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Statistical Causality and Local Uniqueness for Solutions of the Martingale Problem

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Abstract. In this paper we consider the concept of statistical causality between filtrations associated with stopping times, which is based on Granger's definition of causality. Especially, we consider a generalization of a causality relationship "**G** is a cause of **E** within **H**" from fixed to stopping time. Then we apply the given causality concept to local uniqueness for the solution of the martingale problem. Also, we give some applications in finance.

1. Introduction

Many of the systems to which it is natural to apply tests of causality, take place in continuous time (see [4–7, 16–19, 21]). So, in this paper we consider the continuous time processes. Continuous time models become more and more frequent in econometrics, demography, finance.

The paper is organized as follows. After Introduction, in Section 2, we present a generalization of a causality concept "**G** is a cause of **E** within **H**" which is based on Granger's definition of causality (see [7]). The given causality concept can be connected to definitions of weak solutions and local weak solutions of stochastic differential equations driven with semimartingales (see [21, 22]). This concept is closely connected to the extremality of measures and the martingale problem as it is shown in [21]. The given concept of causality was extended from fixed to continuous times in [20].

The last two sections contain our main results. In Section 3 we give a connection between a local weak solution of stochastic differential equation driven with semimartingale (given by the Definition 5 in [22]) and a solution of stopped martingale problem, associated to the same equation.

In Section 4, we relate the given concept of statistical causality which involves stopping times to the local uniqueness for the solutions of the martingale problem.

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2. Preliminary Notions and Definitions

Causality is, in any case, a prediction property and the central question is: is it possible to reduce available information in order to predict a given filtration?

A probabilistic model for a time–dependent system is described by $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t,\ t \in I, I \subseteq R^+\}$ is a "framework" filtration. We suppose that the filtration $(\mathcal{F}_t)$ satisfies the "usual conditions", which means that $(\mathcal{F}_t)$ is right continuous and each $(\mathcal{F}_t)$ is complete. $\mathcal{F}_\infty = \bigvee_{t\in I} \mathcal{F}_t$ is the smallest σ-algebra containing all the $(\mathcal{F}_t)$ (even if $\sup I < +\infty$). An analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_\mathbf{t}\}$, $\mathbf{G} = \{\mathcal{G}_\mathbf{t}\}$ and $\mathbf{E} = \{\mathcal{E}_\mathbf{t}\}$. It is said that the filtration $\mathbf{G}$ is a subfiltration of $\mathbf{H}$ and written as $\mathbf{G} \subseteq \mathbf{H}$, if $\mathcal{G}_t \subseteq \mathcal{H}_t$ for each t.

A family of σ-algebras induced by a stochastic process $\mathbf{X} = \{X_t, t \in I\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$.

The intuitive notion of causality formulated in terms of Hilbert spaces is given in [6] and generalized in [18]. We consider the analogous notion of causality for filtrations.

Definition 2.1. *(see [18] and [21]) It is said that* $\mathbf{G}$ *is a cause of* $\mathbf{E}$ *within* $\mathbf{H}$ *relative to* P *(and written as* $\mathbf{E} \mid\!\!< \mathbf{G}; \mathbf{H}; P$*) if* $\mathcal{E}_\infty \subseteq \mathcal{H}_\infty$, $\mathbf{G} \subseteq \mathbf{H}$ *and if* $(\mathcal{E}_\infty)$ *is conditionally independent of* $(\mathcal{H}_t)$ *given* $(\mathcal{G}_t)$ *for each t, i.e.* $\mathcal{E}_\infty \perp \mathcal{H}_t | \mathcal{G}_t$ *(i.e.* $\mathcal{E}_u \perp \mathcal{H}_t | \mathcal{G}_t$ *holds for each t and each u), or*

$$(\forall A \in \mathcal{E}_\infty) \quad P(A|\mathcal{H}_t) = P(A|\mathcal{G}_t). \tag{1}$$

Intuitively, $\mathbf{E} \mid\!\!< \mathbf{G}; \mathbf{H}; P$ means that, for arbitrary t, information about $\mathcal{E}_\infty$ provided by $(\mathcal{H}_t)$ is not "bigger" than that provided by $(\mathcal{G}_t)$.

If $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{G} \mid\!\!< \mathbf{G}; \mathbf{H}; P$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{H}$. It should be noted that "$\mathbf{G}$ is its own cause" sometimes occurs as a useful assumption in the theory of martingales and stochastic integration (see [2]). It also, should be mentioned that the notion of subordination (as introduced in [25]) is equivalent to the notion of being "its own cause" as defined here. Also, if $\mathbf{G}$ is its own cause within $\mathbf{H}$ we have

$$\mathcal{G}_\infty \perp \mathcal{H}_t | \mathcal{G}_t \text{ for each } t,$$

which is equivalent to Hypothesis $(\mathcal{H})$ introduced in [2]. Namely, Hypothesis $(\mathcal{H})$ implies that $(\mathcal{G}_t)$ has a nice structure with respect to $(\mathcal{H}_t)$: $\mathcal{G}_t = \mathcal{H}_t \cap \mathcal{G}_\infty$ (see Theorem 3 in [2]).

Let us mention that, having in mind classification of causality concepts given in [4], the given causality concept lies in the strong-global group.

