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Nörlund and Riesz Mean of Sequence of Complex Uncertain Variables

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Abstract. In this article we have investigated some properties of the Nörlund and Riesz mean of sequences of complex uncertain variables. Also, we prove results on oscillating sequences of complex uncertain variables.

1. Introduction and Preliminaries

In order to rationally deal with the physical world, the belief degree of fact varies from person to person. The uncertainty of fact is one of the important feature of the real world. The Nörlund and Riesz mean are generalizations of Cesàro mean of order 1 i.e., the arithmetic mean, denoted by $(C, 1)$ method. For details about the matrix maps over sequence spaces, one may refer to Petersen [10]. The Nörlund mean and Riesz mean play a crucial role in the field of sequence spaces. It has been studied from different aspects. Tripathy and Baruah [11] have studied Nörlund and Riesz mean for sequences of fuzzy numbers.

1.1. Preliminary Results

In this subsection, we procure some fundamental concepts and results on uncertainty theory, those will be used throughout the paper.

Definition 1.1. Let $\mathcal{L}$ be a $\sigma-$algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2 (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;

Axiom 3 (Subadditivity Axiom) For every countable sequence of $\{\lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\lambda_j\}.$$

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The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined by Liu [3] as follows:

Axiom 4 (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, 3, ...$ The product uncertain measure $\mathcal{M}$ is an measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, ...$, respectively.

Definition 1.2. An uncertain variable is a measurable function ζ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers,the set

$$\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\}$$

is an event.

Definition 1.3. ([8]) The uncertainty distribution of an uncertain variable ζ is defined by

$$\Phi(x) = \mathcal{M}\{\zeta \leq x\}$$

for any real x.

Definition 1.4. ([8]) The expected value operator of an uncertain variable ζ is defined by

$$E[\zeta] = \int_0^{+\infty} \mathcal{M}\{\zeta \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\zeta \leq r\}dr$$

provided that at least one of the two integrals is finite.

Definition 1.5. Let ζ be an uncertain variable with a finite expected value $E[\zeta]$. Then the variance of ζ is defined by

$$V[\zeta] = E\left[(\zeta - E[\zeta])^2\right]$$

We list some existing results on mean and variance of uncertain variables.

Lemma 1.6. ([8]) *Let ζ be an uncertain variable with uncertainty distribution Φ. If the expected value $E[\zeta]$ exists, then*

$$E[\zeta] = \int_{-\infty}^{+\infty} x d\Phi(x).$$

Lemma 1.7. ([9]) *Let ζ be an uncertain variable with uncertainty distribution Φ. If $f(x)$ is a monotone function such that the expected value $E[f(\zeta)]$ exists, then*

$$E[f(\zeta)] = \int_{-\infty}^{+\infty} f(x)d\Phi(x).$$

Lemma 1.8. ([8]) *Let ζ be an uncertain variable with a finite expected value. Then for any real numbers a and b, we have*

$$V[a\zeta + b] = a^2 V[\zeta].$$

Lemma 1.9. ([13]) *Let ζ be an uncertain variable with a regular uncertainty distribution Φ. If the expected value $E[\zeta]$ exists, then the variance is,*

$$V[\zeta] = \int_0^1 (\Phi^{-1}(r) - E[\zeta])^2 dr.$$

2. Complex Uncertain Variable and Matrix Maps

Now, we procure some concepts on complex uncertain variables (one may refer to Chen, Ning and Wang [1]).

As a complex function on uncertainty space, complex uncertain variable is mainly used to model a complex uncertain quantity.

Definition 2.1. A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

When the range is the set of real numbers, we call it as an uncertain variable, introduced and investigated by Liu [8].

Definition 2.2. The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by $\Phi(c) = \mathcal{M}\{Re(\zeta) \leq Re(c), Im(\zeta) \leq Im(c)\}$ for any complex number c.
An uncertain variable is said to be positive, when it maps from $\mathbb{R}_+ \bigcup\{0\}$(non-negative real numbers) to $[0, 1]$.

