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Extended Partial b-Metric Spaces and some Fixed Point Results

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Abstract. In this paper, we introduce the concept of extended partial b-metric space. We demonstrate a fundamental lemma for the convergence of sequences in such spaces. Then we prove some fixed point results for weakly contractive mappings in the setup of ordered extended partial b-metric spaces. An example is given to verify the effectiveness and applicability of our main results. An application of these results to Volterra-type integral equations is provided at the end.

1. Introduction

The concept of a b-metric space was introduced by Bakhtin [3] and then extensively used by Czerwik [4, 5] and the others.

Definition 1.1. *[4] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

On the other hand, Matthews introduced in 1994 the notion of a partial metric space.

Definition 1.2. *[8] A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:*

(p_1) $x = y$ *if and only if* $p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

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In this case, (X, p) is called a partial metric space.

As a generalization and unification of partial metric and b-metric spaces, Shukla [14] introduced the concept of partial b-metric space. In the following definition, Mustafa et al. [9] modified the concept of partial b-metric space in the sense of Shukla in order to obtain that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 1.3. *[9] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:*

(p_{b1}) $x = y \Longleftrightarrow p_b(x, x) = p_b(x, y) = p_b(y, y)$,

(p_{b2}) $p_b(x, x) \leq p_b(x, y)$,

(p_{b3}) $p_b(x, y) = p_b(y, x)$,

(p_{b4}) $p_b(x, y) \leq s(p_b(x, z) + p_b(z, y) - p_b(z, z)) + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.

The pair (X, p_b) is called a partial b-metric space.

It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converses of these facts do not hold.

In [12], Parvaneh introduced the following notion which he called p-metric space.

Definition 1.4. *Let X be a (nonempty) set. A function $d : X \times X \to R^+$ is a p-metric if there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $t \leq \Omega(t)$ for $t \in [0, +\infty)$, such that for all $x, y, z \in X$, the following conditions hold:*

(1) $d(x, y) = 0$ *iff* $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq \Omega(d(x, y) + d(y, z))$.

In this case, the pair (X, d) is called a p-metric space, or, an extended b-metric space.

It should be noted that the class of p-metric spaces is considerably larger than the class of b-metric spaces, since a b-metric is a p-metric with $\Omega(t) = st$, while a metric is a p-metric, with $\Omega(t) = t$.

Fixed point theorems in partially ordered metric spaces were firstly obtained in 2004 by Ran and Reurings [13], and by Nieto and Lopez [10]. Later, many researchers used this approach.

In this paper, we introduce the notion of extended partial b-metric space (which we also call partial p-metric space). We demonstrate a fundamental lemma for the convergence of sequences in such spaces. Further, we prove some fixed point results for weakly contractive mappings in the setup of ordered extended partial b-metric spaces. An example is provided to verify the effectiveness and applicability of our main results. An application of these results to Volterra-type integral equations is given at the end.

2. Definition and basic properties of partial p-metric spaces

Definition 2.1. *Let X be a (nonempty) set and $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for $t \in [0, +\infty)$. A function $p_p : X \times X \to R^+$ is called an extended partial b-metric, or a partial p-metric if, for all $x, y, z \in X$, the following conditions are satisfied:*

(p_{p1}) $x = y \Longleftrightarrow p_p(x, x) = p_p(x, y) = p_p(y, y)$,

(p_{p2}) $p_p(x, x) \leq p_p(x, y)$,

(p_{p3}) $p_p(x,y) = p_p(y,x)$,

(p_{p4}) $p_p(x,y) - p_p(x,x) \leq \Omega(p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(x,x))$.

The pair (X, p_p) is called a partial p-metric space, or an extended partial b-metric space.

Note that condition (p_{p4}), together with (p_{p3}), implies that also the following holds for all $x, y, z, \in X$:

$$p_p(x,y) - p_p(y,y) \leq \Omega(p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(y,y)).$$

It should be noted that the class of partial p-metric spaces is considerably larger than the class of partial b-metric spaces, since a partial b-metric is a partial p-metric with $\Omega(t) = st$, while a partial metric is a partial p-metric, with $\Omega(t) = t$. We present examples which show that a partial p-metric on X might be neither a partial metric, nor a partial b-metric on X.

Example 2.2. *Let (X, d) be a metric space and $p_p(x,y) = 1 + \xi\big(d(x,y)\big)$ where $\xi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for $t \in [0,+\infty)$ and $\xi(0) = 0$. We will show that p_p is a partial p-metric with $\Omega(t) = \xi(t)$.*

Obviously, conditions (p_{p1})–(p_{p3}) of Definition 2.1 are satisfied. On the other hand, for each $x, y, z \in X$ we obtain

$$\begin{aligned}
p_p(x,y) - p_p(x,x) &= 1 + \xi\big(d(x,y)\big) - 1 \\
&\leq \xi\big(d(x,z) + d(z,y)\big) \\
&\leq \xi\Big(\xi\big(d(x,z)\big) + \xi\big(d(z,y)\big)\Big) \\
&= \xi\Big(1 + \xi\big(d(x,z)\big) + 1 + \xi\big(d(z,y)\big) - 1 - 1\Big) \\
&= \Omega\Big(p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(x,x)\Big).
\end{aligned}$$

Hence, condition (p_{p4}) of Definition 2.1 is fulfilled and p_p is a partial p-metric on X.

In particular, one can take $\xi(t) = e^t - 1$. Then, $p_p(x,y) = e^{d(x,y)}$ is a partial p-metric with $\Omega(t) = e^t - 1$.

