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Singular Value inequalities for Hilbert space Operators

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Abstract. In this paper we show that if A_i, B_i, X_i are Hilbert space operators such that X_i is compact $i = 1, 2, \ldots, n$ and f, g are non-negative continuous functions on $[0, \infty)$ with $f(t)g(t) = t$ for all $t \in [0, \infty)$, also h is non-negative increasing operator convex function on $[0, \infty)$, then

$$h\left(s_j\left(\sum_{i=1}^{n} \omega_i A_i^* X_i^* B_i\right)\right) \leq s_j\left(\sum_{i=1}^{n} \omega_i h(A_i^* f(|X_i^*|)^2 A_i) \oplus \sum_{i=1}^{n} \omega_i h(B_i^* g(|X_i|)^2 B_i)\right)$$

for $j = 1, 2, \ldots$ and $\sum_{i=1}^{n} \omega_i = 1$.

Also, applications of some inequalities are given.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. For a compact operator $X \in \mathcal{B}(\mathcal{H})$, let $s_1(X) \geqslant s_2(X) \geqslant \cdots$ denote the singular values of X; i.e., the eigenvalues of $|X| = (X^*X)^{\frac{1}{2}}$ (The absolute value of X), arranged in decreasing order are repeated according to multiplicity. Note that $s_j(X) = s_j(X^*) = s_j(|X|)$ for $j = 1, 2, \ldots$. For $A, B \in \mathcal{B}(\mathcal{H})$, we utilize the direct sum notation $A \oplus B$ for the block-diagonal operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $\mathcal{H} \oplus \mathcal{H}$. It has been shown in [8] that if X and Y are compact operators, then

$$s_j(X + Y) \leqslant 2s_j(X \oplus Y)$$

for $j = 1, 2, \ldots$.

The usual operator norm of an operator $A \in \mathcal{B}(\mathcal{H})$ is denote by $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. As an immediate consequence of the min-max principle (see e.g.,[2, p. 75]), if A, B, and X are in $\mathcal{B}(\mathcal{H})$ such that X is compact, then

$$s_j(AXB) \leqslant \|A\| \, \|B\| \, s_j(X) \tag{1}$$

for $j = 1, 2, \ldots$. For $1 \leqslant p < \infty$, the Schatten p-norm of compact operator A is defined by $\|A\|_p = (tr|A|^p)^{\frac{1}{p}}$, where tr is the usual trace functional. One can show that

$$\|A \oplus B\| = \max(\|A\|, \|B\|)$$

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and

$$\|A \oplus B\|_p = (\|A\|_p^p + \|B\|_p^p)^{\frac{1}{p}}.$$

In addition to the usual operator norm, which is defined on all of $\mathcal{B}(\mathcal{H})$, we consider unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on a norm ideal contained in the ideal of compact operators, and for the sake of brevity, we will make no explicit mention of this ideal. Thus, when we consider $|||X|||$, we are assuming that the operator X belongs to the norm ideal associated with $|||\cdot|||$. Moreover, each unitarily invariant norm $|||\cdot|||$ is symmetric gauge function of the singular values, and is characterized by equality $|||X||| = |||UXV|||$ for all operator X and for all unitary operators U and $V \in \mathcal{B}(\mathcal{H})$. For the general theory of unitarily invariant norms, we refer to [2] or [7]. It has been shown by Bhatia and Kittaneh in [4] that if A, B are compact operators in $\mathcal{B}(\mathcal{H})$, then

$$2\,|||A^*B||| \leqslant |||AA^* + BB^*||| \tag{2}$$

and

$$|||A^*B + B^*A||| \leqslant |||AA^* + BB^*||| \tag{3}$$

for every unitarily invariant norm.

The inequality (2) has attracted the attention of several mathematicians, and different proof of a stronger version of it have been given. See [3], [8], [9] and [11]. It has been shown by Kittaneh [10] the generalized from of the mixed Schwarz inequality, that if A is an operator in $\mathcal{B}(\mathcal{H})$ and f and g are non-negative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, then

$$|\langle Ax, y\rangle| \leq \left\|f(|A^*|)x\right\| \, \left\|g(|A|)y\right\|$$

for all x and y in $\mathcal{H}$.
In this paper, we generalize inequalities (2) and (3) and present a bound that involves operator A and B.