Remark 2.2. *The condition of Granger causality is actually a condition of transitivity largely used in sequential analysis (in statistics), see e.g. [1] or [8] and in the marginalization of Markov processes (see [5], section 6.4.2).*

Definition 2.1 can be applied to stochastic processes if we consider corresponding induced filtrations. For example, $(\mathcal{F}_t)$-adapted stochastic process X_t is its own cause if $(\mathcal{F}_t^X)$ is its own cause within $(\mathcal{F}_t)$ i.e. if $\mathbf{F}^X \mid\!\!< \mathbf{F}^X; \mathbf{F}; P$ holds.

The process X, which is its own cause, is completely described by its behavior relative to its natural filtration $\mathbf{F}^X$ (see [21]). For example, the process $X = \{X_t, t \in I\}$ is a Markov process with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ if and only if X is a Markov process with respect to $\mathbf{F}^X$ and it is its own cause within $\mathbf{F}$ relative to P. As a consequence, the Brownian motion $W = \{W_t, t \in I\}$ with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is its own cause within $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ relative to probability P.

As a special case, we can consider the set L which contains the right continuous modifications of the processes $L_t = P(A \mid \mathcal{F}_t^X)$, i.e. suppose that we have the set of the form

$$L = \{L_t = P(A|\mathcal{F}_t^X); A \in \mathcal{F}_\infty^X\}. \tag{2}$$

The elements of the set L are $(\mathcal{F}_t, P)$-martingales if and only if $\mathbf{F}^X \mid\!\!< \mathbf{F}^X; \mathbf{F}; P$ holds (see [21]).

The next properties of the causality relationship from Definition 2.1 will be needed later.

Lemma 2.3. ([15]) *In the measurable space* $(\Omega, \mathcal{H})$ *let the filtrations* $\mathbf{E} = \{\mathcal{E}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ *and* $\mathbf{H} = \{\mathcal{H}_t\}$ *be given and let* P *and* Q *be probability measures on* $\mathcal{H}$ *satisfying* $Q \ll P$ *with* $\frac{dQ}{dP}$ *as* $(\mathcal{E}_\infty)$*-measurable. Then*

$$\mathbf{E} \mid\!\!< \mathbf{G}; \mathbf{H}; P \ \text{ implies } \ \mathbf{E} \mid\!\!< \mathbf{G}; \mathbf{H}; Q.$$

The definition of the extremal measure is carried over from [10] section 11.1: P is an extremal measure of the set $\mathcal{M}$ (and is denoted by $P \in ext(\mathcal{M})$, where $ext(\mathcal{M})$ is the set of extreme elements of $\mathcal{M}$), if from $P = aQ + (1-a)Q'$, where $a \in (0,1), Q, Q' \in \mathcal{M}$, it follows that $Q = Q' = P$.

Suppose that H is the set of right continuous modifications of processes

$$H = \{M_t = P(A \mid \mathcal{H}_t), A \in \mathcal{H}_\infty\}. \tag{3}$$

The following result holds.

Proposition 2.4. ([15]) *Let* $(\Omega, \mathcal{G}_\infty, P)$ *be a probability space with a filtration* $\mathbf{G} = \{\mathcal{G}_t\}$. *Let* H *be a set of* $(\mathcal{G}_t, P)$*-martingales. Then the following statements are equivalent*

1) *P is extremal in $\mathcal{M}$, the set of probability measures Q on $(\mathcal{G}_\infty)$ which coincide with P on $\mathcal{G}_{-\infty} = \cap_t \mathcal{G}_t$ and under which all elements of H are $(\mathcal{G}_t, Q)$-martingales.*

2) *For any filtration $\hat{\mathbf{F}} = \{\hat{\mathcal{F}}_t\}$ on an extension $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ of $(\Omega, \mathcal{G}_\infty, P)$ if $\hat{\mathcal{G}}_t \subseteq \hat{\mathcal{F}}_t$ for each t and if all elements of $\hat{H}$ are $(\hat{\mathcal{F}}_t, \hat{P})$-martingales, then*

$$\hat{\mathbf{G}} \mid\!\!< \hat{\mathbf{G}}; \hat{\mathbf{F}}; \hat{P}.$$

In many situations, we observe some system up to some random time, for example, till the time when something happens for the first time. Definition 2.1 is extended from fixed times to stopping times in [20].

The σ-field $(\mathcal{F}_T) = \{A \in \mathcal{F} : A \cap \{T \le t\} \in \mathcal{F}_t\}$ is usually interpreted as the set of events that occurs before or at time T (see [3]). For a process X, we set $X_T(\omega) = X_{T(\omega)}(\omega)$, whenever $T(\omega) < +\infty$. We define the stopped process $X^T = \{X_{t\wedge T}, t \in I\}$ with

$$X_t^T(\omega) = X_{t\wedge T(\omega)}(\omega) = X_t \chi_{\{t<T\}} + X_T \chi_{\{t \ge T\}}.$$

Note that if X is adapted and cadlag and if T is a stopping time, then the stopped process X^T is also adapted.

Let us mention that the truncated filtration $(\mathcal{F}_{t\wedge T})$ is defined as

$$\mathcal{F}_{t\wedge T} = \mathcal{F}_t \cap \mathcal{F}_T = \begin{cases} \mathcal{F}_t, \ t < T, \\ \mathcal{F}_T, \ t \ge T, \end{cases}$$

A martingale stopped at a stopping time is still a martingale. The natural filtration for the stopped martingale $X_{t\wedge T}$ is $\mathbf{F}^{X^T} = (\mathcal{F}^X_{t\wedge T})$, with respect to which the process $X_{t\wedge T}$ is completely described. So, we have the definition of causality which involves the stopping times.