Example 2.3. *Let (X, d) be a metric space and $p_p(x,y) = 1 + \sinh[d(x,y)^2]$. We will show that p_p is a partial p-metric with $\Omega(t) = 2\cosh t \sinh t = \sinh 2t$.*

Obviously, conditions (p_{p1})–(p_{p3}) of Definition 2.1 are satisfied. Using the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ for all $a, b \geq 0$, we obtain that, for each $x, y, z \in X$, the following holds

$$\begin{aligned}
p_p(x,y) - p_p(x,x) &= 1 + \sinh\big(d(x,y)^2\big) - 1 \\
&\leq \sinh\Big[\big(d(x,z) + d(z,y)\big)^2\Big] \leq \sinh\Big[2\big(d(x,z)^2 + d(z,y)^2\big)\Big] \\
&\leq 2\sinh\Big[\sinh d(x,z)^2 + \sinh d(z,y)^2\Big]\cosh\Big[\sinh d(x,z)^2 + \sinh d(z,y)^2\Big] \\
&= 2\sinh[1 + \sinh d(x,z)^2 + 1 + \sinh d(z,y)^2 - 1 - 1] \\
&\quad \times \cosh[1 + \sinh d(x,z)^2 + 1 + \sinh d(z,y)^2 - 1 - 1] \\
&= \Omega(p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(x,x)).
\end{aligned}$$

Hence, condition (p_{p4}) of Definition 2.1 is fulfilled and p_p is a partial p-metric on X.

Note that (X, p_p) is not necessarily a partial metric space. For example, if $X = \mathbb{R}$ is the set of real numbers, $d(x,y) = |x-y|$, then $p_p(x,y) = 1 + \sinh(x-y)^2$ is a partial p-metric on X with $\Omega(t) = \sinh 2t$, but it is not a partial metric on X. Indeed, the ordinary (partial) triangle inequality does not hold. To see this, let $x = 2$, $y = 5$ and $z = \frac{5}{2}$. Then, $p_p(2,5) \approx 4052.54$, $p_p(2,\frac{5}{2}) \approx 1.25$ and $p_p(\frac{5}{2},5) \approx 260.01$, hence, $p_p(2,5) \nleq p_p(2,\frac{5}{2}) + p_p(\frac{5}{2},5) - p_p(\frac{5}{2},\frac{5}{2})$.

Also, p_p is not a partial b-metric. Indeed, if p_p were partial b-metric, then there would exist fixed $s \geq 1$ for which $p_p(x,y) \leq s(p_p(x,z) + p_p(z,y) - p_p(z,z)) + (\frac{1-s}{2})(p_p(x,x) + p_p(y,y))$ for all $x, y, z \geq 0$. However, taking $y = 0$ and $z = 1$, we would have $p_p(x,0) \leq s(p_p(x,1) + 1 + \sinh 1 - 1) + (\frac{1-s}{2})(1+1)$. i.e., $\sinh x^2 \leq s(1 + \sinh(x-1)^2 + \sinh 1) - s$ which cannot hold for fixed s when $x \to +\infty$.

Recall that a real function f is called super-additive if

$$f(s+t) \geq f(s) + f(t)$$

for all $t, s \in D(f)$. If f is a super-additive function, and if $0 \in D(f)$, then $f(0) \leq 0$. Indeed, super-additivity of f yields that $f(s) \leq f(s+t) - f(t)$ for all $s, t \in D(f)$. Taking $s = 0$ one has $f(0) \leq f(0+t) - f(t) = 0$. Also, it is easy to see that $2f(t) \leq f(2t)$ for each $t \in D(f)$.

Proposition 2.4. *Every partial p-metric p_p with a super-additive function Ω, defines a p-metric d_{p_p}, where*

$$d_{p_p}(x,y) = 2p_p(x,y) - p_p(x,x) - p_p(y,y)$$

for all $x, y \in X$.

Proof. Let $x, y, z \in X$. Then we have

$$\begin{aligned} d_{p_p}(x,y) &= 2p_p(x,y) - p_p(x,x) - p_p(y,y) \\ &= p_p(x,y) - p_p(x,x) + p_p(x,y) - p_p(y,y) \\ &\leq \Omega[p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(x,x)] \\ &\quad + \Omega[p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(y,y)] \\ &\leq \Omega[2p_p(x,z) + 2p_p(z,y) - 2p_p(z,z) - p_p(x,x) - p_p(y,y)] \\ &= \Omega[d_{p_p}(x,z) + d_{p_p}(z,y)]. \end{aligned}$$

□

Lemma 2.5. *Let (X, p_p) be a partial p-metric space. Then,*

(A) if $p_p(x,y) = 0$, then $x = y$;

(B) if $x \neq y$, then $p_p(x,y) > 0$.

The concepts of p_p-convergence, p_p-Cauchyness and p_p-completeness are the same as in the setting of a partial b-metric [9]. The following lemma shows the relationship between these concepts in two spaces (X, p_p) and (X, d_{p_p}). The proof is similar to the ones of Lemma 2.2 in [11] and Lemma 1 in [9].

Lemma 2.6. *Let (X, p_p) be a partial p-metric space with super-additive function Ω.*

1. *A sequence $\{x_n\}$ is a p_p-Cauchy sequence in (X, p_p) if and only if it is a p-Cauchy sequence in the p-metric space (X, d_{p_p}).*
2. *The space (X, p_p) is p_p-complete if and only if the p-metric space (X, d_{p_p}) is p-complete. Moreover, $\lim_{n\to\infty} d_{p_p}(x, x_n) = 0$ if and only if*

$$\lim_{n\to\infty} p_p(x, x_n) = \lim_{n,m\to\infty} p_p(x_n, x_m) = p_p(x,x).$$

The following useful lemma (adapted according to [2]) will be applied in proving our main results.

