t)|$;
4. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a bounded selfadjoint operator A. If A is a bounded selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A),$$

in the operator order in $B(H)$.
A real valued continuous function f on an interval $\mathcal{I}$ is said to be operator convex function, if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B),$$

in the operator order, for all $t \in [0, 1]$ and for every self adjoint operator A and B on a Hilbert space H whose spectra are contained in I. For more information, see [5].
Dragomir in [3] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

Theorem 2.7. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\begin{aligned} & f\left(\frac{A+B}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ & \leq \int_0^1 f((1-t)A + tB)\mathrm{d}t \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{2.5}$$

For the reader's convenience, we recall the definitions of the Gamma function $\Gamma(.)$ and Beta function $\mathbb{B}(.,.)$ respectively.

$$\Gamma(x) = \int_0^{\infty} e^{-x} t^{x-1} \mathrm{d}t,$$

$$\mathbb{B}(x,y) = \int_0^{1} t^{x-1}(1-t)^{y-1}\, \mathrm{d}t.$$

It is known that [8]

$$\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

3. New Notions

We are now in a position to define new notion of (h_1, h_2)-preinvex functions on co-ordinates and also discuss special cases.

Definition 3.1. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the continuous function $f : \Delta \to \mathbb{R}$ is said to be operator (h_1, h_2)-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2 if for every $A, B \in X_1$ and $C, D \in X_2$ and $t, \lambda \in [0,1]$*

$$\begin{aligned} & f(A + t\eta_1(B,A), C + \lambda\eta_2(D,C)) \\ & \le h_1(1-t)h_2(1-\lambda)f(A,C) + h_1(1-t)h_2(\lambda)f(A,D) \\ & \quad + h_1(t)h_2(1-\lambda)f(B,C) + h_1(t)h_2(\lambda)f(B,D), \end{aligned} \tag{3.1}$$

in the operator order in $B(\mathbb{R}^n)_{sa} \times B(\mathbb{R}^n)_{sa}$.

We now discuss some special cases:

1. For $h_1(t) = t^s$ and $h_2(\lambda) = \lambda^s$ we have Definition 3.2 of Breckner type of operator s-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2.
2. For $h_1(t) = t$ and $h_2(\lambda) = \lambda$ we have Definition 3.3 for operator preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2.
3. For $h_1(t) = t^{-s}$ and $h_2(\lambda) = \lambda^{-s}$ we have Definition 3.4 of Godunova-Levin type of operator s-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2.
4. For $h_1(t) = 1$ and $h_2(\lambda) = 1$ we have Definition 3.5 of operator P-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2.

Definition 3.2. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the continuous function $f : \Delta \to \mathbb{R}$ is said to be Breckner type of operator s-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2 if for every $A, B \in X_1$ and $C, D \in X_2$, $t, \lambda \in [0,1]$ and $s \in (0,1]$, the following inequality holds*

$$\begin{aligned} & f(A + t\eta_1(B,A), C + \lambda\eta_2(D,C)) \\ & \le (1-t)^s(1-\lambda)^s f(A,C) + (1-t)^s\lambda^s f(A,D) + t^s(1-\lambda)^s f(B,C) + t^s\lambda^s f(B,D), \end{aligned}$$

in the operator order in $B(\mathbb{R}^n)_{sa} \times B(\mathbb{R}^n)_{sa}$.

If $s = 1$ we have definition of operator preinvex on the co-ordinates.

Definition 3.3. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the continuous function $f : \Delta \to \mathbb{R}$ is said to be operator preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2 if for every $A, B \in X_1$ and $C, D \in X_2$ and $t, \lambda \in [0, 1]$, the following inequality holds*

$$\begin{aligned} & f\left(A + t\eta_1(B, A), C + \lambda\eta_2(D, C)\right) \\ & \leq (1-t)(1-\lambda)f(A, C) + (1-t)\lambda f(A, D) + t(1-\lambda)f(B, C) + t\lambda f(B, D), \end{aligned}$$

in the operator order in $B(\mathbb{R}^n)_{sa} \times B(\mathbb{R}^n)_{sa}$.

Definition 3.4. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the continuous function $f : \Delta \to \mathbb{R}$ is said to be Godunova-Levin type of operator s-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2 if for every $A, B \in X_1$ and $C, D \in X_2$, $t, \lambda \in (0, 1)$ and $s \in [0, 1]$, the following inequality holds*

$$\begin{aligned} & f\left(A + t\eta_1(B, A), C + \lambda\eta_2(D, C)\right) \\ & \leq (1-t)^{-s}(1-\lambda)^{-s}f(A, C) + (1-t)^{-s}\lambda^{-s}f(A, D) \\ & \quad + t^{-s}(1-\lambda)^{-s}f(B, C) + t^{-s}\lambda^{-s}f(B, D), \end{aligned}$$

in the operator order in $B(\mathbb{R}^n)_{sa} \times B(\mathbb{R}^n)_{sa}$.