Definition 2.5. ([20]) *Let* $\mathbf{H} = \{\mathcal{H}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ *and* $\mathbf{E} = \{\mathcal{E}_t\}, t \in I$, *be given filtrations on the probability space* $(\Omega, \mathcal{F}, P)$ *and let* T *be a stopping time with respect to filtration* $\mathbf{E}$. *The filtration* $\mathbf{G}^T$ *entirely causes* $\mathbf{E}^T$ *within* $\mathbf{H}^T$ *relative to* P *(and written as* $\mathbf{E}^T \mid\!\!< \mathbf{G}^T; \mathbf{H}^T; P$*) if* $\mathbf{E}^T \subseteq \mathbf{H}^T$, $\mathbf{G}^T \subseteq \mathbf{H}^T$ *and if* $\mathcal{E}_T$ *is conditionally independent of* $\mathcal{H}_{t\wedge T}$ *given* $\mathcal{G}_{t\wedge T}$ *for each* t, *i.e.* $(\forall t) \quad \mathcal{E}_T \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{G}_{t\wedge\tau}$, *or*

$$(\forall t \in I)(\forall A \in \mathcal{E}_T) \quad P(A \mid \mathcal{H}_{t\wedge T}) = P(A \mid \mathcal{G}_{t\wedge T}). \tag{4}$$

The concept of causality given in Definition 2.5 includes the stopped filtrations. Namely, the causality relationship is defined up to a specified stopping time T.

Let us consider the following example.

Example 2.6. *In order to show a simple example of application of Hypothesis ($\mathcal{H}$) (introduced in [2, 26]), we set $\mathcal{G}_t = \mathcal{G}_{t\wedge T}$ in relation $\mathbf{G} \models \mathbf{G}; \mathbf{H}; P$, where T is a stopping time with respect to $\mathbf{G}$, so $\mathbf{G}^T \models \mathbf{G}^T; \mathbf{H}^T; P$ holds. Then, we have*

$$\mathcal{G}_T \perp \mathcal{H}_{t\wedge T} | \mathcal{G}_{t\wedge T}$$

and, intuitively, this means that for predicting a filtration ($\mathcal{G}_T$), associated to ($\mathcal{G}_t$)-stopping time T, it is enough to have all information from the stopped (truncated) filtration $\mathbf{G}^T = \{\mathcal{G}_{t\wedge T}\}$. Also, due to Hypothesis ($\mathcal{H}$) in [2], every martingale with respect to ($\mathcal{G}_{t\wedge T}$) is ($\mathcal{H}_{t\wedge T}$)-martingale and conversely, i.e we have the preservation of the martingale property with respect to the enlarged filtration ($\mathcal{H}_{t\wedge T}$).

Indeed, if N is the set of right continuous modifications of martingales

$$N = \{N_t^T = P(A \mid \mathcal{G}_{t\wedge T}), A \in \mathcal{G}_T\}, \tag{5}$$

then from the causality relation $\mathbf{G}^T \models \mathbf{G}^T; \mathbf{H}^T; P$ we have

$$\mathcal{G}_T \perp \mathcal{H}_{t\wedge T} | \mathcal{G}_{t\wedge T} \quad \textit{or, equivalently,} \quad (\forall A \in \mathcal{G}_T) \;\; P(A \mid \mathcal{H}_{t\wedge T}) = P(A \mid \mathcal{G}_{t\wedge T}).$$

Obviously, if N_t^T is ($\mathcal{G}_{t\wedge T}$)-martingale, then

$$\begin{aligned} E(N_T \mid \mathcal{H}_{t\wedge T}) &= E(P(A \mid \mathcal{G}_T) \mid \mathcal{H}_{t\wedge T}) = E(E(\chi_A \mid \mathcal{G}_T) \mid \mathcal{H}_{t\wedge T}) = E(\chi_A \mid \mathcal{H}_{t\wedge T}) = P(A \mid \mathcal{H}_{t\wedge T}) \\ &= P(A \mid \mathcal{G}_{t\wedge T}) = N_t^T \end{aligned}$$

and $\{N_t^T\}$ is ($\mathcal{H}_{t\wedge T}$)-martingale.

Conversely, if every element of the set (5) is a martingale with respect to filtrations ($\mathcal{H}_{t\wedge T}$) and ($\mathcal{G}_{t\wedge T}$), we have

$$\begin{aligned} E(N_T \mid \mathcal{H}_{t\wedge T}) &= E(N_T \mid \mathcal{G}_{t\wedge T}) \\ E(P(A \mid \mathcal{G}_T) \mid \mathcal{H}_{t\wedge T}) &= E(P(A \mid \mathcal{G}_T) \mid \mathcal{G}_{t\wedge T}) \\ E(\chi_A \mid \mathcal{H}_{t\wedge T}) &= E(\chi_A \mid \mathcal{G}_{t\wedge T}), \end{aligned}$$

(since χ_A is ($\mathcal{G}_T$)-measurable function), then it follows that $P(A \mid \mathcal{H}_{t\wedge T}) = P(A \mid \mathcal{G}_{t\wedge T})$, for every $A \in (\mathcal{G}_T)$, or equivalently, that $\mathbf{G}^T \models \mathbf{G}^T; \mathbf{H}^T; P$ holds.