Complex uncertain sequence is a sequence of complex uncertain variables. We discuss about different types of convergence concepts of complex uncertain sequences; convergence almost surely(a.s.), convergence in measure, convergence in mean, convergence in distribution and convergence uniformly almost surely(u.a.s.). The following definitions are due to Chen, Ning and Wang [1].

Definition 2.3. The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent almost surely (a.s.) to* ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that $\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0$, for every $\gamma \in \Lambda$. In that case we write $\zeta_n \to \zeta$, a.s..

Definition 2.4. The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in measure to* ζ if for a given $\varepsilon > 0$, $\lim_{n\to\infty} \mathcal{M}\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0$.

Definition 2.5. The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in mean to* ζ if $\lim_{n\to\infty} E[\|\zeta_n - \zeta\|] = 0$.

Definition 2.6. Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. We say the complex uncertain sequence $\{\zeta_n\}$ *converges in distribution to* ζ if $\lim_{n\to\infty} \Phi_n(c) = \Phi(c)$ for all $c \in \mathbb{C}$, at which $\Phi(c)$ is continuous.

Definition 2.7. The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent uniformly almost surely(u.a.s.) to* ζ if there exists a sequence of events $\{E'_k\}$, $\mathcal{M}\{E'_k\} \to 0$ such that $\{\zeta_n\}$ converges uniformly to ζ in $\Gamma - E'_k$, for any fixed $k \in \mathbb{N}$.

2.1. Matrix Maps

Let $A = (a_{nk})$ be an infinite matrix mapping from a sequence space E into a sequence space F, then for $\xi = \{\xi_n\} \in E$, the A-transform of $\{\xi_n\}$ is the sequence $(A_n(\xi))$, where $A_n\xi = \Sigma_{k=1}^{\infty} a_{nk}\xi_k$ for each $n \in \mathbb{N}$, provided the summation exists for each $n \in \mathbb{N}$.
The following well known result contains the necessary and sufficient conditions for the regularity of a matrix map known as Silverman-Toeplitz conditions (one may refer to Petersen [10]).

Lemma 2.8. *The matrix $A = (a_{mn})$ is regular or limit preserving if and only if it satisfies the following conditions:*

1. *there exists a constant K such that $\sum_{n=1}^{\infty} |a_{mn}| < K$ for every m;*
2. *for every n, $\lim_{m\to\infty} a_{mn} = 0$;*
3. $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = 1$.

3. Main Results

We recall the definitions of Nörlund and Riesz mean of the classical sequence spaces and some related results. Throughout $\{p_n\}$ denotes sequence of non-negative real numbers, those are not all 0, in particular $p_1 > 0$ and $P_n = p_1 + p_2 + ... + p_n$ (for $n = 1, 2, ...$);

Definition 3.1. The transformation defined by

$$t_m = \frac{p_m s_1 + ... + p_1 s_m}{P_m}$$

is called the Nörlund mean (N, p_n) or simply the (N, p_n) mean of the sequence $\{s_m\}$.

The matrix of the (N, p_n) mean is given by

$$a_{mn} = \begin{cases} \frac{p_{m-n+1}}{P_m}, & (n \leq m) \\ 0, & (n > m) \end{cases}$$

Definition 3.2. The transformation defined by

$$t_m = \frac{p_1 s_1 + ... + p_m s_m}{P_m}$$

is called the Riesz mean or simply the (R, p_n) mean of the sequence $\{s_m\}$.

The matrix of the (R, p_n) mean is given by

$$a_{mn} = \begin{cases} \frac{p_n}{P_m}, & (n \leq m) \\ 0, & (n > m) \end{cases}$$

The following results are well-known.