Definition 3.5. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the continuous function $f : \Delta \to \mathbb{R}$ is said to be operator P-preinvex on the co-ordinates with respect to η_1 on X_1 and η_2 on X_2 if for every $A, B \in X_1$ and $C, D \in X_2$ and $t, \lambda \in [0, 1]$, the following inequality holds*

$$f\left(A + t\eta_1(B, A), C + \lambda\eta_2(D, C)\right) \leq f(A, C) + f(A, D) + f(B, C) + f(B, D),$$

in the operator order in $B(\mathbb{R}^n)_{sa} \times B(\mathbb{R}^n)_{sa}$.

4. Results and Discussions

In this section, we derive our main results.

Lemma 4.1. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the function $f : \Delta \to \mathbb{R}$ be a continuous function. Suppose that η_1 and η_2 satisfy condition (C) on X_1 and on X_2 respectively. Then for every $A, B \in X_1$ and $V = A + \eta_1(B, A)$, $C, D \in X_2$ and $W = C + \eta_2(D, C)$ the function f is operator (h_1, h_2)-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} if and only if the function*

$$\varphi_{x,A,B,C,D}(t, \lambda) = \langle f\left(A + t\eta_1(B, A), C + \lambda\eta_2(D, C)\right) x, x\rangle \tag{4.1}$$

is (h_1, h_2)-convex on the co-ordinates on $[0, 1]^2$ for every $x \in H$ with $\| x \|= 1$.

Proof. Suppose that f is operator (h_1, h_2)-preinvex on the co-ordinates for operators $[A, B] = \{A + t\eta_1(B, A)\}\times$

$[C,D]=\{C+\lambda\eta_2(D,C)\}$. Then for any $t_1,t_2,\lambda_1,\lambda_2\in[0,1]$ and $\alpha_i,\beta_i\geqslant 0$ with $\alpha_i+\beta_i=1$ for $i=1,2$ we drive

$$\begin{aligned}
&\varphi_{x,A,B,C,D}(\alpha_1t_1+\beta_1t_2,\alpha_2\lambda_1+\beta_2\lambda_2)\\
&=\langle f\left(A+(\alpha_1t_1+\beta_1t_2)\eta_1(B,A),C+(\alpha_2\lambda_1+\beta_2\lambda_2)\eta_2(D,C)\right)x,x\rangle\\
&=\langle f\left(\alpha_1\left(A+t_1\eta_1(B,A)\right)+\beta_1\left(A+t_2\eta_1(B,A)\right),\right.\\
&\quad\left.\alpha_2\left(C+\lambda_1\eta_2(D,C)\right)+\beta_2\left(C+\lambda_2\eta_2(D,C)\right)\right)x,x\rangle\\
&\leq h_1(\alpha_1)h_2(\alpha_2)\langle f\left(A+t_1\eta_1(B,A),C+\lambda_1\eta_2(D,C)\right)x,x\rangle\\
&\quad+h_1(\alpha_1)h_2(\beta_2)\langle f\left(A+t_1\eta_1(B,A),C+\lambda_2\eta_2(D,C)\right)x,x\rangle\\
&\quad+h_1(\beta_1)h_2(\alpha_2)\langle f\left(A+t_2\eta_1(B,A),C+\lambda_1\eta_2(D,C)\right)x,x\rangle\\
&\quad+h_1(\beta_1)h_2(\beta_2)\langle f\left(A+t_2\eta_1(B,A),C+\lambda_2\eta_2(D,C)\right)x,x\rangle\\
&=h_1(\alpha_1)h_2(\alpha_2)\varphi_{x,A,B,C,D}(t_1,\lambda_1)+h_1(\alpha_1)h_2(\beta_2)\varphi_{x,A,B,C,D}(t_1,\lambda_2)\\
&\quad+h_1(\beta_1)h_2(\alpha_2)\varphi_{x,A,B,C,D}(t_2,\lambda_1)+h_1(\beta_1)h_2(\beta_2)\varphi_{x,A,B,C,D}(t_2,\lambda_2).
\end{aligned}$$