3. Martingal problem and stopped martingale problem

We now consider stochastic differential equation of the form:

$$\begin{cases} dX_t = u_t(X) dZ_t \\ X_0 = x, \end{cases} \tag{6}$$

where the driving process $\mathbf{Z} = \{Z_t, t \in I\}$ is an m-dimensional semimartingale ($Z_0 = 0$) and the coefficient $u_t(X)$ is an $n \times m$-dimensional predictable functional.

The definition of the weak solution of the stochastic differential equation (6) driven with semimartingales (using the concept of causality) is introduced in [15].

Definition 3.1. *([15]) Set of objects $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Z_t, P)$ is regular weak solution of the stochastic differential equation (6) if:*

1. *$\mu_Z(A) = P(Z \in A)$ coincides with a predetermined measure on the function space where Z_t take values,*
2. *X_t and Z_t satisfy the equation (6),*

3. *for every $t \geq 0$, $\mathbf{F}^Z \not\leftarrow \mathbf{F}^Z; \mathbf{F}; P$ holds, i.e.*

$$\forall A \in \mathcal{F}^Z_\infty \quad P(A \mid \mathcal{F}_t) = P(A \mid \mathcal{F}^Z_t).$$

Definition 3.2. *([15]) The regular weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Z_t, P)$ of the equation (6) is weakly unique if there is no other measure $Q \neq P$ on $\mathcal{F}^{Z,X}_\infty$, so that $(\Omega, \mathcal{F}^{Z,X}_\infty, \mathcal{F}^{Z,X}_t, X_t, Z_t, Q)$ is a weak solution.*

Local weak solutions of the given stochastic differential equations were investigated in [13, 14] and [22] (see Definition 5).

Suppose that $(\Omega, \mathcal{F}, \mathcal{F}^{Z,X}_t)$ is a filtered probability space (no measure on it, yet) and that $\mathcal{F}^Z$ is a sub-σ-field of $(\mathcal{F}^{Z,X})$, called the initial σ-field.

We consider the definition of the martingale problem given by Jacod and Shiryaev.

Definition 3.3. *([11]) Assume that on the basic space $(\Omega, \mathcal{F})$ a right continuous filtration $\mathbf{F}^{Z,X} = (\mathcal{F}^{Z,X}_t)_{t\geq 0}$ with $(\mathcal{F}^{Z,X}_\infty) = (\mathcal{F})$ and a processes $Z_t = (Z_t)_{t\geq 0}$, $X_t = (X_t)_{t\geq 0}$ are given. A solution of the martingale problem $(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$ is a probability measure P on $(\Omega, \mathcal{F})$ such that:*

(i) the restriction $P|_{\mathcal{F}^Z} = P_Z$

(ii) X_t is a local martingale on the basis $(\Omega, \mathcal{F}, \mathcal{F}^{Z,X}_t, P)$ with (A, C, ν) as its predictable characteristics.

The process X from the previous definition is called the solution-process and the law P the solution-measure. The definition of the martingale problem can be found in the [27], but we use the definition from [11], because it is related to the characteristics of semimartingales. The collection of all solutions of the martingale problem is denoted by $\Gamma_s(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$ (the used notation is from [9]). Now, the measure P is defined as a solution of the martingale problem and the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}^{Z,X}_t, P)$ is complete.

Remark 3.4. *The solution process is often required to have continuous or cadlag paths. In this case the solution of the martingale problem corresponds to a probability measure on the Borel sets of the canonical space.*

Definition 3.5. *([11]) The solution of the martingale problem $(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$ is weakly unique if any two solution measures P and Q (where $P, Q \in \Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$) of the martingale problem coincide on the σ-field $(\mathcal{F}^{Z,X}_\infty)$.*

Remark 3.6. *The uniqueness of solutions for the martingale problems is directly connected to the Markov property of their solutions.*

Let us mention that the uniqueness of solutions of the martingale problems means that any two solutions have the same finite-dimensional distributions (in one dimensional case, all marginal distributions coincide).

For stochastic differential equations driven by the Wiener process, the equivalence between the problem of finding a weak solution of the equation and the solution of martingale problem associated with the equation is shown in [11, 12]. This equivalence for stochastic differential equations of the form (6), which are driven by semimartingales, is considered in [23].

Theorem 3.4 from [23] shows that the existence of an extremal solution of the martingale problem can be related to the concept of causality. The following theorem shows that the existence of extremal solution of the martingale problem and of weakly unique solution of the martingale problem can be related to the concept of causality.

Theorem 3.7. *Let P be a solution of the martingale problem ($P \in \Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$). Then the following statements are equivalent:*

(1) $\mathbf{F}^{Z,X} \ltimes \mathbf{F}^{Z,X}; \mathbf{F}; P$;

(2) the measure P is an extremal solution of the martingale problem ($P \in ext(\mathcal{M}_x)$);

(3) the measure P is a weakly unique solution of the martingale problem;

(4) if $X \in R^d$ and $u_t(X)$ is predictable functional, then the weak solution of the equation (6) is weakly unique.

Proof. The equvalence between statements (1) an (2) is proved as Theorem 3.4 in [23].