Lemma 3.3. *The Nörlund mean (N, p_n) is regular if and only if $\frac{p_n}{P_n} \to 0$, as $n \to \infty$.*

Lemma 3.4. *The Riesz mean (R, p_n) is regular if and only if $P_m \to \infty$, as $m \to \infty$.*

Now we introduce the notion of Nörlund and Riesz mean of the sequences of complex uncertain variable. Let $\{\xi_i\}$ be a sequence of complex uncertain variables in the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Throughout $p = \{p_n\}$ denotes a sequence of positive complex uncertain variables and

$$P_n = P_n(\gamma) = \sum_{i=1}^{n} \mathcal{M}(\{p_i(\gamma)\})$$

Definition 3.5. Let $p = \{p_n\}$ be a sequence of complex uncertain variables in the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. The transformation given by

$$t_n(\gamma) = \frac{\xi_n(\gamma)p_1 + ... + \xi_1(\gamma)p_n}{P_n} \qquad \forall \gamma \in \Gamma,$$

is the Nörlund mean of the complex uncertain sequence $\{\xi_n\}$.

Definition 3.6. Let $p = \{p_n\}$ be a sequence of complex uncertain variables in the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. The transformation given by

$$t_n(\gamma) = \frac{\xi_1(\gamma)p_1 + ... + \xi_n(\gamma)p_n}{P_n} \qquad \forall \gamma \in \Gamma,$$

is the Riesz mean of the complex uncertain sequence $\{\xi_n\}$.

Definition 3.7. The sequence $\{\xi_n\}$ of complex uncertain variables in the space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be slowly oscillating if $\| \mathcal{M}(\xi_m(\gamma)) - \mathcal{M}(\xi_n(\gamma)) \| \to 0$, as $m, n \to \infty$ with $1 \leq \frac{m}{n} \to 1$.

Theorem 3.8. *The Nörlund mean (N, p_n) is regular if and only if $\frac{\mathcal{M}(\{p_n(\gamma):\gamma\in\Gamma\})}{\mathcal{M}(\{P_n(\gamma):\gamma\in\Gamma\})} \to 0$ as $n \to \infty$.*

Proof. Let the sequence $\{\xi_n\}$ be convergent to ξ, then there exists a constant $H > 0$ such that $\| \xi_n(\gamma) \| < H$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Since $\{\xi_n\}$ is convergent, so for a given $\varepsilon > 0$ there exists an integer n_0 such that $\| \xi_n(\gamma) - \xi(\gamma) \| < \frac{\varepsilon}{2n_0H}$ for $n > n_0, \quad \gamma \in \Gamma$.

Then we have,

$$\begin{aligned}
\|t_n(\gamma) - \xi(\gamma)\| &\leq \|\frac{p_1(\xi_n(\gamma) - \xi(\gamma)) + p_2(\xi_{n-1}(\gamma) - \xi(\gamma)) + ... + p_{n_0}(\xi_{n-n_0}(\gamma) - \xi(\gamma))}{P_n(\gamma)}\| \\
&\qquad + \|\frac{p_{n_0+1}(\xi_{n_0+1}(\gamma) - \xi(\gamma))... + p_n(\xi_1(\gamma) - \xi(\gamma))}{P_n(\gamma)}\| \\
&\leq \frac{p_1\|\xi_n(\gamma) - \xi(\gamma)\| + p_2\|\xi_{n-1}(\gamma) - \xi(\gamma)\| + ... + p_{n-n_0}\|\xi_{n-n_0}(\gamma) - \xi(\gamma)\|}{P_n(\gamma)} \\
&\qquad + ... + \frac{p_{n-n_0+1}\|\xi_{n-n_0+1} - \xi(\gamma)\|l + ... + p_n\|\xi_1(\gamma) - \xi(\gamma)\|}{P_n(\gamma)} \\
&\leq \frac{(p_n + p_{n-1} + ... + p_{n-n_0})\frac{\varepsilon}{2}}{P_n(\gamma)} + \frac{p_{n-n_0+1}}{P_n(\gamma)}\|\xi_{n-n_0+1}(\gamma) - \xi(\gamma)\| + ... + \frac{p_n}{P_n(\gamma)}\|\xi_1(\gamma) - \xi(\gamma)\| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2n_0H}.Hn_0 \\
&= \varepsilon
\end{aligned}$$