It shows that $\varphi_{x,A,B,C,D}$ is (h_1,h_2)-convex on the co-ordinates on $[0,1]^2$.
Let now $\varphi_{x,A,B,C,D}$ is (h_1,h_2)-convex on the co-ordinates on $[0,1]^2$ and $C_1:=A+t_1\eta_1(B,A)\in P_{AV}$, $C_2:=A+t_2\eta_1(B,A)\in P_{AV}$, $D_1:=C+\lambda_1\eta_2(D,C)\in P_{CW}$, respectively $D_2:=C+\lambda_2\eta_2(D,C)\in P_{CW}$. Fixed $\alpha,\beta\in[0,1]$. By (4.1)

$$\begin{aligned}
&\langle f\left(C_1+\alpha\eta_1(C_2,C_1),D_1+\beta\eta_2(D_2,D_1)\right)x,x\rangle\\
&=\langle f\left(A+(\alpha t_2+(1-\alpha)t_1)\eta_1(B,A),C+(\beta\lambda_2+(1-\beta)\lambda_1)\eta_2(D,C)\right)x,x\rangle\\
&=\varphi_{x,A,B,C,D}(\alpha t_2+(1-\alpha)t_1,\beta\lambda_2+(1-\beta)\lambda_1)\\
&\leq h_1(1-\alpha)h_2(1-\beta)\varphi_{x,A,B,C,D}(t_1,\lambda_1)+h_1(1-\alpha)h_2(\beta)\varphi_{x,A,B,C,D}(t_1,\lambda_2)\\
&+h_1(\alpha)h_2(1-\beta)\varphi_{x,A,B,C,D}(t_2,\lambda_1)+h_1(\alpha)h_2(\beta)\varphi_{x,A,B,C,D}(t_2,\lambda_2)\\
&=h_1(1-\alpha)h_2(1-\beta)\langle f\left(A+t_1\eta_1(B,A),C+\lambda_1\eta_2(D,C)\right)x,x\rangle\\
&+h_1(1-\alpha)h_2(\beta)\langle f\left(A+t_1\eta_1(B,A),C+\lambda_2\eta_2(D,C)\right)x,x\rangle\\
&+h_1(\alpha)h_2(1-\beta)\langle f\left(A+t_2\eta_1(B,A),C+\lambda_1\eta_2(D,C)\right)x,x\rangle\\
&+h_1(\alpha)h_2(\beta)\langle f\left(A+t_2\eta_1(B,A),C+\lambda_2\eta_2(D,C)\right)x,x\rangle\\
&=h_1(1-\alpha)h_2(1-\beta)\langle f\left(C_1,D_1\right)x,x\rangle+h_1(1-\alpha)h_2(\beta)\langle f\left(C_1,D_2\right)x,x\rangle\\
&+h_1(\alpha)h_2(1-\beta)\langle f\left(C_2,D_1\right)x,x\rangle+h_1(\alpha)h_2(\beta)\langle f\left(C_2,D_2\right)x,x\rangle.
\end{aligned}$$

Hence, f is operator (h_1,h_2)-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and respect to η_2 on η_2-path P_{CW}. This completes the proof. □

Lemma 4.2. *Let $\Delta=X_1\times X_2\subset\mathbb{R}^n\times\mathbb{R}^n$ and $X_1,X_2\subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1:X_1\times X_1\to B(\mathbb{R}^n)_{sa}$ and $\eta_2:X_2\times X_2\to B(\mathbb{R}^n)_{sa}$. Then, the function $f:\Delta\to\mathbb{R}$ be a continuous function. Suppose that η_1 and η_2 satisfy condition (C) on X_1 respectively, on X_2. Then for every $A,B\in X_1$ and $V=A+\eta_1(B,A)$, $C,D\in X_2$ and $W=C+\eta_2(D,C)$ the function f is operator s-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} if and only if the function*

$$\varphi_{x,A,B,C,D}(t,\lambda)=\langle f\left(A+t\eta_1(B,A),C+\lambda\eta_2(D,C)\right)x,x\rangle,$$

is s-convex on the co-ordinates on $[0,1]^2$ for every $x\in H$ with $\|x\|=1$.

Proof. Suppose that f is s-operator preinvex on the co-ordinates for operators $[A,B]=\{A+t\eta_1(B,A)\}\times$

$[C, D] = \{C + \lambda\eta_2(D, C)\}$. Then for any $t_1, t_2, \lambda_1, \lambda_2 \in [0, 1]$ and $\alpha_i, \beta_i \geqslant 0$ with $\alpha_i + \beta_i = 1$ for $i = 1, 2$ we drive