(2) $\Longrightarrow$ (3) Let P be a solution of the martingale problem and an extremal measure of the set $\mathcal{M}_x$. We need to prove that the solution of the martingale problem is weakly unique, or equivalently, that for another solution measure of martingale problem $Q_1 \in \mathcal{M}_x$ ($Q_1 \in \Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$), it follows that $P = Q_1$ on $(\mathcal{F}_\infty^{Z,X})$. Suppose that $P \neq Q_1$ and define the probability measure Q_2 on $(\mathcal{F}_\infty^{Z,X})$ by $P = aQ_1 + (1-a)Q_2$. Obviously, Q_1 and Q_2 coincide with P on $(\mathcal{F}_{-\infty}^{Z,X})$. Elements of the set L (of the form (2)) are $(\mathcal{F}_t^{Z,X}, P)$-martingales (because P is a solution measure of the martingale problem), and from $Q_1 \in \mathcal{M}_x$ according to Lemma 2.3, we have that elements of L are $(\mathcal{F}_t^{Z,X}, Q_1)$-martingales, too. Hence, they are $(\mathcal{F}_t^{Z,X}, Q_2)$-martingales. It follows (by an ordinary change of variables formula, see e.g [24]) that $Q_1 f^{-1}, Q_2 f^{-1} \in \mathcal{M}_x$, hence P is not extremal in $\mathcal{M}_x$. So, we proved that $P = Q_1$ on $(\mathcal{F}_\infty^{Z,X})$, and that P is a weakly unique solution of the martingale problem.

(3) $\Longrightarrow$ (4) Let the measure P be a weakly unique solution measure of the martingale problem ($P \in \Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$). According to Theorem 3.4 in [23], $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Z_t, P)$ is a regular weak solution of the equation (6). Suppose that the solution $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Z_t, P)$ is not weakly unique solution, i.e. that $(\Omega, \mathcal{F}_\infty^{Z,X}, \mathcal{F}_t^{Z,X}, Q, Z_t, X_t)$ is another solution. Due to Theorem 3.4 in [23], the measure Q is another solution of the martingale problem associated to the equation (6). But, by Definition 3.2 we have $P \neq Q$ on the filtration $(\mathcal{F}_\infty^{Z,X})$, which is contradiction. Hence $(\Omega, \mathcal{F}, \mathcal{F}_t, P, Z_t, X_t)$ is weakly unique solution of the equation (6).

(4) $\Longrightarrow$ (1) Follows directly by Theorem 4.3 in [21]. $\square$

Now, we consider the stopped martingale problem.

Suppose that triplet (A, C, ν) is a local characteristic of the semimartingale X. If T is a $(\mathcal{F}_t^X)$-stopping time, X^T, A^T, C^T are stopped processes and the "stopped random measure" ν^T is defined as

$$\nu^T(\omega, ds, dz) = \nu(\omega, ds, dz) I_{\{s \leq T(\omega)\}}.$$

The definition for the local weak solution of the stochastic differential equation (6) was investigated in [13, 22].

Definition 3.8. *([22]) A set of objects $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t, T)$ is a local weak solution of equation (6) if:*

1) $\mu_Z(A) = P\{Z \in A\}$ coincides with predetermined measure on the function space where Z takes values;

2) $\mathbf{F}^Z \ltimes \mathbf{F}^Z; \mathbf{F}^T; P$ *or*

$$\forall A \in (\mathcal{F}_\infty^Z) \qquad P(A \mid \mathcal{F}_t^Z) = P(A \mid \mathcal{F}_{t \wedge T}),$$

3) T is $(\mathcal{F}_t^{Z,X})$-stopping time (called the lifetime of X);

4) X is adapted and satisfy the equation (6).

The notion of stopped martingale problem was introduced in [11, 13].

Definition 3.9. *([11]) Assume that on the basic space $(\Omega, \mathcal{F})$ a right continuous filtration $\mathbf{F}^{Z,X} = (\mathcal{F}_t^{Z,X})_{t \geq 0}$ with $(\mathcal{F}_\infty^{Z,X}) = (\mathcal{F})$ and a processes $X_t = (X_t)_{t \geq 0}, Z_t = (Z_t)_{t \geq 0}$ are given. A probability measure P on $(\Omega, \mathcal{F})$ is called a solution of the stopped martingale problem $(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$ for the $(\mathcal{F}_t^X)$-stopping time T such that:*

(i) the restriction $P|_{\mathcal{F}^Z} = P_Z$,

(ii) X^T *is a martingale on the basis* $(\Omega, \mathcal{F}, \mathcal{F}_t^{(Z,X)^T}, P)$ *with* (A^T, C^T, ν^T) *as its predictable characteristics.*

The following theorem gives a connection between the local weak solution of the equation (6) and the solution of the stopped martingale problem.

Theorem 3.10. *For every local weak solution* $(\Omega, \mathcal{F}, F, P, X_t, Z_t, T)$ *of the equation (6), P is a solution of the stopped martingale problem, i.e.* $P \in \Gamma_m(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$.