2. Main Results

In this section, we establish generalized singular value inequalities for Hilbert space operator. The following Lemmas are essential role in our analysis. The first one on the Mixed schwarz inequality has been proven by Kittaneh [10].

Lemma 2.1. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, such that A and B are positive, then $T = \begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive in $B(\mathcal{H} \oplus \mathcal{H})$ if and only if $|\langle Cx, y\rangle|^2 \leqslant \langle Ax, x\rangle\langle By, y\rangle$ for all x, y in $\mathcal{H}$.*

We prove the second one by Lemma 2.1.

Lemma 2.2. *Let A, B and X be operators in $\mathcal{B}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, \infty)$ that satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, then*

$$\begin{bmatrix} A^*f(|X^*|)^2A & A^*X^*B \\ B^*XA & B^*g(|X|)^2B \end{bmatrix}$$

is positive.

Proof. For any x, y in $\mathcal{H}$

$$|\langle B^*XAx, y\rangle| = |\langle XAx, By\rangle| \leqslant \langle f(|X^*|)^2Ax, Ax\rangle^{\frac{1}{2}} \langle g(|X|)^2By, By\rangle^{\frac{1}{2}}.$$

By Lemma 2.1

$$\begin{bmatrix} A^*f(|X^*|)^2A & A^*X^*B \\ B^*XA & B^*g(|X|)^2B \end{bmatrix} \geqslant 0.$$

□

The third Lemma is Theorem 2.1 in [1].

Lemma 2.3. *Let A, B and C be compact operators in $\mathcal{B}(\mathcal{H})$ and $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is positive. Then $s_j(C) \leqslant s_j(A \oplus B)$ for $j = 1, 2, \ldots$.*

Now we establish a general singular value inequality, from which singular value inequalities for products of operators follow as special cases.

Theorem 2.4. *Let A_i, B_i and X_i be operators in $\mathcal{B}(\mathcal{H})$, where X_i is compact operator $i = 1, 2, \ldots, n$ and let f and g be non-negative functions on $[0, \infty)$, which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, also let h be a non-negative increasing operator convex function on $[0, \infty)$. Then*

$$h\left(s_j\left(\sum_{i=1}^{n} \omega_i A_i^* X_i^* B_i\right)\right) \leq s_j\left(\sum_{i=1}^{n} \omega_i h(A_i^* f(|X_i^*|)^2 A_i) \oplus \sum_{i=1}^{n} \omega_i h(B_i^* g(|X_i|)^2 B_i)\right)$$

for $j = 1, 2, \ldots$ and positive real numbers ω_i such that $\sum_{i=1}^{n} \omega_i = 1$.

Proof. The matrix for $i = 1, 2, \ldots, n$

$$\begin{bmatrix} A_i^* f(|X_i^*|)^2 A_i & A_i^* X_i^* B_i \\ B_i^* X_i A_i & B_i^* g(|X_i|)^2 B_i \end{bmatrix} \geqslant 0 \quad \text{(by Lemma 2.2)}.$$

So, by Lemma 2.3 and some property of singular values, we have

$$s_j\left(\sum_{i=1}^{n} A_i^* X_i^* B_i\right) \leqslant s_j\left(\sum_{i=1}^{n} A_i^* f(|X_i^*|)^2 A_i \oplus \sum_{i=1}^{n} B_i^* g(|X_i|)^2 B_i\right). \tag{4}$$

for $j = 1, 2, \ldots$. Now consider the non-negative increasing operator convex function h on $[0, \infty)$ and in the inequality (4), put $\sqrt{\omega_i} A_i$, $\sqrt{\omega_i} B_i$ instead of A_i, B_i respectively . It follows that