$$\begin{aligned}
&\varphi_{x,A,B,C,D}(\alpha_1 t_1 + \beta_1 t_2, \alpha_2\lambda_1 + \beta_2\lambda_2) \\
&= \langle f\,(A + (\alpha_1 t_1 + \beta_1 t_2)\eta_1(B, A), C + (\alpha_2\lambda_1 + \beta_2\lambda_2)\eta_2(D, C))\, x, x\rangle \\
&= \langle f\,(\alpha_1\,(A + t_1\eta_1(B, A)) + \beta_1\,(A + t_2\eta_1(B, A)), \\
&\quad \alpha_2\,(C + \lambda_1\eta_2(D, C)) + \beta_2\,(C + \lambda_2\eta_2(D, C)))\, x, x\rangle \\
&\leq \alpha_1^s\alpha_2^s\langle f\,(A + t_1\eta_1(B, A), C + \lambda_1\eta_2(D, C))\, x, x\rangle \\
&\quad + \alpha_1^s\beta_2^s\langle f\,(A + t_1\eta_1(B, A), C + \lambda_2\eta_2(D, C))\, x, x\rangle \\
&\quad + \alpha_2^s\beta_1^s\langle f\,(A + t_2\eta_1(B, A), C + \lambda_1\eta_2(D, C))\, x, x\rangle \\
&\quad + \beta_1^s\beta_2^s\langle f\,(A + t_2\eta_1(B, A), C + \lambda_2\eta_2(D, C))\, x, x\rangle \\
&= \alpha_1^s\alpha_2^s\varphi_{x,A,B,C,D}(t_1, \lambda_1) + \alpha_1^s\beta_2^s\varphi_{x,A,B,C,D}(t_1, \lambda_2) \\
&\quad + \alpha_2^s\beta_1^s\varphi_{x,A,B,C,D}(t_2, \lambda_1) + \beta_1^s\beta_2^s\varphi_{x,A,B,C,D}(t_2, \lambda_2).
\end{aligned}$$

It shows that $\varphi_{x,A,B,C,D}$ be s-convex on the co-ordinates on $[0, 1]^2$.
Let now $\varphi_{x,A,B,C,D}$ is s-convex on the co-ordinates on $[0, 1]^2$ and $C_1 := A+t_1\eta_1(B, A) \in P_{AV}$, $C_2 := A+t_2\eta_1(B, A) \in P_{AV}$, $D_1 := C + \lambda_1\eta_2(D, C) \in P_{CW}$, respectively $D_2 := C + \lambda_2\eta_2(D, C) \in P_{CW}$. Fixed $\alpha, \beta \in [0, 1]$. By (4.1), we have

$$\begin{aligned}
&\langle f\,(C_1 + \alpha\eta_1(C_2, C_1), D_1 + \beta\eta_2(D_2, D_1))\, x, x\rangle \\
&= \langle f\,(A + (\alpha t_2 + (1-\alpha)t_1)\eta_1(B, A), C + (\beta\lambda_2 + (1-\beta)\lambda_1)\eta_2(D, C))\, x, x\rangle \\
&= \varphi_{x,A,B,C,D}(\alpha t_2 + (1-\alpha)t_1, \beta\lambda_2 + (1-\beta)\lambda_1) \\
&\leq (1-\alpha)^s(1-\beta)^s\varphi_{x,A,B,C,D}(t_1, \lambda_1) + (1-\alpha)^s\beta^s\varphi_{x,A,B,C,D}(t_1, \lambda_2) \\
&\quad + \alpha^s(1-\beta)^s\varphi_{x,A,B,C,D}(t_2, \lambda_1) + \alpha^s\beta^s\varphi_{x,A,B,C,D}(t_2, \lambda_2) \\
&= (1-\alpha)^s(1-\beta)^s\langle f\,(A + t_1\eta_1(B, A), C + \lambda_1\eta_2(D, C))\, x, x\rangle \\
&\quad + (1-\alpha)^s\beta^s\langle f\,(A + t_1\eta_1(B, A), C + \lambda_2\eta_2(D, C))\, x, x\rangle \\
&\quad + \alpha^s(1-\beta)^s\langle f\,(A + t_2\eta_1(B, A), C + \lambda_1\eta_2(D, C))\, x, x\rangle \\
&\quad + \alpha^s\beta^s\langle f\,(A + t_2\eta_1(B, A), C + \lambda_2\eta_2(D, C))\, x, x\rangle \\
&= (1-\alpha)^s(1-\beta)^s\langle f\,(C_1, D_1)\, x, x\rangle + (1-\alpha)^s\beta^s\langle f\,(C_1, D_2)\, x, x\rangle \\
&\quad + \alpha^s(1-\beta)^s\langle f\,(C_2, D_1)\, x, x\rangle + \alpha^s\beta^s\langle f\,(C_2, D_2)\, x, x\rangle.
\end{aligned}$$

Hence, f is operator s-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and respect to η_2 on η_2-path P_{CW}. This completes the proof. □

If $s = 1$ we have a similar result for the operator preinvex on the co-ordinates.