Proof. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t, T)$ is a local weak solution of equation (6). Due to Theorem 3 in [22], $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ is a regular weak solution of the same equation (simply if $T = \infty$). According to Theorem 5.3 in [21], P is a solution of the martingale problem, i.e. $P \in \Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$, and by assumption T is $(\mathcal{F}_t^X)$-stopping time. So, on the basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ semimartingale X is a solution with characteristics (A, C, ν). X^T is a semimartingale with characteristics (A^T, C^T, ν^T). According to Definition 3.9, P is a solution of the stopped martingale problem, i.e. $P \in \Gamma_m(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$. □

Let us mention that every solution of the stopped martingale problem is not a local weak solution of the associated stochastic differential equation. According to Definition 5 in [22] and Definition 3.9, we need to find conditions under which the concept of causality from the filtration $\mathcal{F}_T$ should be extended to the filtration $(\mathcal{F}_\infty)$. This is not the case in generally, but this problem can be overcome if we suppose that a measure P is extremal on $\mathbf{F}$.

Let L^T be a set of processes defined according with (2) as

$$L^T = \{L_{t\wedge T} = P(A \mid \mathcal{F}_{t\wedge T}^X),\ A \in \mathcal{F}_T^X\}. \tag{7}$$

Let $\mathcal{M}_x^T$ be a set of all measures Q for which the process $L_{t\wedge T}$ (of the form (7)) is $(\mathcal{F}_{t\wedge T}, P)$-martingale. Now, we consider connections between the extremal solutions of the stopped martingale problem and the relation "being its own cause" for stopped processes and stopped filtrations.

Theorem 3.11. *Let T be a* $(\mathcal{F}_t^X)$*-stopping time and* $P \in \Gamma_m(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$*. The measure P is extremal solution of the stopped martingale problem* ($P \in ext(\mathcal{M}_x^T)$) *if and only if* $\mathbf{F}^{(Z,X)^T}$ *is its own cause within* $\mathbf{F}^T$*, i.e.*

$$\mathbf{F}^{(Z,X)^T} \mid\!\!< \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P.$$

Proof. Suppose that P is an extremal solution of the stopped martingale problem for the $(\mathcal{F}_t^X)$-stopping time T, i.e. $P \in \Gamma_m(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$. Therefore, elements of the set L^T (defined as in (7)) are $(\mathcal{F}_{t\wedge T}^{Z,X}, P)$-martingales. On the other hand, $P \in ext(\mathcal{M}_x^T)$ ($ext(\mathcal{M}_x^T)$ is a set of extremal measures, for which the process $L_{t\wedge T}$ is $(\mathcal{F}_{t\wedge T}, P)$-martingale). Now, from Proposition 2.1,
it follows that

$$\mathbf{F}^{(Z,X)^T} \mid\!\!< \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P.$$

Conversely, suppose that $\mathbf{F}^{(Z,X)^T} \mid\!\!< \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P$, holds, or equivalently,

$$\forall A \in \mathcal{F}_T^{Z,X} \quad P(A \mid \mathcal{F}_{t\wedge T}) = P(A \mid \mathcal{F}_{t\wedge T}^{Z,X}). \tag{8}$$

Let the measure P be a solution of the stopped martingale problem. Then the elements of the set L^T of the form (7) are $(\mathcal{F}_{t\wedge T}^{Z,X}, P)$-martingales, i.e. (according to Hypothesis $\mathcal{H}$) $\mathbf{F}^{X^T} \mid\!\!< \mathbf{F}^{X^T}; \mathbf{F}^{(Z,X)^T}; P$ holds. So, we have

$$\forall A \in \mathcal{F}_T^X \quad P(A \mid \mathcal{F}_{t\wedge T}^{Z,X}) = P(A \mid \mathcal{F}_{t\wedge T}^X). \tag{9}$$

From (8), (9) and $\mathcal{F}_T^X \subseteq \mathcal{F}_T^{Z,X}$, for all $A \in \mathcal{F}_T^X$ it follows that

$$P(A \mid \mathcal{F}_{t\wedge T}^X) = P(A \mid \mathcal{F}_{t\wedge T}^{Z,X}) = P(A \mid \mathcal{F}_{t\wedge T}).$$

So, elements of the set L^T are $(\mathcal{F}_{t\wedge T}, P)$-martingales and $\mathbf{F}^{X^T} \mid\!\!< \mathbf{F}^{X^T}; \mathbf{F}^T; P$ holds, or equivalently,

$$\forall A \in \mathcal{F}_T^X \quad P(A \mid \mathcal{F}_{t\wedge T}) = P(A \mid \mathcal{F}_{t\wedge T}^X).$$

Now, we can apply Proposition 2.4. Hence, the measure P is an extremal measure in the set $\mathcal{M}_x^T$ (the set of measures Q for which the process $L_{t\wedge T}$, is $(\mathcal{F}_{t\wedge T}, Q)$-martingale) and P is the extremal solution of the stopped martingale problem. □

The following martingale problem may be considered to be a stock price model, where the same event not only changes the return process X_t^1 but also increases the volatility X_t^2 of the stock. In this example the volatility can be interpreted as an arrival rate of price changes.

