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RD-phantom and *RD*-Ext-phantom Morphisms

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Abstract. A morphism f of left R-modules is called an RD-phantom morphism if the induced morphism $\mathrm{Tor}_1(R/aR, f) = 0$ for any $a \in R$. Similarly, a morphism g of left R-modules is said to be an RD-Ext-phantom morphism if the induced morphism $\mathrm{Ext}^1(R/Ra, g) = 0$ for any $a \in R$. It is proven that a morphism f is an RD-phantom morphism if and only if the pullback of any short exact sequence along f is an RD-exact sequence; a morphism g is an RD-Ext-phantom morphism if and only if the pushout of any short exact sequence along g is an RD-exact sequence. We also characterize Prüfer domains, left P-coherent rings, left PP rings, von Neumann regular rings in terms of RD-phantom and RD-Ext-phantom morphisms. Finally, we prove that every module has an epic RD-phantom cover with the kernel RD-injective and has a monic RD-Ext-phantom preenvelope with the cokernel RD-projective.

1. Introduction

The notion of purity has a substantial role in module theory and also in model theory. Warfield is the first to use the terminology RD-purity but this relative purity is the first purity used in theory of Abelian groups and in the theory of modules over PID (see [19, Notes on chapter I, p. 55-56]). RD-purity has been an object of deep study in the literature [8, 16, 19, 26, 33, 37]. Recall that an exact sequence $0 \to A \to B \to C \to 0$ of left R-modules is called *RD-exact* [37] if for every $a \in R$, the sequence $0 \to (R/aR)\otimes A \to (R/aR)\otimes B \to (R/aR)\otimes C \to 0$ is exact, or equivalently, if the sequence $0 \to \mathrm{Hom}(R/Ra, A) \to \mathrm{Hom}(R/Ra, B) \to \mathrm{Hom}(R/Ra, C) \to 0$ is exact. RD-purity has a close relationship with torsionfree and divisible modules, where a left R-module M is called *torsionfree* [11] if $\mathrm{Tor}_1(R/aR, M) = 0$ for any $a \in R$; a left R-module N is said to be *divisible* [11] if $\mathrm{Ext}^1(R/Ra, N) = 0$ for any $a \in R$. It is known that a module M is torsionfree if and only if every exact sequence $0 \to A \to B \to M \to 0$ is RD-exact; a module N is divisible if and only if every exact sequence $0 \to N \to B \to C \to 0$ is RD-exact.

On the other hand, ideal approximation theory has been recently introduced and developed by Fu, Guil Asensio, Herzog and Torrecillas in [17]. This theory is a generalization of the classical theory of covers and envelopes (approximation theory) initiated by Enochs, Auslander and Smalø [1, 12] since it need to be set forth in terms of morphisms instead of objects. An important instance is about the approximation by the ideal of phantom morphisms. The study of phantom morphisms has its roots in topology in the study of maps between CW-complexes [30] and was first introduced into the setting of a triangulated category

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by Neeman [32]. The theory of phantom morphisms was also developed in the stable category of a finite group ring by Benson and Gnacadja [3–5, 20]. Later the definition of a phantom morphism was generalized by Herzog to the category R-Mod of left R-modules over any associative ring R [22].

In the present paper, we first introduce the concepts of RD-phantom and RD-Ext-phantom morphisms, which may be viewed as the morphism versions of torsionfree and divisible modules. Some characterizations of RD-phantom and RD-Ext-phantom morphisms are given. Then we characterize Prüfer domains, left P-coherent rings, left PP rings, von Neumann regular rings in terms of RD-phantom and RD-Ext-phantom morphisms. Finally, we prove that every R-module has an epic RD-phantom cover with the kernel RD-injective and has a monic RD-Ext-phantom preenvelope with the cokernel RD-projective.

We next recall some notions and facts needed in the sequel.

A left R-module M is said to be *RD-projective* [37] if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact. By [37, Corollary 1], M is RD-projective if and only if M is a direct summand of a direct sum of cyclically presented left R-modules, where a left R-module is called *cyclically presented* if it is isomorphic to R/Rr for some $r \in R$.

A left R-module N is called *RD-injective* [37] if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \to 0$ is exact.

According to [8], a right R-module F is called *RD-flat* if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ is exact. By [8, Proposition I.1], F is RD-flat if and only if F is a direct limit of finite direct sums of cyclically presented right R-modules.