Lemma 4.3. *Let $\Delta = X_1 \times X_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $X_1, X_2 \subseteq B(\mathbb{R}^n)_{sa}$ be two invex sets with respect to $\eta_1 : X_1 \times X_1 \to B(\mathbb{R}^n)_{sa}$ and $\eta_2 : X_2 \times X_2 \to B(\mathbb{R}^n)_{sa}$. Then, the function $f : \Delta \to \mathbb{R}$ be a continuous function. Suppose that η_1 and η_2 satisfy condition (C) on X_1 respectively, on X_2. Then for every $A, B \in X_1$ and $V = A + \eta_1(B, A)$, $C, D \in X_2$ and $W = C + \eta_2(D, C)$ the function f is operator preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} if and only if the function*

$$\varphi_{x,A,B,C,D}(t, \lambda) = \langle f\,(A + t\eta_1(B, A), C + \lambda\eta_2(D, C))\, x, x\rangle$$

is convex on the co-ordinates on $[0, 1]^2$ for every $x \in H$ with $\| x \|= 1$.

We now derive two dimensional version of Hermite-Hadamard type inequalities.

Theorem 4.4. *Suppose that a continuous function $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is operator preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-pathP_{CW} for all 2-tuples of selfadjoint operators in the domain of f acting* *on any Hilbert spaces X_1, X_2. If η satisfies Condition C, we have following inequalities*

$$\begin{aligned}
& f\left(\frac{2A+\eta_1(B,A)}{2}, \frac{2C+\eta_2(D,C)}{2}\right) \\
& \leq \frac{1}{2}\left[\int_0^1 f\left(\frac{2A+\eta_1(B,A)}{2}, C+\lambda\eta_2(D,C)\right)\mathrm{d}\lambda + \int_0^1 f\left(A+t\eta_1(B,A), \frac{2C+\eta_2(D,C)}{2}\right)\mathrm{d}t\right] \\
& \leq \int_0^1 \int_0^1 f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C)\mathrm{d}t\mathrm{d}\lambda \\
& \leq \frac{1}{4}\left[\int_0^1 f(A, C+\lambda\eta_2(D,C))\mathrm{d}\lambda + \int_0^1 f(B, C+\lambda\eta_2(D,C))\mathrm{d}\lambda\right. \\
& \quad \left. + \int_0^1 f(A+t\eta_1(B,A), C)\mathrm{d}t + \int_0^1 f(A+t\eta_1(B,A), D)\mathrm{d}t\right] \\
& \leq \frac{f(A,C)+F(A,D)+f(B,C)+f(B,D)}{2},
\end{aligned} \tag{4.2}$$

where $(A,C),(A,D),(B,C),(B,D) \in B(X_1)\times B(X_2)$ with spectra in Δ.

Proof. Since the spectrum of $A+t\eta_1(B,A)$ and $C+\lambda\eta_2(D,C)$ are contained in the intervals $[A, A+t\eta_1(B,A)]$ and $[C+\lambda\eta_2(D,C)]$ respectively, and f is continuous, the operator valued integrals $\int_0^1 f(A, C+\lambda\eta_2(D,C))\mathrm{d}\lambda$, $\int_0^1 f(B, C+\lambda\eta_2(D,C))\mathrm{d}\lambda$, $\int_0^1 f(A+t\eta_1(B,A), C)\mathrm{d}t$ and $\int_0^1 f(A+t\eta_1(B,A), D)\mathrm{d}t$ exists. This implies

$$\begin{aligned}
& f\left(\frac{2A+\eta_1(B,A)}{2}, C+\lambda\eta_2(D,C)\right) \leq \int_0^1 f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))\mathrm{d}t \\
& \leq \frac{f(A, C+\lambda\eta_2(D,C))+f(B, C+\lambda\eta_2(D,C))}{2}.
\end{aligned}$$

Integrating this inequality on $[0,1]$ over λ, we deduce

$$\begin{aligned}
& \int_0^1 f\left(\frac{2A+\eta_1(B,A)}{2}, C+\lambda\eta_2(D,C)\right)\mathrm{d}\lambda \leq \int_0^1 \int_0^1 f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))\mathrm{d}t\mathrm{d}\lambda \\
& \leq \frac{1}{2}\left[\int_0^1 f(A, C+\lambda\eta_2(D,C))\mathrm{d}\lambda + \int_0^1 f(B, C+\lambda\eta_2(D,C))\mathrm{d}\lambda\right].
\end{aligned} \tag{4.3}$$

By a similar argument we get

$$\begin{aligned}
& \int_0^1 f\left(A+t\eta_1(B,A), \frac{2C+\eta_2(D,C)}{2}\right)\mathrm{d}t \leq \int_0^1 \int_0^1 f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))\mathrm{d}t\mathrm{d}\lambda \\
& \leq \frac{1}{2}\left[\int_0^1 f\left(A+t\eta_1(B,A), C\right)\mathrm{d}t + \int_0^1 f\left(A+t\eta_1(B,A), D\right)\mathrm{d}t\right].
\end{aligned} \tag{4.4}$$