$$\begin{aligned}
h\left(s_j\left(\sum_{i=1}^{n} \omega_i A_i^* X_i^* B_i\right)\right) &\leqslant h\left(s_j\left(\sum_{i=1}^{n} \omega_i A_i^* f(|X_i^*|)^2 A_i \oplus \sum_{i=1}^{n} \omega_i B_i^* g(|X_i|)^2 B_i\right)\right) \\
&= h\left(s_j\left(\sum_{i=1}^{n} \omega_i (A_i^* f(|X_i^*|)^2 A_i \oplus B_i^* g(|X_i|)^2 B_i)\right)\right) \\
&= s_j\left(h\left(\sum_{i=1}^{n} \omega_i (A_i^* f(|X_i^*|)^2 A_i \oplus B_i^* g(|X_i|)^2 B_i)\right)\right) \\
&\quad \text{(by elementary functional calculus)} \\
&\leqslant s_j\left(\sum_{i=1}^{n} \omega_i h(A_i^* f(|X_i^*|)^2 A_i \oplus B_i^* g(|X_i|)^2 B_i)\right) \\
&\quad (h \text{ is operator convex}) \\
&= s_j\left(\sum_{i=1}^{n} \omega_i h(A_i^* f(|X_i^*|)^2 A_i) \oplus \sum_{i=1}^{n} \omega_i h(B_i^* g(|X_i|)^2 B_i)\right)
\end{aligned}$$

for $j = 1, 2, \ldots$. □

Corollary 2.5. *Let $A_i, B_i, X_i \in \mathcal{B}(\mathcal{H})$ such that X_i is compact for $i = 1, 2, \ldots, n$. Then*

$$s_j^r\left(\sum_{i=1}^{n} \omega_i A_i^* X_i^* B_i\right) \leq s_j\left(\sum_{i=1}^{n} \omega_i \left(A_i^* f(|X_i^*|)^2 A_i\right)^r \oplus \sum_{i=1}^{n} \omega_i \left(B_i^* g(|X_i|)^2 B_i\right)^r\right)$$

for $j = 1, 2, \ldots$.

Proof. Let $h(x) = x^r$ $(1 \le r \le 2)$ in Theorem 2.4, and we get the result. $\square$

A particular case of Theorem 2.4 and Corollary 2.5 can be seen as follows.

Corollary 2.6. *Let $U_i, V_i, X_i \in \mathcal{B}(\mathcal{H})$ such that U_i, V_i are unitary and X_i is compact for $i = 1, 2, \dots, n$. Then*

$$s_j^r\left(\sum_{i=1}^n \omega_i U_i^* X_i^* V_i\right) \le s_j\left(\sum_{i=1}^n \omega_i U_i^* f(|X_i^*|)^{2r} U_i \oplus \sum_{i=1}^n \omega_i V_i^* g(|X_i|)^{2r} V_i\right)$$

$j = 1, 2, \dots$. In particular,

$$s_j^r\left(\sum_{i=1}^n U_i^* X_i^* V_i\right) \le s_j\left(\sum_{i=1}^n U_i^* |X_i^*|^r U_i \oplus \sum_{i=1}^n V_i^* |X_i|^r V_i\right)$$

$j = 1, 2, \dots$.

Proof. The result follows from Theorem 2.4 by letting $h(x) = x^r$ $(1 \le r \le 2)$, $A_i = U_i$ and $B_i = V_i$ unitary operators for $i = 1, 2, \dots, n$, and using Lemma 1.6 in [6]. The particular case follows by letting $f(t) = g(t) = t^{\frac{1}{2}}$ and $\omega_i = \frac{1}{n}$ for $i = 1, 2, \dots, n$. $\square$

As an application of the inequality (4), we get the following corollaries.