Lemma 2.7. *Let (X, p_p) be a partial p-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have*

$$\begin{aligned} &\Omega^{-1}\Big(\Omega^{-1}[p_p(x,y) - p_p(x,x)] - 2p_p(x,x)\Big) - p_p(y,y) \\ &\quad \leq \liminf_{n\to\infty} p_p(x_n, y_n) \leq \limsup_{n\to\infty} p_p(x_n, y_n) \\ &\quad \leq \Omega\Big(2p_p(x,x) + \Omega[p_p(x,y) + p_p(y,y)]\Big) + p_p(x,x). \end{aligned}$$

In particular, if $p_p(x,y)=0$*, then we have* $\lim_{n\to\infty} p_p(x_n,y_n)=0$.

Moreover, for each $z\in X$ *we have*

$$\begin{aligned}\Omega^{-1}[p_p(x,z)-p_p(x,x)]-p_p(x,x)\\ \le \liminf_{n\to\infty} p_p(x_n,z)\le \limsup_{n\to\infty} p_p(x_n,z)\\ \le \Omega[p_p(x,x)+p_p(x,z)]+p_p(x,x).\end{aligned}$$

In particular, if $p_p(x,z)=0$*, then we have* $\lim_{n\to\infty} p_p(x_n,z)=0$.

Proof. Using property (p_{p4}) of the partial p-metric space and properties of function Ω, it is easy to see that

$$\begin{aligned}p_p(x,y)-p_p(x,x) &\le \Omega(p_p(x,x_n)+p_p(x_n,y))\\ &\le \Omega(p_p(x,x_n)+\Omega[p_p(x_n,y_n)+p_p(y_n,y)]+p_p(x_n,x_n))\end{aligned}$$

and

$$\begin{aligned}p_p(x_n,y_n)-p_p(x_n,x_n) &\le \Omega(p_p(x_n,x)+p_p(x,y_n))\\ &\le \Omega\Big(p_p(x_n,x)+\Omega[p_p(x,y)+p_p(y,y_n)]+p_p(x,x)\Big).\end{aligned}$$

Taking the lower limit as $n\to\infty$ in the first inequality one has

$$p_p(x,y)-p_p(x,x)\le \Omega\Big(p_p(x,x)+\Omega[\liminf_{n\to\infty} p_p(x_n,y_n)+p_p(y,y)]+p_p(x,x)\Big),$$

which yields that

$$\Omega^{-1}\Big[\Omega^{-1}[p_p(x,y)-p_p(x,x)]-2p_p(x,x)\Big]-p_p(y,y)\le \liminf_{n\to\infty} p_p(x_n,y_n).$$

Taking the upper limit as $n\to\infty$ in the second inequality we obtain

$$\limsup_{n\to\infty} p_p(x_n,y_n)\le \Omega\Big(p_p(x,x)+\Omega[p_p(x,y)+p_p(y,y)]+p_p(x,x)\Big)+p_p(x,x).$$

If $p_p(x,y)=0$, then $p_p(x,x)=0$ and $p_p(y,y)=0$. Therefore, we have $\lim_{n\to\infty} p_p(x_n,y_n)=0$.

Now, suppose that $\{x_n\}$ is convergent to x and $z\in X$. Again, using the triangle inequality in the partial p-metric space, it is easy to see that

$$p_p(x,z)-p_p(x,x)\le \Omega\Big(p_p(x,x_n)+p_p(x_n,z)\Big)$$

and

$$p_p(x_n,z)-p_p(x_n,x_n)\le \Omega\Big(p_p(x_n,x)+p_p(x,z)\Big).$$

Taking the lower limit as $n\to\infty$ in the first inequality one has

$$\Omega^{-1}[p_p(x,z)-p_p(x,x)]-p_p(x,x)\le \liminf_{n\to\infty} p_p(x_n,z),$$

and taking the upper limit as $n\to\infty$ in the second inequality we obtain

$$\limsup_{n\to\infty} p_p(x_n,z)\le \Omega[p_p(x,x)+p_p(x,z)]+p_p(x,x).$$

□

A triplet $(X,\preceq,p_p)$ will be called an ordered partial p-metric space (ordered PPMS, for short) if $(X,\preceq)$ is a partially ordered set and p_p is a partial p-metric on X.

Recall that a function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function [7], if the following properties are satisfied:

1. ψ is continuous and nondecreasing;
2. $\psi(t)=0$ if and only if $t=0$.

3. Fixed point results in ordered partial p-metric spaces

Definition 3.1. *Let $(X, \preceq, p_p)$ be an ordered partial p-metric space with function Ω and let $f : X \to X$ be a mapping. Set*

$$M^f(x, y) = \max\Big\{p_p(x, y), p_p(x, fx) + p_p(y, fy), p_p(x, fy) - p_p(x, x), p_p(y, fx)\Big\}.$$

We say that f is a $(\psi, \varphi)_\Omega$-weakly contractive mapping, if there exist two altering distance functions ψ and φ such that

$$\psi\Big(\Omega^2(2p_p(fx, fy))\Big) \leq \psi(M^f(x, y)) - \varphi(M^f(x, y)) \tag{1}$$

for all comparable elements $x, y \in X$.

First, we prove the following result.