Summing the inequalities (4.3) and (4.4) and dividing by 2, we get the second and the third inequalities in

(4.2)

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1 f\left(\frac{2A+\eta_2(B,A)}{2}, C+\lambda\eta_2(D,C)\right)\mathrm{d}\lambda + \int_0^1 f\left(A+t\eta_1(B,A), \frac{2C+\eta_2(D,C)}{2}\right)\mathrm{d}t\right]\\
&\le \int_0^1\int_0^1 f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C)\mathrm{d}t\mathrm{d}\lambda\\
&\le \frac{1}{4}\left[\int_0^1 f(A, C+\lambda\eta_2(D,C))\mathrm{d}\lambda + \int_0^1 f(B, C+\lambda\eta_2(D,C))\mathrm{d}\lambda\right.\\
&\quad\left. + \int_0^1 f(A+t\eta_1(B,A), C)\mathrm{d}t + \int_0^1 f(A+t\eta_1(B,A), D)\mathrm{d}t\right]
\end{aligned}$$

Also one can observe

$$\begin{aligned}
&\int_0^1 f\left(\frac{2A+\eta_1(B,A)}{2}, C+\lambda\eta_2(D,C)\right)\mathrm{d}\lambda \ge f\left(\frac{2A+\eta_1(B,A)}{2}, \frac{2C+\eta_2(D,C)}{2}\right),\\
&\int_0^1 f\left(A+t\eta_1(B,A), \frac{2C+\eta_2(D,C)}{2}\right)\mathrm{d}t \ge f\left(\frac{2A+\eta_1(B,A)}{2}, \frac{2C+\eta_1(D,C)}{2}\right)
\end{aligned}$$

which give, by addition, the first inequality in (4.2). Finally, by the same inequality we can also state

$$\begin{aligned}
&\int_0^1 f\left(A, C+\lambda\eta_2(D,C)\right)\mathrm{d}\lambda \le \frac{f(A,C)+f(A,D)}{2},\\
&\int_0^1 f\left(B, C+\lambda\eta_2(D,C)\right)\mathrm{d}\lambda \le \frac{f(B,C)+f(B,D)}{2},\\
&\int_0^1 f\left(A+t\eta_1(B,A), C\right)\mathrm{d}t \le \frac{f(A,C)+f(B,C)}{2},\\
&\int_0^1 f\left(A+t\eta_1(B,A), D\right)\mathrm{d}t \le \frac{f(A,D)+f(B,D)}{2},
\end{aligned}$$

which give, by addition, the last inequality in (4.2). Thus the proof is complete. □

Theorem 4.5. *Suppose that continuous functions $f, g : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ are operators (h_1, h_2)-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} for all 2-tuples of selfadjoint operators in the domain of f, g acting on any Hilbert spaces X_1, X_2. Then for any selfadjoint operators $(A,C), (A,D), (B,C), (B,D) \in B(X_1)\times B(X_2)$, we have the inequalities*

$$\begin{aligned}
&\int_0^1\int_0^1 \langle f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))x, x\rangle\langle g(A+t\eta_1(B,A), C+\eta_2\lambda(D,C))x, x\rangle\mathrm{d}t\mathrm{d}\lambda\\
&\le \langle f(A,C)x,x\rangle\langle g(A,C)x,x\rangle\int_0^1\int_0^1 h_1^2(1-t)h_2^2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda\\
&\quad+\langle f(A,C)x,x\rangle\langle g(A,D)x,x\rangle\int_0^1\int_0^1 h_1^2(1-t)h_2(\lambda)h_2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda
\end{aligned}$$

$$+ \langle f(A,C)x,x\rangle\langle g(B,C)x,x\rangle \int_0^1\int_0^1 h_1(t)h_1(1-t)h_2^2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(A,C)x,x\rangle\langle g(B,D)x,x\rangle \int_0^1\int_0^1 h_1(t)h_1(1-t)h_2(\lambda)h_2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(A,D)x,x\rangle\langle g(A,C)x,x\rangle \int_0^1\int_0^1 h_1^2(1-t)h_2(\lambda)h_2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(A,D)x,x\rangle\langle g(A,D)x,x\rangle \int_0^1\int_0^1 h_1^2(1-t)h_2^2(\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(A,D)x,x\rangle\langle g(B,C)x,x\rangle \int_0^1\int_0^1 h_1(t)h_1(1-t)h_2(\lambda)h_2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(A,D)x,x\rangle\langle g(B,D)x,x\rangle \int_0^1\int_0^1 h_1(t)h_1(1-t)h_2^2(\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(B,C)x,x\rangle\langle g(A,C)x,x\rangle \int_0^1\int_0^1 h_1(t)h_1(1-t)h_2^2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(B,C)x,x\rangle\langle g(A,D)x,x\rangle \int_0^1\int_0^1 h_1(t)h_1(1-t)h_2(\lambda)h_2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(B,C)x,x\rangle\langle g(B,C)x,x\rangle \int_0^1\int_0^1 h_1^2(t)h_2^2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda$$
$$+ \langle f(B,C)x,x\rangle\langle g(B,D)x,x\rangle \int_0^1\int_0^1 h_1^2(t)h_2(\lambda)h_2(1-\lambda)\mathrm{d}t\mathrm{d}\lambda. \tag{4.5}$$