Corollary 2.7. *Let $A_i, B_i, X_i \in \mathcal{B}(\mathcal{H})$ such that X_i is compact for $i = 1, 2, \dots, n$. Then*

$$\left|\left|\left|\sum_{i=1}^n A_i^* X_i^* B_i\right|\right|\right| \le \left|\left|\left|\sum_{i=1}^n A_i^* f(|X_i^*|)^2 A_i \oplus \sum_{i=1}^n B_i^* g(|X_i|)^2 B_i\right|\right|\right| \tag{5}$$

for every unitarily invariant norm. In particular,

$$\left\|\sum_{i=1}^n A_i^* X_i^* B_i\right\| \le \max\left(\left\|\sum_{i=1}^n A_i^* f(|X_i^*|)^2 A_i\right\|, \left\|\sum_{i=1}^n B_i^* g(|X_i|)^2 B_i\right\|\right)$$

and

$$\left\|\sum_{i=1}^n A_i^* X_i^* B_i\right\|_p \le \left(\left\|\sum_{i=1}^n A_i^* f(|X_i^*|)^2 A_i\right\|_p^p + \left\|\sum_{i=1}^n B_i^* g(|X_i|)^2 B_i\right\|_p^p\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$.

Remark 2.8. *Let A and B be in $\mathcal{M}_n(\mathbb{C})$. Putting $n = 2$, $f(t) = g(t) = t^{\frac{1}{2}}$, $A_1 = A$, $A_2 = B$, $B_1 = B$, $B_2 = A$ and $X_1^* = X_2^* = I$ in the inequality (5), we get*

$$|||(A^*B + B^*A) \oplus 0||| \le |||(A^*A + B^*B) \oplus (A^*A + B^*B)|||$$

for every unitarily invariant norm, which was given by Hirzallah and Kittaneh in [8].

Corollary 2.9. *Let $X_i \in \mathcal{B}(\mathcal{H})$ be compact for $i = 1, 2, \dots, n$. Then*

$$\left|\left|\left|\sum_{i=1}^n X_i^*\right|\right|\right| \leqslant \left|\left|\left|\sum_{i=1}^n (|X_i^*| \oplus |X_i|)\right|\right|\right|$$

for every unitarily invariant norm.

Proof. Put $A_i^* = A_i = B_i^* = B_i = I$ for $i = 1, 2, \dots, n$, and $f(t) = g(t) = t^{\frac{1}{2}}$ in the inequality (5) and we get the result. $\square$

Corollary 2.10. *Let $A_i, X \in \mathcal{B}(\mathcal{H})$, $i = 1, 2, \dots, n$ such that X is a compact positive operator. Then*

$$\left|\left|\left|A_1XA_2^* + A_2XA_3^* + \cdots + A_nXA_1^*\right|\right|\right| \leqslant \sum_{i=1}^{n} \left|\left|\left|\left(X^{\frac{1}{2}}|A_i|^2X^{\frac{1}{2}}\right) \oplus \left(X^{\frac{1}{2}}|A_i|^2X^{\frac{1}{2}}\right)\right|\right|\right| \tag{6}$$

for every unitarily invariant norm.

Proof. Put $A_1^* = A_1$, $B_1 = A_2^*$, $A_2^* = A_2$, $B_2 = A_3^*, \dots$, $A_n^* = A_n$, $B_n = A_1^*$, $X_i^* = X$ for $i = 1, 2, ..., n$ and $f(t) = g(t) = t^{\frac{1}{2}}$ for all $t \in [0, \infty)$ in the inequality (5) and we have

$$\begin{aligned}
&\left|\left|\left|A_1XA_2^* + A_2XA_3^* + \cdots + A_nXA_1^*\right|\right|\right| \\
\leqslant &\left|\left|\left|\sum_{i=1}^{n} A_i^*XA_i \oplus \sum_{i=1}^{n} A_i^*XA_i\right|\right|\right| \\
\leqslant &\sum_{i=1}^{n} \left|\left|\left|\begin{bmatrix} A_i^* & 0 \\ 0 & A_i^* \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix}\right|\right|\right| \\
= &\sum_{i=1}^{n} \left|\left|\left|\left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} A_i^* & 0 \\ 0 & A_i^* \end{bmatrix}\right)^* \left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} A_i^* & 0 \\ 0 & A_i^* \end{bmatrix}\right)\right|\right|\right| \\
= &\sum_{i=1}^{n} \left|\left|\left|\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} |A_i| & 0 \\ 0 & |A_i| \end{bmatrix}^2\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}^{\frac{1}{2}}\right|\right|\right| \qquad (\text{since } \|T^*T\| = \|TT^*\|) \\
= &\sum_{i=1}^{n} \left|\left|\left|\left(X^{\frac{1}{2}}|A_i|^2X^{\frac{1}{2}}\right) \oplus \left(X^{\frac{1}{2}}|A_i|^2X^{\frac{1}{2}}\right)\right|\right|\right|
\end{aligned}$$