Example 3.12. *(see [12]) Fix parameters $\mu \in \mathbf{R}, \alpha, \beta, \sigma, x_0 \in \mathbf{R}_+^*, x_1 \in (\alpha, \infty)$. Let φ be a measure on $(\mathbf{R}, \mathcal{B})$ with λ-density $x \to \frac{1}{|x|}e^{-|x|}$. We define the measure $P(((\bar{\omega}^1, \bar{\omega}^2), t), \cdot)$ as the image of $h(\bar{\omega}_t^2)\varphi$, where $h : \mathbf{R} \to \mathbf{R}$, such that $h(x) \geq \frac{\alpha}{2}$ for any $x \in \mathbf{R}$, and $h(x) = x$ for any $x \in (\alpha, \infty)$. Moreover, define the drift by $b_t(\bar{\omega}) = (\mu, -(\bar{\omega}^2 - \alpha)\beta + \int x_2 P((\bar{\omega}, t), d(x_1, x_2)))$ for any $(\bar{\omega}, t) = ((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbf{D}^2 \times \mathbf{R}_+$. Then, for the martingale problem $\Gamma_m(Y, \mathcal{F}_t^Y, \epsilon_Y, A, 0, \nu)$ (where A_t and ν are defined in ([12])), the process $Y = (X^1, X^2)$ is its own cause.*

For a solution process (X^1, X^2), we interpret X^2 as a volatility. It increases due to positive jumps (which also affect X^1) and is pulled back towards the lower bound α by the drift term $-(\bar{\omega}_t^2 - \alpha)\beta$. Therefore, X^2 always stays above α.

Whereas in this example, the volatility can be interpreted as a measure of the average size of the stock price jumps.

Example 3.13. *(see [12]) Fix parameters $\mu \in \mathbf{R}, \alpha, \beta, \sigma, x_0 \in \mathbf{R}_+^*, x_1 \in (\alpha, \infty)$, and let φ, h be as in the previous example. For any $((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbf{R}^2 \times \mathbf{R}_+$ we define the measure $P(((\bar{\omega}^1, \bar{\omega}^2), t), \cdot)$ as the image of φ under the mapping $\mathbf{R} \to \mathbf{R}^2, x \to (\sigma h(\bar{\omega}_t^2)x, h(\bar{\omega}_t^2)|x|)$. Moreover, define b as in Example 3.1. Then the martingale problem $\Gamma_m(Y, \mathcal{F}_t^Y, \epsilon_{y_0}, A, 0, \nu)$ has a unique solution measure, that is, by Theorem 4.1, process $Y = (X^1, X^2)$ is its own cause.*

4. Local uniqueness and statistical causality

Now, we consider the concept of local uniqueness, a form of uniqueness that is stronger than ordinary uniqueness. It is important for studying absolutely continuity or singularity questions for solutions of the martingale problem and it can be applied to a local weak solutions, too.

Let $(\mathcal{F}_t^0)$ be a filtration which is not right continuous and we have that

$$(\mathcal{F}_{t-}) \subset (\mathcal{F}_t^0) \subset (\mathcal{F}_t), \quad t > 0$$

where $(\mathcal{F}_t)^0 = \bigcap_{s>t} \mathcal{F}_s^0$, $\mathcal{F}_s^0 = (\mathcal{F}_s^{Z,X})^0$ and $(\mathcal{F})^0 = (\mathcal{F}_\infty)^0 = \bigvee_t (\mathcal{F}_t)^0$. Then, the filtration $\mathbf{F}^0 = \{(\mathcal{F}_t)^0, t \in I\}$ is the smallest filtration such that (Z, X) is adapted. The next definition considers stopping times with respect to the filtration $(\mathcal{F}_t)^0$.

Definition 4.1. *([11]) A strict stopping time (or a stopping time relative to $\mathcal{F}_t^0$) is a map $T : \Omega \to \mathbf{R}_+$ such that $\{T \leq t\} \in \mathcal{F}_t^0$ for all $t \in \mathbf{R}_+$. If T is a strict stopping time, then $(\mathcal{F}_T^0)$ denotes the σ-field of all $A \in \mathcal{F}$ such that $A \cap \{T \leq t\}$ belongs to $(\mathcal{F}_t^0)$ for all $t \in \mathbf{R}_+$.*

Remark 4.2. *A strict stopping time is a stopping time. Also, $\mathcal{F}_{T-} \subset \mathcal{F}_T^0 \subset \mathcal{F}_T$ on the set $\{T > 0\}$, where $\mathcal{F}_{T-} = \{A \in \mathcal{F}; A \cap \{T < t\} \in \mathcal{F}_t, \forall t > 0\}$.*

Let us consider the martingale problem $\Gamma_m(X, \mathbf{F}^{Z,X} P_Z, A, C, \nu)$ associated with the equation (6). The local weak uniqueness (that is, the local uniqueness in the sense of distributions) for the martingale problem is given by Jacod and Shiryaev in [11].

Definition 4.3. *([11]) Assume that $(\mathcal{F}_t^{Z,X})^0$ is not right continuous. We say that the local uniqueness holds for the solution of the martingale problem $\Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$ if, for every strict stopping time T, any two solutions P and P' of the stopped martingale problem $\Gamma_m(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$ coincide on the σ-field $(\mathcal{F}_T^{Z,X})^0$.*

The similiar concept to the local uniqueness is given in [13], named as "stopped uniqueness" and it is not defined for strict stopping times, but for stopping times relative to the filtration $(\mathcal{F}_t)$ which satisfies the usual conditions.

Remark 4.4. *Note that the local uniqueness implies the uniqueness (if we take $T = \infty$). If the uniqueness holds, then every solution of the stopped martingale problem can be extended (beyond the stopping time) to a solution of the (local) martingale problem.*

From the definition of the local uniqueness, it is evident that it is not easy to be checked, unless it is proved that this property can be implied by the uniqueness. The local uniqueness can be connected with the concept of causality given in Definition 2.5 as it is proved in [23] (see Theorem 4.4). In this paper, we consider connections between the local uniqueness of the solution of the martingale problem and the relation "being its own cause" for stopped processes and stopped filtrations.