Proof. The demonstration is immediate taking into account that $f,g : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ are operators (h_1,h_2)-preinvex on the co-ordinates and integrating inequality over $t,\lambda \in [0,1]$. □

The coming theorems correspond to special cases $h_1(t) = t^s, h_2(\lambda) = \lambda^s$, $h_1(t) = t^{-s}, h_2(\lambda) = \lambda^{-s}$ and $h_1(t) = t, h_2(\lambda) = \lambda$ respectively.

Theorem 4.6. *Suppose that continuous functions $f,g : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ are Breckner type of operators s-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} for all 2-tuples of selfadjoint operators in the domain of f,g acting on any Hilbert spaces X_1, X_2. Then for any selfadjoint operators $(A,C),(A,D),(B,C),(B,D) \in B(X_1)\times B(X_2)$ we have the inequalities*

$$\int_0^1\int_0^1 \langle f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))x,x\rangle\langle g(A+t\eta_1(B,A), C+\eta_2\lambda(D,C))x,x\rangle\mathrm{d}t\mathrm{d}\lambda$$
$$\le \frac{1}{(2s+1)^2}\cdot U(A,B,C,D) + [\mathbb{B}(s+1,s+1)]^2\cdot V(A,B,C,D) + \frac{\mathbb{B}(s+1,s+1)}{2s+1}\cdot P(A,B,C,D), \tag{4.6}$$

holds for any $x \in H$ with $\| x \| = 1$, where

$$U(A,B,C,D) = \langle f(A,C)x,x\rangle\langle g(A,C)x,x\rangle + \langle f(A,D)x,x\rangle\langle g(A,D)x,x\rangle + \langle f(B,C)x,x\rangle\langle g(B,C)x,x\rangle + \langle f(B,D)x,x\rangle\langle g(B,D)x,x\rangle \tag{4.7}$$

$$V(A,B,C,D) = \langle f(A,C)x,x\rangle\langle g(B,D)x,x\rangle + \langle f(A,D)x,x\rangle\langle g(B,C)x,x\rangle + \langle f(B,C)x,x\rangle\langle g(A,D)x,x\rangle + \langle f(B,D)x,x\rangle\langle g(A,C)x,x\rangle \tag{4.8}$$

$$\begin{aligned} P(A,B,C,D) =& \langle f(A,C)x,x\rangle\langle g(A,D)x,x\rangle + \langle f(A,C)x,x\rangle\langle g(B,C)x,x\rangle \\ &+ \langle f(A,D)x,x\rangle\langle g(A,C)x,x\rangle + \langle f(A,D)x,x\rangle\langle g(B,D)x,x\rangle \\ &+ \langle f(B,C)x,x\rangle\langle g(A,C)x,x\rangle + \langle f(B,C)x,x\rangle\langle g(B,D)x,x\rangle \\ &+ \langle f(B,D)x,x\rangle\langle g(A,D)x,x\rangle + \langle f(B,D)x,x\rangle\langle g(B,C)x,x\rangle. \end{aligned} \tag{4.9}$$

Proof. Since f, g are operator Breckner type of s-preinvex on the co-ordinates, for every $t, \lambda \in [0,1]$ we have