Theorem 3.2. *Let $(X, \preceq, p_p)$ be an ordered p_p-complete PPMS with super-additive function Ω. Let $f : X \to X$ be a non-decreasing continuous mapping and suppose that f is a $(\psi, \varphi)_\Omega$-weakly contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

Proof. Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. Let (x_n) be the sequence in X such that $x_{n+1} = fx_n$, for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$. By induction, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (1), we have

$$\begin{aligned}\psi(\Omega^2(2p_p(x_n, x_{n+1}))) &= \psi(\Omega^2(2p_p(fx_{n-1}, fx_n)))\\ &\leq \psi(M^f(x_{n-1}, x_n)) - \varphi(M^f(x_{n-1}, x_n)),\end{aligned} \tag{2}$$

where

$$\begin{aligned}M^f(x_{n-1}, x_n) &= \max\Big\{p_p(x_{n-1}, x_n), p_p(x_{n-1}, fx_{n-1}) + p_p(x_n, fx_n),\\ &\qquad p_p(x_{n-1}, fx_n) - p_p(x_{n-1}, x_{n-1}), p_p(x_n, fx_{n-1})\Big\}\\ &= \max\Big\{p_p(x_{n-1}, x_n), p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1}),\\ &\qquad p_p(x_{n-1}, x_{n+1}) - p_p(x_{n-1}, x_{n-1}), p_p(x_n, x_n)\Big\}\\ &\leq \max\Big\{p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1}),\\ &\qquad \Omega(p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1})), p_p(x_n, x_n)\Big\}\\ &= \Omega(p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1})).\end{aligned} \tag{3}$$

From (2) and (3) and the properties of ψ and φ, we get

$$\begin{aligned}\psi(\Omega^2(2p_p(x_n, x_{n+1}))) &\leq \psi\Big(\Omega(p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1}))\Big)\\ &\qquad - \varphi\Big(\max\Big\{p_p(x_{n-1}, x_n), p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1}),\\ &\qquad p_p(x_{n-1}, x_{n+1}) - p_p(x_{n-1}, x_{n-1}), p_p(x_n, x_n)\Big\}\Big)\\ &< \psi\Big(\Omega(p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1}))\Big).\end{aligned} \tag{4}$$

By the properties of functions ψ and Ω, it follows that

$$2p_p(x_n, x_{n+1}) \leq \Omega(2p_p(x_n, x_{n+1})) < p_p(x_{n-1}, x_n) + p_p(x_n, x_{n+1}),$$

i.e.

$$p_p(x_n, x_{n+1}) < p_p(x_{n-1}, x_n).$$

Therefore, $\{p_p(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is a decreasing sequence of positive numbers. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p_p(x_n, x_{n+1}) = r.$$

Letting $n \to \infty$ in (4), we get

$$\begin{aligned}&\psi(\Omega^2(2r)) \leq \psi(\Omega(2r))\\ &\quad - \varphi\Big(\max\Big\{r, r + r, \liminf_{n\to\infty}[p_p(x_{n-1}, x_{n+1}) - p_p(x_{n-1}, x_{n-1})], \liminf_{n\to\infty} p_p(x_n, x_n)\Big\}\Big)\\ &\leq \psi(\Omega(2r)),\end{aligned}$$

which is only possible if $\Omega(2r) \leq 2r$. Thus, according to the assumptions on Ω, we have

$$r = \lim_{n\to\infty} p_p(x_n, x_n) = \lim_{n\to\infty} p_p(x_n, x_{n+1}) = 0. \tag{5}$$

Next, we show that $\{x_n\}$ is a p_p-Cauchy sequence in X. For this, we have to show that $\{x_n\}$ is a p-Cauchy sequence in (X, d_{p_p}) (see Lemma 2.6). Suppose the contrary, that is, $\{x_n\}$ is not a p-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d_{p_p}(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{6}$$

This means that

$$d_{p_p}(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{7}$$

From (6) and using the triangular inequality, we get

$$\varepsilon \leq d_{p_p}(x_{m_i}, x_{n_i}) \leq \Omega\Big(d_{p_p}(x_{m_i}, x_{n_i-1}) + d_{p_p}(x_{n_i-1}, x_{n_i})\Big). \tag{8}$$

Taking the upper limit as $i \to \infty$, and using (7), we get

$$\Omega^{-1}(\varepsilon) \leq \limsup_{i\to\infty} d_{p_p}(x_{m_i}, x_{n_i-1}) \leq \varepsilon. \tag{9}$$

Also, from (8) and (9),

$$\varepsilon \leq \liminf_{i\to\infty} d_{p_p}(x_{m_i}, x_{n_i}) \leq \limsup_{i\to\infty} d_{p_p}(x_{m_i}, x_{n_i}) \leq \Omega(\varepsilon). \tag{10}$$

Further,

$$d_{p_p}(x_{m_i}, x_{n_i}) \leq \Omega\Big(d_{p_p}(x_{m_i}, x_{m_i+1}) + d_{p_p}(x_{m_i+1}, x_{n_i})\Big)$$

and hence,

$$\limsup_{i\to\infty} d_{p_p}(x_{m_i+1}, x_{n_i}) \geq \Omega^{-1}(\varepsilon). \tag{11}$$

Finally,

$$d_{p_p}(x_{m_i+1}, x_{n_i-1}) \leq \Omega\Big(d_{p_p}(x_{m_i+1}, x_{m_i}) + d_{p_p}(x_{m_i}, x_{n_i-1})\Big)$$

and hence,

$$\limsup_{i\to\infty} d_{p_p}(x_{m_i+1}, x_{n_i-1}) \leq \Omega(\varepsilon). \tag{12}$$

On the other hand, by the definition of d_{p_p} and (5)

$$\limsup_{i\to\infty} d_{p_p}(x_{m_i}, x_{n_i-1}) = 2\limsup_{i\to\infty} p_p(x_{m_i}, x_{n_i-1}). \tag{13}$$