Let L^T be a set of processes defined according with (2) as

$$L^T = \{L_{t\wedge T} = P(A \mid \mathcal{F}_{t\wedge T}^{Z,X}), A \in \mathcal{F}_T^{Z,X}\}. \tag{10}$$

Let $\mathcal{M}^0$ be the set of all probability measures Q on $(\mathcal{F}_T^{Z,X})^0$, under which all elements of the set M^T are $(\mathcal{F}_{t\wedge T}^{Z,X})^0$-martingales. In the following theorem we give conditions for the local uniqueness of the solution of the martingale problem.

Theorem 4.5. *Let T be a strict stopping time with respect to the filtration $(\mathcal{F}_t^{Z,X})^0$. The solution of the martingale problem $\Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$ is locally unique if and only if the process $(Z, X)^T$ is its own cause within $\mathbf{F}^T$, i.e.*

$$\mathbf{F}^{(Z,X)^T} \models \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P.$$

Proof. Suppose that T is a strict stopping time and the measure P is a solution of the martingale problem $\Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$, for which the local uniqueness holds. According Theorem 4.4 in [23], we have that $\mathbf{F}^{(Z,X)} \models \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P$ holds. Since $\mathbf{F}^{(Z,X)^T} \subseteq \mathbf{F}^{(Z,X)}$, it follows that $\mathbf{F}^{(Z,X)^T} \models \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P$.

Conversely, let T be a strict stopping time and P be a solution of the martingale problem, $P \in \Gamma_m(X, \mathbf{F}^{Z,X}, P_Z,$
$A, C, \nu)$. The measure P will be a solution of the stopped martingale problem $\Gamma_m(X^T, \mathbf{F}^{(Z,X)^T}, P_Z, A^T, C^T, \nu^T)$, too. Namely, if X is $(\mathcal{F}_t^{Z,X}, P)$-martingale with characteristics (A, C, ν), then X^T is $(\mathcal{F}_{t\wedge T}^{Z,X}, P)$-martingale with characteristics (A^T, C^T, ν^T), which satisfy the definition for the solution of the stopped martingale problem in [11, 13].

By the assumption, such a measure P satisfies the causality condition

$$\mathbf{F}^{(Z,X)^T} \models \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P, \quad \text{or equivalently,}$$

$$\forall A \in \mathcal{F}_T^{Z,X} \quad P(A \mid \mathcal{F}_{T\wedge t}^{Z,X}) = P(A \mid \mathcal{F}_{t\wedge T}).$$

Now, from

$$E(L_T \mid \mathcal{F}_{t\wedge T}) = E(P(A \mid \mathcal{F}_T^{Z,X}) \mid \mathcal{F}_{t\wedge T}) = E(E(\chi_A \mid \mathcal{F}_T^{Z,X}) \mid \mathcal{F}_{t\wedge T}) = E(\chi_A \mid \mathcal{F}_{t\wedge T}) = E(\chi_A \mid \mathcal{F}_{t\wedge T}^{Z,X}) = L_{t\wedge T}$$

it follows that the elements of the set $L_{t\wedge T}$ are $(\mathcal{F}_{t\wedge T}, P)$-martingales.

According to Proposition 2.2 in [15], the measure P is extremal on the $(\mathcal{F}_T^{Z,X})$. In other words, if there are another measures Q_1 and Q_2, for which $P = aQ_1 + (1 - a)Q_2$ holds, we have $P = Q_1 = Q_2$ on $(\mathcal{F}_T^{Z,X})$ and this measures are the solutions of the same stopped martingale problem. Due to the inclusion $(\mathcal{F}_T^{Z,X})^0 \subseteq (\mathcal{F}_T^{Z,X})$, obviously $P = Q_1 = Q_2$ on $(\mathcal{F}_T^{Z,X})^0$, so the solution of the martingale problem $\Gamma_m(X, \mathbf{F}^{Z,X}, P_Z, A, C, \nu)$ is locally unique. $\square$

As a consequence of Theorem 3.10, Theorem 4.5 and Theorem 3.1 in [23], we have the following result.

Theorem 4.6. *Let T be a stopping time with respect to the filtration $(\mathcal{F}_t^{Z,X})^0$. A local weak solution of the stochastic differential equation (6) is locally unique if and only if*

$$\mathbf{F}^{(Z,X)^T} \ltimes \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P.$$

The following theorem holds.

Theorem 4.7. *A weakly unique solution $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ of the equation (6) is locally unique.*

Proof. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ be a weak solution of the equation (6). According to Theorem 3.7, the solution of the associated martingale problem is weakly unique, or $\mathbf{F}^{Z,X} \ltimes \mathbf{F}^{Z,X}; \mathbf{F}; P$ holds. Due to Remark 1 in [20], we have that $\mathbf{F}^{(Z,X)^T} \ltimes \mathbf{F}^{(Z,X)^T}; \mathbf{F}^T; P$ holds. By Theorem 4.5, the solution of the martingale problem is locally unique. According to the equivalence between the weak solution of the equation (6) and the solution of the associated martingale problem (see Theorem 3.3 in [23]), it follows that the weak solution of the stochastic differential equation is locally unique for the $(\mathcal{F}_t^X)$-stopping time T. $\square$

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