The converse part can be established using the technique for classical case. □

Theorem 3.9. *The Riesz mean (R, p_n) is regular if and only if $\mathcal{M}(\{P_n(\gamma) : \gamma \in \Gamma\}) \to \infty$, as $n \to \infty$.*

Proof. Let the sequence $\{\xi_n\}$ convergent to ξ, then there exists a constant $H > 0$ such that $\|\xi_n\| < H$ for all $n \in \mathbb{N}$ and for a given $\varepsilon > 0$, there is an integer n_0 such that $\|\xi_n(\gamma) - \xi(\gamma)\| < \frac{\varepsilon}{2}$ for all $n > n_0$.

Now by hypothesis $P_n \to \infty$, as $n \to \infty$.

Let $\frac{p_1 2H + p_2 2H + ... + p_{n_0} 2H}{P_n} < \frac{\varepsilon}{2}$, for $n \geq n_0$.

Then we have,

$$\begin{aligned}
\|t_n - \gamma\| &= \|\frac{p_1(\xi_1(\gamma) - \xi(\gamma)) + ... + p_{n_0}(\xi_{n_0}(\gamma) - \xi(\gamma))}{P_n(\gamma)}\| + \\
&\quad ... + \|\frac{p_{n_0+1}(\xi_{n_0+1}(\gamma) - \xi(\gamma)) + ... + p_n(\xi_n(\gamma) - \xi(\gamma))}{P_n(\gamma)}\| \\
&\leq \frac{p_1 + p_2 + ... + p_{n_0}}{P_n(\gamma)}.\frac{\varepsilon}{2} + \frac{p_{n_0+1}\frac{\varepsilon}{2} + ... + p_n\frac{\varepsilon}{2}}{P_n(\gamma)} \\
&\leq \frac{\varepsilon}{2Hn_0}.\frac{Hn_0}{P_n(\gamma)} + \frac{\varepsilon}{2} \quad [By\ Lemma\ 3.4\ ] \\
&= \varepsilon
\end{aligned}$$

□

Theorem 3.10. *If $\{\xi_n\}$ is (R, p) summable to ξ in $(\Gamma, \mathcal{L}, \mathcal{M})$ and is slowly oscillating, then it converges to ξ in $(\Gamma, \mathcal{L}, \mathcal{M})$.*

Proof. Without loss of generality, one may assume that $\mathcal{M}(\xi(\gamma)) = 0$. Suppose, $\lim_{n\to\infty} \|\mathcal{M}(\xi_n(\gamma)) - 0\| > 0$. Then there exists $\alpha > 0$ and a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ with $\|\mathcal{M}(\xi_{n_i}(\gamma)) - \mathcal{M}(\xi(\gamma))\| \geq \alpha$ for all $i \in \mathbb{N}$.

Let $\{\xi_n\}$ be slowly oscillating, then $\{\xi_{n_i}\}$ as a subsequence of $\{\xi_n\}$ is also slowly oscillating. Then for a given $\delta > 0$, there exists $g_0 \in \mathbb{N}$ such that $g_0 \leq n \leq m < (1+\delta)n$, $\|\mathcal{M}(\xi_m) - \mathcal{M}(\xi_n)\| < \frac{\alpha}{2}$.

Since $\{\xi_n\}$ is summable to 0, let

$\sigma_n(\gamma) = \frac{1}{P_n} \sum_{k=1}^{n} p_k \xi_n(\gamma)$ be convergent to 0 in $(\Gamma, \mathcal{L}, \mathcal{M})$.