Hence, by (7), (9) and (13),

$$\frac{\Omega^{-1}(\varepsilon)}{2} \leq \limsup_{i\to\infty} p_p(x_{m_i}, x_{n_i-1}) \leq \frac{\varepsilon}{2}. \tag{14}$$

Similarly, according to (10)–(12) and (13)

$$\frac{\varepsilon}{2} \leq \liminf_{i\to\infty} p_p(x_{m_i}, x_{n_i}) \leq \limsup_{i\to\infty} p_p(x_{m_i}, x_{n_i}) \leq \frac{\Omega(\varepsilon)}{2}. \tag{15}$$

$$\limsup_{i\to\infty} p_p(x_{m_i+1}, x_{n_i}) \geq \frac{\Omega^{-1}(\varepsilon)}{2}. \tag{16}$$

$$\limsup_{i\to\infty} p_p(x_{m_i+1}, x_{n_i-1}) \leq \frac{\Omega(\varepsilon)}{2}. \tag{17}$$

From (1), we have

$$\begin{aligned}\psi(\Omega^2(2p_p(x_{m_i+1}, x_{n_i}))) &= \psi(\Omega^2(2p_p(fx_{m_i}, fx_{n_i-1}))) \\ &\leq \psi(M^f(x_{m_i}, x_{n_i-1})) - \varphi(M^f(x_{m_i}, x_{n_i-1})),\end{aligned} \tag{18}$$

where

$$\begin{aligned}M^f(x_{m_i}, x_{n_i-1}) &= \max\Big\{p_p(x_{m_i}, x_{n_i-1}), p_p(x_{m_i}, fx_{m_i}) + p_p(x_{n_i-1}, fx_{n_i-1}), \\ &\qquad p_p(x_{m_i}, fx_{n_i-1}) - p_p(x_{m_i}, x_{m_i}), p_p(fx_{m_i}, x_{n_i-1})\Big\} \\ &= \max\Big\{p_p(x_{m_i}, x_{n_i-1}), p_p(x_{m_i}, x_{m_i+1}) + p_p(x_{n_i-1}, x_{n_i}), \\ &\qquad p_p(x_{m_i}, x_{n_i}) - p_p(x_{m_i}, x_{m_i}), p_p(x_{m_i+1}, x_{n_i-1})\Big\}.\end{aligned} \tag{19}$$

Taking the upper limit as $i \to \infty$ in (19) and using (5), (14), (16) and (17), we get

$$\begin{aligned}\limsup_{i\to\infty} M^f(x_{m_i}, x_{n_i-1}) &= \max\Big\{\limsup_{i\to\infty} p_p(x_{m_i}, x_{n_i-1}), 0 + 0, \\ &\qquad \limsup_{i\to\infty} p_p(x_{m_i}, x_{n_i}), \limsup_{i\to\infty} p_p(x_{m_i+1}, x_{n_i-1})\Big\} \\ &\leq \max\Big\{\frac{\varepsilon}{2}, \frac{\Omega(\varepsilon)}{2}, \frac{\Omega(\varepsilon)}{2}\Big\} = \frac{\Omega(\varepsilon)}{2}.\end{aligned} \tag{20}$$

Now, taking the upper limit as $i \to \infty$ in (18) and using (14) and (20), we have

$$\begin{aligned}\psi\Big(\frac{\Omega(\varepsilon)}{2}\Big) \leq \psi\Big(\Omega(\varepsilon)\Big) &\leq \psi\Big(\Omega^2(2\limsup_{i\to\infty} p_p(x_{m_i+1}, x_{n_i}))\Big) \\ &\leq \psi(\limsup_{i\to\infty} M^f(x_{m_i}, x_{n_i-1})) - \liminf_{i\to\infty} \varphi(M^f(x_{m_i}, x_{n_i-1})) \\ &\leq \psi\Big(\frac{\Omega(\varepsilon)}{2}\Big) - \varphi\Big(\liminf_{i\to\infty} M^f(x_{m_i}, x_{n_i-1})\Big),\end{aligned}$$

which further implies that

$$\varphi\Big(\liminf_{i\to\infty} M^f(x_{m_i}, x_{n_i-1})\Big) = 0,$$

so $\liminf_{i\to\infty} M^f(x_{m_i}, x_{n_i-1}) = 0$, a contradiction with (19) and (15).

Thus, we have proved that $\{x_n\}$ is a p-Cauchy sequence in the p-metric space (X, d_{p_p}). Since (X, p_p) is p_p-complete, then from Lemma 2.6, (X, d_{p_p}) is a p-complete p-metric space. Therefore, the sequence $\{x_n\}$ converges to some $z \in X$, that is, $\lim_{n\to\infty} d_{p_p}(x_n, z) = 0$. Again, from Lemma 2.6,

$$\lim_{n\to\infty} p_p(z, x_n) = \lim_{n\to\infty} p_p(x_n, x_n) = p_p(z, z).$$

On the other hand, (5) yields that

$$\lim_{n\to\infty} p_p(z, x_n) = \lim_{n\to\infty} p_p(x_n, x_n) = p_p(z, z) = 0.$$

Using the triangular inequality, we get

$$p_p(z, fz) - p_p(z, z) \le \Omega\Big(p_p(z, fx_n) + p_p(fx_n, fz)\Big).$$

Letting $n \to \infty$ and using the continuity of f and Ω, and $p_p(z, z) = 0$, we get

$$p_p(z, fz) \le \Omega\Big(\lim_{n\to\infty} p_p(z, x_{n+1}) + \lim_{n\to\infty} p_p(fx_n, fz)\Big) = \Omega(p_p(fz, fz)). \tag{21}$$