$$\begin{aligned}
&\langle f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))x,x\rangle\langle g(A+t\eta_1(B,A), C+\eta_2\lambda(D,C))x,x\rangle \\
&\leq \langle((1-t)^s(1-\lambda)^s f(A,C)+(1-t)^s\lambda^s f(A,D)+t^s(1-\lambda)^s f(B,C)+t^s\lambda^s f(B,D))\,x,x\rangle \\
&\cdot \langle((1-t)^s(1-\lambda)^s g(A,C)+(1-t)^s\lambda^s g(A,D)+t^s(1-\lambda)^s g(B,C)+t^s\lambda^s g(B,D))\,x,x\rangle \\
&= (1-t)^{2s}(1-\lambda)^{2s}\langle f(A,C)x,x\rangle\langle g(A,C)x,x\rangle \\
&\quad + (1-t)^{2s}\lambda^s(1-\lambda)^s\langle f(A,C)x,x\rangle\langle g(A,D)x,x\rangle \\
&\quad + t^s(1-t)^s(1-\lambda)^{2s}\langle f(A,C)x,x\rangle\langle g(B,C)x,x\rangle \\
&\quad + t^s(1-t)^s\lambda^s(1-\lambda)^s\langle f(A,C)x,x\rangle\langle g(B,D)x,x\rangle \\
&\quad + (1-t)^{2s}\lambda^s(1-\lambda)^s\langle f(A,D)x,x\rangle\langle g(A,C)x,x\rangle \\
&\quad + (1-t)^{2s}\lambda^{2s}\langle f(A,D)x,x\rangle\langle g(A,D)x,x\rangle \\
&\quad + t^s(1-t)^s\lambda^s(1-\lambda)^s\langle f(A,D)x,x\rangle\langle g(B,C)x,x\rangle \\
&\quad + t^s(1-t)^s\lambda^{2s}\langle f(A,D)x,x\rangle\langle g(B,D)x,x\rangle \\
&\quad + t^s(1-t)^s(1-\lambda)^{2s}\langle f(B,C)x,x\rangle\langle g(A,C)x,x\rangle \\
&\quad + t^s(1-t)^s\lambda^s(1-\lambda)^s\langle f(B,C)x,x\rangle\langle g(A,D)x,x\rangle \\
&\quad + t^{2s}(1-\lambda)^{2s}\langle f(B,C)x,x\rangle\langle g(B,C)x,x\rangle \\
&\quad + t^{2s}\lambda^s(1-\lambda)^s\langle f(B,C)x,x\rangle\langle g(B,D)x,x\rangle \\
&\quad + t^s(1-t)^s\lambda^s(1-\lambda)^s\langle f(B,D)x,x\rangle\langle g(A,C)x,x\rangle \\
&\quad + t^s(1-t)^s\lambda^{2s}\langle f(B,D)x,x\rangle\langle g(A,D)x,x\rangle \\
&\quad + t^{2s}\lambda^s(1-\lambda)^s\langle f(B,D)x,x\rangle\langle g(B,C)x,x\rangle \\
&\quad + t^{2s}\lambda^{2s}\langle f(B,D)x,x\rangle\langle g(B,D)x,x\rangle.
\end{aligned} \tag{4.10}$$

Integrating both sides of (4.10) over $t, \lambda \in [0,1]$, we get the required inequalities (4.6), which completes the proof of Theorem 4.6. □

For $s = 1$ we have similar theorem of Theorem 4.6 for f, g are operators preinvex on the co-ordinates.

Theorem 4.7. *Suppose that continuous functions $f, g : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ are operators preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} for all 2-tuples of selfadjoint operators in the domain of f, g acting on any Hilbert spaces X_1, X_2. Then for any selfadjoint operators $(A,C), (A,D), (B,C), (B,D) \in B(X_1) \times B(X_2)$ we have the inequalities*

$$\begin{aligned}
&\int_0^1\int_0^1 \langle f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))x,x\rangle\langle g(A+t\eta_1(B,A), C+\eta_2\lambda(D,C))x,x\rangle \mathrm{d}t\mathrm{d}\lambda \\
&\leq \frac{4U(A,B,C,D)+V(A,B,C,D)+2P(A,B,C,D)}{36}
\end{aligned} \tag{4.11}$$

holds for any $x \in H$ with $\| x \| = 1$, where $U(A,B,C,D), V(A,B,C,D), P(A,B,C,D)$ are given to (4.7), (4.8) and (4.9).

Theorem 4.8. *Suppose that continuous functions $f, g : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ are Godunova-Levin type of operators s-preinvex on the co-ordinates with respect to η_1 on η_1-path P_{AV} and η_2 on η_2-path P_{CW} for all 2-tuples of self-adjoint operators in the domain of f, g acting on any Hilbert spaces X_1, X_2. Then for any selfadjoint operators $(A,C), (A,D), (B,C), (B,D) \in B(X_1) \times B(X_2)$ we have the inequalities*

$$\begin{aligned}
&\int_0^1\int_0^1 \langle f(A+t\eta_1(B,A), C+\lambda\eta_2(D,C))x,x\rangle\langle g(A+t\eta_1(B,A), C+\eta_2\lambda(D,C))x,x\rangle \mathrm{d}t\mathrm{d}\lambda \\
&\leq \frac{1}{(1-2s)^2}\cdot U(A,B,C,D) + [\mathbb{B}(1-s,1-s)]^2 \cdot V(A,B,C,D) + \frac{\mathbb{B}(1-s,1-s)}{2s+1}\cdot P(A,B,C,D),
\end{aligned}$$

holds for any $x \in H$ with $\| x \|= 1$, where $U(A, B, C, D)$, $V(A, B, C, D)$ and $P(A, B, C, D)$ are given by (4.7), (4.8) and (4.9) respectively.

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