for every unitarily invariant norm. □

Remark 2.11. *In $\mathcal{M}_n(\mathbb{C})$ if taken $X_i = I$, for $i = 1, 2, \dots, n$ in (6), then*

$$\left\|A_1A_2^* + A_2A_3^* + \cdots + A_nA_1^*\right\|_p \leqslant \sum_{i=1}^{n} \||A_i|^2 \oplus |A_i|^2\|_p = 2^{\frac{1}{p}} \sum_{i=1}^{n} \|A_i\|_p^2$$

for $1 \leqslant p < \infty$, which is another version of the inequality in [5, Corollary 2.3].

Corollary 2.12. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$. Then*

$$s_j(A^*A - B^*B) \leqslant s_j\left(\left(|A|^2 + |B|^2\right) \oplus \left(|A|^2 + |B|^2\right)\right)$$

for $j = 1, 2, \dots, n$.

Proof. Put $n = 2$, $f(t) = g(t) = t^{\frac{1}{2}}$, $A_1^* = (A + B)^*$, $B_1 = (A - B)$, $A_2^* = (A - B)^*$, $B_2 = (A + B)$, $X_1^* = X_2^* = I$, in the inequality (4), also $2(A^*A - B^*B) = (A + B)^*(A - B) + (A - B)^*(A + B)$.
We have

$$\begin{aligned}
2s_j(A^*A - B^*B) &= s_j\Big((A + B)^* (A - B) + (A - B)^* (A + B) \Big) \\
&\leqslant s_j\Big(((A + B)^*(A + B) + (A - B)^*(A - B)) \\
&\qquad \oplus ((A - B)^*(A - B) + (A + B)^*(A + B)) \Big) \\
&= s_j\Big(2(A^*A + B^*B) \oplus 2(A^*A + B^*B)\Big) \\
&= 2s_j\left(\left(|A|^2 + |B|^2\right) \oplus \left(|A|^2 + |B|^2\right)\right)
\end{aligned}$$

for $j = 1, 2, \dots$. This completes the proof. □

Based on Theorem 2.4 and the inequality (1), we have the following singular value of the operator AX^*B.

Theorem 2.13. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ such that X is compact, A and B are self-adjoint with $|A| \leqslant a$ and $|B| \leqslant b$ for some real numbers a and b. Then*

$$s_j(AX^*B) \leqslant \left(\max\{a,b\}\right)^2 s_j(X \oplus X) \tag{7}$$

for $j = 1, 2, \ldots$.

Proof. Let

$$C = \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}.$$

Note that C is positive and $C \leqslant \max\{a,b\}$, therefore

$$\|C\| \leqslant \max\{a,b\}.$$

Hence

$$\begin{aligned} s_j(AX^*B) &\leqslant s_j(A|X^*|A \oplus B|X|B) && \text{(by Theorem 2.4)} \\ &\leqslant \|C\|^2 s_j(X \oplus X) && \text{(by the inequality (1))} \\ &\leqslant \left(\max\{a,b\}\right)^2 s_j(X \oplus X) \end{aligned}$$

for $j = 1, 2, \ldots$. □

Corollary 2.14. *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$ such that X and Y are compact and A, B are as in Theorem 2.13, then*

$$s_j(AX^*B + AY^*B) \leqslant \left(\max\{a,b\}\right)^2 s_j((X+Y) \oplus (X+Y)). \tag{8}$$

Proof. The inequality (8) follows from the inequality (7) by replacing the operator X by $X + Y$. □

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