Note that from (1), we have

$$\psi\Big(\Omega(2p_p(fz, fz))\Big) \le \psi\Big(\Omega^2(2p_p(fz, fz))\Big) \le \psi(M^f(z, z)) - \varphi(M^f(z, z)), \tag{22}$$

where

$$\begin{aligned} M^f(z, z) &= \max\Big\{p_p(z, z), p_p(z, fz) + p_p(z, fz), p_p(z, fz) - p_p(z, z), p_p(z, fz)\Big\} \\ &= 2p_p(fz, z). \end{aligned}$$

Suppose that $fz \ne z$, i.e., $p_p(fz, z) > 0$. Then, by the properties of φ, we get from (22)

$$\psi(\Omega(2p_p(fz, fz))) < \psi(2p_p(fz, z)).$$

Now, using properties of ψ and super-additivity of Ω, we have

$$2\Omega(p_p(fz, fz)) \le \Omega(2p_p(fz, fz)) < 2p_p(fz, z).$$

Finally, (21) implies that $2\Omega(p_p(fz, fz)) < 2\Omega(p_p(fz, fz))$, a contradiction. Hence, we have $p_p(fz, z) = 0$, and so $fz = z$. Thus, z is a fixed point of f. □

An ordered PPMS $(X, \preceq, p_p)$ is said to have sequential limit comparison (s.l.c.) property if for every nondecreasing sequence $\{x_n\}$ in X, the convergence of $\{x_n\}$ to some $x \in X$ yields that $x_n \preceq x$ for all $n \in \mathbb{N}$. We will show that the continuity of f in Theorem 3.2 can be replaced by s.l.c. property of $(X, \preceq, p_p)$.

Theorem 3.3. *Under the hypotheses of Theorem 3.2, without the continuity assumption on f, assume that $(X, \preceq, p_p)$ has the s.l.c. property. Then f has a fixed point in X.*

Proof. Following similar arguments as those given in the proof of Theorem 3.2, we construct a non-decreasing sequence $\{x_n\}$ in X such that $x_n \to z$, for some $z \in X$. Using the s.l.c. property on X, we have $x_n \preceq z$, for all $n \in \mathbb{N}$. Now, we show that $fz = z$. By (1), we have

$$\begin{aligned}\psi\Big(\Omega^2(2p_p(x_{n+1}, fz))\Big) &= \psi\Big(\Omega^2(2p_p(fx_n, fz))\Big)\\ &\leq \psi(M^f(x_n, z)) - \varphi(M^f(x_n, z)),\end{aligned} \tag{23}$$

where

$$\begin{aligned}&M^f(x_n, z)\\ &\quad= \max\Big\{p_p(x_n, z), p_p(x_n, fx_n) + p_p(z, fz), p_p(x_n, fz) - p_p(x_n, x_n), p_p(fx_n, z)\Big\}\\ &\quad= \max\Big\{p_p(x_n, z), p_p(x_n, x_{n+1}) + p_p(z, fz), p_p(x_n, fz) - p_p(x_n, x_n), p_p(x_{n+1}, z)\Big\}.\end{aligned} \tag{24}$$

Letting $n \to \infty$ in (24) and using Lemma 2.7, we get

$$\begin{aligned}\Omega^{-1}[p_p(z, fz)] &= \min\Big\{p_p(z, fz), \Omega^{-1}[p_p(z, fz) - p_p(z, z)] - p_p(z, z)\Big\}\\ &\leq \liminf_{i\to\infty} M^f(x_n, z) \leq \limsup_{i\to\infty} M^f(x_n, z)\\ &\leq \max\Big\{p_p(z, fz), \Omega[p_p(z, z) + p_p(z, fz)] + p_p(z, z)\Big\}\\ &= \Omega[p_p(z, fz)].\end{aligned} \tag{25}$$

Again, taking the upper limit as $n \to \infty$ in (23) and using Lemma 2.7 and (25) we get

$$\begin{aligned}\psi(\Omega^2[\Omega^{-1}[p_p(z, fz)]]) &\leq \psi(\Omega^2[\limsup_{n\to\infty} p_p(x_{n+1}, fz)])\\ &\leq \psi(\Omega^2[2\limsup_{n\to\infty} p_p(x_{n+1}, fz)])\\ &\leq \psi(\limsup_{n\to\infty} M^f(x_n, z)) - \liminf_{n\to\infty} \varphi(M^f(x_n, z))\\ &\leq \psi(\Omega[p_p(z, fz)]) - \varphi(\liminf_{n\to\infty} M^f(x_n, z)).\end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M^f(x_n, z)) \leq 0$, i.e., $\liminf_{n\to\infty} M^f(x_n, z) = 0$. Thus, from (25) we get $fz = z$ and hence z is a fixed point of f. □

Corollary 3.4. *Let $(X, \preceq, p_p)$ be a p_p-complete ordered PPMS with super-additive function Ω, and let $f : X \to X$ be a non-decreasing mapping. Let f be continuous, or $(X, \preceq, p_p)$ possesses the s.l.c. property. Suppose that there exists $k \in [0, 1)$ such that*

$$\Omega^2(2p_p(fx, fy)) \leq k \max\Big\{p_p(x, y), p_p(x, fx) + p_p(y, fy), p_p(x, fy) - p_p(x, x), p_p(y, fx)\Big\},$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Follows from Theorems 3.2 and 3.3 by taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$, for all $t \in [0, +\infty)$. □

Corollary 3.5. *Let $(X, \preceq, p_p)$ be a p_p-complete ordered PPMS with super-additive function Ω, and let $f : X \to X$ be a non-decreasing mapping. Let f be continuous, or $(X, \preceq, p_p)$ possesses the s.l.c. property. Suppose that there exist coefficients $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + \delta \in [0, 1)$ such that*

$$\Omega^2(2p_p(fx, fy)) \leq \alpha p_p(x, y) + \beta[p_p(x, fx) + p_p(y, fy)] + \gamma[p_p(x, fy) - p_p(x, x)] + \delta p_p(y, fx),$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Taking $p_p(x,y) = 1 + \sinh(d(x,y)^2)$ where $(X, \preceq, d)$ is a complete ordered metric space and according to Example 2.3 and Corollary 3.6 we have the following result.