Given a ring R, we denote by R-Mor the category whose objects are left R-module morphisms and the morphism from a left R-module morphism $M_1 \xrightarrow{f} M_2$ to a left R-module morphism $N_1 \xrightarrow{g} N_2$ is a pair of left R-module morphisms $(M_1 \xrightarrow{d} N_1, M_2 \xrightarrow{s} N_2)$ such that the following diagram is commutative:

$$\begin{array}{ccc} M_1 & \xrightarrow{d} & N_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M_2 & \xrightarrow{s} & N_2. \end{array}$$

The category R-Mor is also denoted by $\mathcal{A}_2$ in [15], which means the category of all representations of the quiver $\mathbb{A}_2$ by left R-modules, where $\mathbb{A}_2$ is the quiver with two vertices v_1, v_2 and an edge $a : v_1 \to v_2$. It is well known that the category R-Mor is a locally finitely presented Grothendieck category. A morphism $f : E^1 \to E^2$ in R-Mor is injective if and only if E^1 and E^2 are injective left R-modules and f is a split epimorphism. A morphism $g : P_1 \to P_2$ in R-Mor is projective if and only if P_1 and P_2 are projective left R-modules and g is a split monomorphism.

In an additive category with direct limits, an object A is said to be *finitely presented* provided that the functor $\mathrm{Hom}(A, -)$ commutes with direct limits. Recall that an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in a locally finitely presented additive category C is *pure exact* [9] if it induces an exact sequence of Abelian groups $0 \to \mathrm{Hom}_C(A, X) \xrightarrow{f_*} \mathrm{Hom}_C(A, Y) \xrightarrow{g_*} \mathrm{Hom}_C(A, Z) \to 0$ for every finitely presented object A of C. Since both R-Mod and R-Mor are locally finitely presented Grothendieck categories, we can get the notions of purity in R-Mod and R-Mor.

Let $\mathcal{A}$ be any category and C a class of objects in $\mathcal{A}$. Following [12, 14], we say that a morphism $\phi : X \to Y$ in $\mathcal{A}$ is a *C-precover* of Y if $X \in C$ and, for any morphism $f : Z \to Y$ with $Z \in C$, there is a morphism $g : Z \to X$ such that $\phi g = f$. A C-precover $\phi : X \to Y$ is said to be a *C-cover of* Y if every endomorphism $g : X \to X$ such that $\phi g = \phi$ is an isomorphism. Dually we have the definitions of a C-preenvelope and a C-envelope. C-covers (C-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

Throughout this paper, all rings are associative with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+. $\mathrm{Hom}(M, N)$ and $M \otimes N$ will mean $\mathrm{Hom}_R(M, N)$ and $M \otimes_R N$ respectively, and similarly for derived functors $\mathrm{Ext}^1(M, N)$ and $\mathrm{Tor}_1(M, N)$. For unexplained concepts and notations, we refer the reader to [13, 21, 25, 34].

2. *RD*-phantom and *RD*-Ext-phantom morphisms

Definition 2.1. A morphism $f : M \to N$ of left R-modules is called an *RD-phantom morphism* if the induced morphism $\mathrm{Tor}_1(R/aR, f) : \mathrm{Tor}_1(R/aR, M) \to \mathrm{Tor}_1(R/aR, N)$ is 0 for any $a \in R$.

Similarly, a morphism $g : X \to Y$ of left R-modules is said to be an *RD*-Ext-*phantom morphism* if the induced morphism $\mathrm{Ext}^1(R/Ra, g) : \mathrm{Ext}^1(R/Ra, X) \to \mathrm{Ext}^1(R/Ra, Y)$ is 0 for any $a \in R$.

In the context of modules, an *RD*-phantom (resp. *RD*-Ext-phantom) morphism is the morphism version of a torsionfree (resp. divisible) module.

Let $f : M \to N$ be a morphism in R-Mod. Then the pullback of an exact sequence $0 \to B \to Q \to N \to 0$ along f induces a morphism $\mathrm{Ext}^1(f, B) : \mathrm{Ext}^1(N, B) \to \mathrm{Ext}^1(M, B)$ of Abelian groups. Dually, the pushout of an exact sequence $0 \to M \to H \to C \to 0$ along f induces a morphism $\mathrm{Ext}^1(C, f) : \mathrm{Ext}^1(C, M) \to \mathrm{Ext}^1(C, N)$ of Abelian groups.

The next theorem shows that there exists a close relationship between *RD*-phantom morphisms and *RD*-purity.