Now, for $\mathcal{M}(m_i) \geq \mathcal{M}(n_i)$, we have

$$\begin{aligned}\sigma_{m_i} - \frac{P_{n_i}}{P_{m_i}}\sigma_{n_i} &= \frac{1}{P_{m_i}} \sum_{k=1}^{m_i} p_k \xi_k(\gamma) - \frac{P_{n_i}}{P_{m_i}} \sum_{k=1}^{m_i} p_k \xi_k(\gamma) \\ &= \frac{1}{P_{m_i}} \sum_{k=n_i+1}^{m_i} p_k \xi_k(\gamma)\end{aligned}$$

Thus for all $n_i > g - 1$ and $n_i < m < m_i = [(1+\delta)n_i]$, where $[x]$ denote the integral part of x.
We have

$$\begin{aligned}\|\mathcal{M}(0) - \mathcal{M}(\xi_m(\gamma))\| &\geq \|\mathcal{M}(0) - \mathcal{M}(\xi_n(\gamma))\| - \|\mathcal{M}(\xi_n(\gamma)) - \mathcal{M}(\xi_m(\gamma))\| \\ &\geq \alpha - \frac{\alpha}{2}\end{aligned}$$

Further,we have

$$\begin{aligned}&\|\mathcal{M}(\sigma_{m_i}(\gamma)) - \mathcal{M}(\sigma_{n_i}(\gamma))\| + \|\mathcal{M}(\sigma_{n_i}(\gamma)) - \tfrac{P_{n_i}}{P_{m_i}}\mathcal{M}(\sigma_{n_i}(\gamma))\| \\ &\geq \|\mathcal{M}(\sigma_{m_i}(\gamma)) - \tfrac{P_{n_i}}{P_{m_i}}\mathcal{M}(\sigma_{n_i}(\gamma))\| \\ &\geq \|\tfrac{1}{P_{m_i}}\Sigma_{k=n_i+1}^{m_i} p_k(\gamma)\xi_k(\gamma) - 0\| \\ &\geq \|\tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}\xi_{n_i}(\gamma) - 0\| - \|\Sigma_{k=n_i+1}^{m_i} \tfrac{p_k\xi_k - p_{n_i}\xi_{n_i}}{P_{m_i}} - 0\| \\ &\geq \tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}\|\xi_{n_i}(\gamma) - 0\| - \tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}\|\xi_k(\gamma) - \xi_{n_i}(\gamma)\| \\ &\geq \tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}\alpha - \geq \tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}\tfrac{\alpha}{2} \\ &\geq \tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}(\alpha - \tfrac{\alpha}{2}) \\ &\geq \tfrac{p_{m_i} - p_{n_i}}{P_{m_i}}\tfrac{\delta}{1+\delta} \\ &\geq 0.\end{aligned}$$

Thus for all $m_i > n_i > N$, we have

$$\begin{aligned}&\|\mathcal{M}(\sigma_{m_i}(\gamma)) - \mathcal{M}(\sigma_{n_i}(\gamma))\| + \|\mathcal{M}(\sigma_{n_i}(\gamma)) - \tfrac{P_{n_i}}{P_{m_i}}\mathcal{M}(\sigma_{n_i}(\gamma))\| \\ &\geq \|\mathcal{M}(\sigma_{m_i}(\gamma)) - \tfrac{P_{n_i}}{P_{m_i}}\mathcal{M}(\sigma_{n_i}(\gamma))\| \\ &\geq \tfrac{\alpha}{2}\tfrac{\delta}{1+\delta}\end{aligned}$$

Therefore, $0 = \lim \|\mathcal{M}(\sigma_{m_i}(\gamma)) - \frac{P_{n_i}}{P_{m_i}}\mathcal{M}(\sigma_{n_i}(\gamma))\| > \frac{\alpha}{2}\frac{\delta}{1+\delta} > 0$ which contradicts that $\{\xi_i(\gamma)\}$ converges in $(\Gamma, \mathcal{L}, \mathcal{M})$.

This establishes the result. □

4. Conclusion

In this article we have established some results on the Nörlund and Riesz mean of sequences of complex uncertain variables. For data analysis weighted means play a crucial role. Thus the results established can be applied for data analysis.

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