Corollary 3.6. *Let $(X, \preceq, d)$ be a complete ordered metric space and let $f : X \to X$ be a non-decreasing mapping. Let f be continuous, or $(X, \preceq, d)$ possesses the s.l.c. property. Suppose that there exists a coefficient $\alpha \in [0,1)$ such that*

$$\sinh\Big[2\sinh\big[4 + 4\sinh(d(fx, fy)^2)\big]\Big] \le \alpha\Big[1 + \sinh(d(x,y)^2)\Big],$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Remark 3.7. *In Theorems 3.2 and 3.3, it can be proved in a standard way that f has a unique fixed point provided that all fixed points of f are comparable.*

The usability of these results is demonstrated by the following example.

Example 3.8. *Let $X = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ be equipped with the following partial order $\preceq$:*

$$\preceq := \{(0,0), (\tfrac{1}{2}, \tfrac{1}{2}), (1,1), (\tfrac{3}{2}, \tfrac{3}{2}), (2,1), (2,2)\}.$$

Define a partial p-metric $p_p : X \times X \to \mathbb{R}^+$ by

$$p_p(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1 + \sinh[(x+y)^2], & \text{if } x \neq y. \end{cases}$$

It is easy to see that (X, p_p) is a p_p-complete PPMS, with $\Omega(t) = \sinh 2t$ (which is super-additive).

Define a self-map f by

$$f = \begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ 0 & 1 & 1 & \frac{1}{2} & 1 \end{pmatrix}.$$

We see that f is a non-decreasing mapping and that f is continuous.

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \sqrt[3]{t^2}$ and $\varphi(t) = \frac{1}{3}\sqrt[4]{t^3}$. In order to check that f is a $(\psi,\varphi)_\Omega$-weakly contractive mapping, only the cases $x = 1$, $y = 2$ and $x = 1$, $y = 2$ are nontrivial. Then,

$$\begin{aligned} M^f(1,2) &= \max\Big\{p_p(1,2), p_p(1,f1) + p_p(2,f2), p_p(1,f2) - p_p(1,1), p_p(2,f1)\Big\} \\ &= \max\Big\{p_p(1,2), p_p(1,1) + p_p(2,1), 0, p_p(2,1)\Big\} \\ &= p_p(1,1) + p_p(2,1) \\ &= 1 + \sinh 9 \approx 4052.54 \\ &= M^f(2,1). \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi(\Omega^2(2p_p(f1,f2))) &= \psi(\Omega^2(2\cdot 0)) = \psi(\sinh 2(\sinh 2 \cdot 0)) = 0 \\ &\le 254.23 - 169.34 \\ &\approx \psi(M^f(1,2)) - \varphi(M^f(1,2)). \end{aligned}$$

Thus, all the conditions of Theorem 3.2 are satisfied and hence f has a fixed point. Indeed, 0 and 1 are two fixed points of f. Note that the set $(\{0,1\}, \preceq)$ is not well ordered (i.e., elements 0 and 1 are not comparable).

Note that if the same example is considered in the space without order, then the contractive condition is not satisfied. For example,

$$\begin{aligned} M^f(0,\tfrac{3}{2}) &= \max\left\{p_p(0,\tfrac{3}{2}), p_p(0,f0)+p_p(\tfrac{3}{2},f\tfrac{3}{2}), p_p(0,f\tfrac{3}{2}), p_p(\tfrac{3}{2},f0)\right\} \\ &= \max\left\{p_p(0,\tfrac{3}{2}), p_p(0,0)+p_p(\tfrac{3}{2},\tfrac{1}{2}), p_p(0,\tfrac{1}{2}), p_p(\tfrac{3}{2},0)\right\} \\ &= p_p(\tfrac{3}{2},\tfrac{1}{2}) = 1+\sinh 4 \approx 28.29. \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi(\Omega^2(2p_p(f0,f\tfrac{3}{2}))) &= \psi(\Omega^2(2\cdot[1+\sinh\tfrac{1}{4}])) \approx \psi(95942.58) = 2095.76 \\ &\nleq 9.28 - 4.092 \\ &\approx \psi(M^f(0,\tfrac{3}{2})) - \varphi(M^f(0,\tfrac{3}{2})) \end{aligned}$$

(the same effect would be obtained with arbitrary altering distance functions ψ and φ).

4. Existence theorem for solutions of a Volterra-type integral equation

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 6] and references therein). In this section, we apply our result to the existence of a solution of an integral equation. Consider the integral equation

$$x(t) = p(t) + \int_0^t f(t,r,x(r))\,dr, \quad t \in I = [0,T], \tag{26}$$

where $p : I \to \mathbb{R}$ and $f : I \times I \times \mathbb{R} \to \mathbb{R}$ are given functions. The purpose of this section is to provide an existence theorem for solutions of the equation (26) that belongs to $X = C(I,\mathbb{R})$ (the set of continuous real functions defined on I), via the result obtained in Theorem 3.3.