Theorem 2.2. *The following conditions are equivalent for a morphism $f : M \to N$ in R-Mod:*

1. *f is an RD-phantom morphism.*
2. *For any left R-module B,* $\mathrm{Ext}^1(f, B) : \mathrm{Ext}^1(N, B) \to \mathrm{Ext}^1(M, B)$ *takes values in the subgroup consisting of RD-exact sequences.*
3. *For each morphism $g : A \to M$ with A an RD-projective left R-module, the composition fg factors through a projective left R-module.*
4. *For each morphism $g : R/Ra \to M$ with $a \in R$, the composition fg factors through a projective left R-module.*
5. $\mathrm{Ext}^1(f, W) = 0$ *for every RD-injective left R-module W.*
6. $\mathrm{Tor}_1(F, f) = 0$ *for every RD-flat right R-module F.*

Proof. (1) ⇒ (2) Let $\eta : 0 \to B \to Q \to N \to 0$ be any exact sequence of left R-modules. We get the pullback η' of η along f:

$$\begin{array}{lccccccccc}
\eta' : & 0 & \longrightarrow & B & \xrightarrow{\lambda} & H & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\eta : & 0 & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

For any $a \in R$, we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Tor}_1(R/aR, M) & \xrightarrow{\theta} & (R/aR) \otimes B & \xrightarrow{(R/aR)\otimes\lambda} & (R/aR) \otimes H \\
\downarrow{\scriptstyle \mathrm{Tor}_1(R/aR,f)} & & \| & & \downarrow \\
\mathrm{Tor}_1(R/aR, N) & \xrightarrow{\xi} & (R/aR) \otimes B & \longrightarrow & (R/aR) \otimes Q.
\end{array}$$

Since $\mathrm{Tor}_1(R/aR, f) = 0$, we have $\theta = \xi \mathrm{Tor}_1(R/aR, f) = 0$. So $(R/aR) \otimes \lambda$ is a monomorphism. Thus η' is an *RD*-exact sequence.

(2) ⇒ (3) There exists an exact sequence $\zeta : 0 \to C \to P \xrightarrow{\mu} N \to 0$ of left R-modules with P projective, which yields the pullback of ζ along f:

$$\begin{array}{lccccccccc}
\zeta' : & 0 & \longrightarrow & C & \longrightarrow & H & \xrightarrow{\omega} & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle f} & & \\
\zeta : & 0 & \longrightarrow & C & \longrightarrow & P & \xrightarrow{\mu} & N & \longrightarrow & 0.
\end{array}$$

Then ζ' is an RD-exact sequence by (2). For each morphism $g : A \to M$ with A an RD-projective left R-module, there exists $\tau : A \to H$ such that $\omega\tau = g$. So $fg = f\omega\tau = \mu(\delta\tau)$, which implies that fg factors through the projective left R-module P.

(3) ⇒ (4) is trivial.

(4) ⇒ (1) By [19, VI Lemma 12.1, p. 240], there exists an RD-exact sequence of left R-modules $0 \to K \to X \xrightarrow{\rho} M \to 0$ with $X = \oplus_{i\in I} R/Rr_i$. Also there exists an exact sequence $0 \to C \to P \to N \to 0$ with P projective. It is easy to verify that $f\rho$ factors through a projective left R-module by (4). So one obtains the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\iota} & X & \xrightarrow{\rho} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

For any $a \in R$, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Tor}_1(R/aR, X) & \longrightarrow & \mathrm{Tor}_1(R/aR, M) & \xrightarrow{\gamma} & (R/aR)\otimes K & \xrightarrow{(R/aR)\otimes\iota} & (R/aR)\otimes X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Tor}_1(R/aR,f)} & & \downarrow{\scriptstyle (R/aR)\otimes\varphi} & & \downarrow \\ 0 = \mathrm{Tor}_1(R/aR, P) & \longrightarrow & \mathrm{Tor}_1(R/aR, N) & \xrightarrow{\psi} & (R/aR)\otimes C & \longrightarrow & (R/aR)\otimes P. \end{array}$$

Since $(R/aR) \otimes \iota$ is a monomorphism, we get $\gamma = 0$. So

$$\psi\mathrm{Tor}_1(R/aR, f) = ((R/aR) \otimes \varphi)\gamma = 0.$$

But ψ is a monomorphism, hence $\mathrm{Tor}_1(R/aR, f) = 0$. Thus f is an RD-phantom morphism.

(2) ⇒ (5) Let W be any RD-injective left R-module and $\Delta : 0 \to W \to Z \to N \to 0$ any exact sequence of left R-modules. We get the pullback Δ' of Δ along f:

$$\begin{array}{cccccccccc} \Delta' : & 0 & \longrightarrow & W & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ \Delta : & 0 & \longrightarrow & W & \longrightarrow & Z & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

By (2), Δ' is an RD-exact sequence and so is split since W is RD-injective. Thus $\mathrm{Ext}^1(f, W)([\Delta]) = [\Delta'] = 0$. It follows that $\mathrm{Ext}^1(f, W) : \mathrm{Ext}^1(N, W) \to \mathrm{Ext}^1(M, W)$ is 0.