We endow X with the partial order $\preceq$ given by

$$x \preceq y \iff x(t) \le y(t), \quad \text{for all } t \in I.$$

For $x \in X$ define

$$\|x\|_\tau = \max_{t\in I} |x(t)|e^{-\tau t},$$

where $\tau \ge 1$ is taken arbitrary. Notice that $\|\cdot\|_\tau$ is a norm equivalent to the maximum norm and $(X, \|\cdot\|_\tau)$ is a Banach space. The metric induced by this norm is given by

$$d_\tau(x,y) = \|x-y\|_\tau = \max_{t\in I} |x(t)-y(t)|e^{-\tau t},$$

for all $x, y \in X$.

Now, let $\xi : [0,+\infty) \to [0,+\infty)$ be a strictly increasing continuous function with $t \le \xi(t)$ and consider X endowed with the partial p-metric given by

$$\rho_\tau(x,y) = 1 + \xi\Big(d_\tau(x,y)\Big), \text{ for } x, y \in X$$

(see Example 2.2). Obviously, (X, ρ_τ) is p_p-complete. It is easy to prove (see, e.g., [10]) that $(X, \preceq, p_p)$ has the s.l.c. property.

Define $F : X \to X$ by

$$F(x(t)) = p(t) + \int_0^T f(t,r,x(r))\,dr, \quad x \in X,\ t \in I.$$

Clearly, a function $u \in X$ is a solution of (26) if and only if it is a fixed point of F.

We will consider the equation (26) under the following assumptions:

(i) $p : I \to \mathbb{R}$ and $f : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

(ii) if $x \preceq y$, then

$$f(t, r, x(r)) \leq f(t, r, y(r)), \text{ for all } t, r \in I.$$

(iii) For all $x, y \in X$ with $x \preceq y$, and for all $t \in I$,

$$\xi^2\Big(2 + 2\xi\Big(e^{\tau T} \int_0^T \Big|\Big(f(t, r, x(r)) - f(t, r, y(r))\Big)e^{-\tau t}\Big|\, dr\Big)\Big) \leq \ln(1 + d_\tau(x, y)).$$

(iv) There exists a continuous function $x_0 : I \to \mathbb{R}$ such that

$$x_0(t) \leq p(t) + \int_0^t f(t, r, x_0(r))\, dr, \quad t \in I.$$

Theorem 4.1. *Under assumptions (i)–(iv), the equation* (26) *has a solution in* X*, where* $X = C([0, T], \mathbb{R})$.

Proof. It follows from (ii) that the mapping F is non-decreasing w.r.t. $\preceq$. Now, we have, for all $t \in I$,

$$\begin{aligned}
&\xi^2\Big(2 + 2\xi\Big(\Big|Fx(t) - Fy(t)\Big|\Big)\Big) \\
&\leq \xi^2\Big(2 + 2\xi\Big(\int_0^T \Big|f(t, r, x(r)) - f(t, r, y(r))\Big|\, dr\Big)\Big) \\
&\leq \xi^2\Big(2 + 2\xi\Big(e^{\tau T} \int_0^T \Big|\Big(f(t, r, x(r)) - f(t, r, y(r))\Big)e^{-\tau t}\Big|\, dr\Big)\Big) \\
&\leq \ln(1 + d_\tau(x, y)) \leq \ln(1 + \xi(d_\tau(x, y))) \\
&\leq \ln(1 + M^F(x, y)) = M^F(x, y) - \Big(M^F(x, y) - \ln(1 + M^F(x, y))\Big),
\end{aligned}$$

where

$$M^F(x, y) = \max\Big\{\rho_\tau(x, y), \rho_\tau(x, Fx) + \rho_\tau(y, Fy), \rho_\tau(y, Fx) - \rho_\tau(Fx, Fx), \rho_\tau(x, Fy)\Big\}.$$

Hence, taking $\psi(t) = t$, $\varphi(t) = t - \ln(1 + t)$ and $\Omega = \xi$, we get that

$$\psi(\Omega^2(2\rho_\tau(Fx, Fy))) \leq \psi(M^F(x, y)) - \varphi(M^F(x, y)).$$

Let x_0 be the function appearing in assumption (iv). Then we get $x_0 \preceq F(x_0)$. Thus, all the assumptions of Theorem 3.3 are fulfilled and we deduce the existence of $u \in X$ such that $u = F(u)$. □

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References

[1] Ravi P. Agarwal, N. Hussain and M.A. Taoudi, *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstr. Appl. Anal., **2012** (2012), Article ID 245872, 15 pp.

[2] A. Aghajani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca. **64**, 4 (2014), 941–960.

[3] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, In: *Functional Analysis*, vol. 30, pp. 26–37, Ulianowsk Gos. Ped. Inst. Ulianowsk (1989).
[4] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostrav., **1** (1993), 5–11.
[5] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276.
[6] N. Hussain, A.R. Khan and Ravi P. Agarwal, *Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces*, J. Nonlinear Convex Anal., **11**(3) (2010), 475-489.
[7] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., **30** (1984), 1–9.
[8] S.G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183–197.
[9] Z. Mustafa, J.R. Roshan, V. Parvaneh, Z. Kadelburg, *Common fixed point results in ordered partial b-metric spaces*, J. Inequal. Appl. **2013**:562 (2013).
[10] J.J. Nieto and R.R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239.
[11] S. Oltra and O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste, **36** (2004), 17-26.
[12] V. Parvaneh and S.J. Hosseini Ghoncheh, *Fixed points of $(\psi,\varphi)_\Omega$-contractive mappings in ordered p-metric spaces*, submitted.
[13] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443.
[14] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math. **11** (2014), 703–711.