(5) ⇒ (2) Let $\Lambda : 0 \to B \to V \to N \to 0$ be an exact sequence of left R-modules. Then we have the following pullback Λ' of Λ along f:

$$\begin{array}{cccccccccc} \Lambda' : & 0 & \longrightarrow & B & \xrightarrow{\tau} & U & \longrightarrow & M & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ \Lambda : & 0 & \longrightarrow & B & \longrightarrow & V & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

From [19, XIII Theorem 1.6, p. 425], there exists an RD-exact sequence $0 \to B \xrightarrow{h} W \to L \to 0$, where W is an RD-injective left R-module. So we get the following pushout Υ of Λ' along h:

$$\begin{array}{cccccccccc} \Lambda' : & 0 & \longrightarrow & B & \xrightarrow{\tau} & U & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle h} & & \downarrow & & \| & & \\ \Upsilon : & 0 & \longrightarrow & W & \xrightarrow{\chi} & H & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

From [17, Proposition 3], the composition of morphisms $\mathrm{Ext}^1(M,h)\mathrm{Ext}^1(f,B) : \mathrm{Ext}^1(N,B) \to \mathrm{Ext}^1(M,B) \to \mathrm{Ext}^1(M,W)$ is equal to the composition of morphisms $\mathrm{Ext}^1(f,W)\mathrm{Ext}^1(N,h) : \mathrm{Ext}^1(N,B) \to \mathrm{Ext}^1(N,W) \to \mathrm{Ext}^1(M,W)$.

By (5), one obtains that

$$\mathrm{Ext}^1(M,h)\mathrm{Ext}^1(f,B) = \mathrm{Ext}^1(f,W)\mathrm{Ext}^1(N,h) = 0.$$

Therefore $[\Upsilon] = \mathrm{Ext}^1(M,h)([\Lambda']) = \mathrm{Ext}^1(M,h)\mathrm{Ext}^1(f,B)([\Lambda]) = 0$. So the exact sequence $\Upsilon : 0 \to W \xrightarrow{\chi} H \to M \to 0$ is split. It follows that $((R/aR)\otimes\chi)((R/aR)\otimes h)$ is a monomorphism for any $a \in R$. Thus $(R/aR)\otimes\tau$ is a monomorphism by the above diagram. Hence $\Lambda' : 0 \to B \xrightarrow{\tau} U \to M \to 0$ is *RD*-exact.

(1) ⇔ (6) holds by the fact that every *RD*-flat right *R*-module is a direct limit of finite direct sums of cyclically presented right *R*-modules by [8, Proposition I.1]. □

Recall that a left *R*-module *M* is *Warfield cotorsion* [19, 21] if $\mathrm{Ext}^1(T,M) = 0$ for every torsionfree left *R*-module *T*. Clearly, any *RD*-injective left *R*-module is Warfield cotorsion.

Recall that a ring *R* is *left PP* [11] if every principal left ideal of *R* is projective.

If a morphism *f* in *R*-Mod factors through a torsionfree left *R*-module, then *f* is clearly an *RD*-phantom morphism. Conversely, we have the following result.

Proposition 2.3. *If a ring R satisfies one of the following conditions:*

1. *R is a commutative PP ring;*
2. *Every Warfield cotorsion left R-module is RD-injective,*

then any RD-phantom morphism in R-Mod factors through a torsionfree R-module.

Proof. Suppose that $f : M \to N$ is an *RD*-phantom morphism in *R*-Mod. By [21, Theorem 4.1.1(b)] and Wakamutsu's Lemma, *N* has a torsionfree cover $g : T \to N$ with $\ker(g)$ Warfield cotorsion. So we get the following pullback:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(g) & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \ker(g) & \longrightarrow & T & \xrightarrow{g} & N & \longrightarrow & 0. \end{array}$$

By Theorem 2.2, the exact sequence $0 \to \ker(g) \to U \to M \to 0$ is *RD*-exact.

(1) Assume that *R* is a commutative *PP* ring. Then $\ker(g)$ is torsionfree by [11, Proposition 3.6]. This implies that $\ker(g)$ is *RD*-injective by the proof of [19, XIII, Lemma 8.1, p. 458].

(2) Assume that *R* satisfies that every Warfield cotorsion left *R*-module is *RD*-injective. Then $\ker(g)$ is *RD*-injective.

In either case, the exact sequence $0 \to \ker(g) \to U \to M \to 0$ is split. It is easy to verify that *f* factors through the torsionfree *R*-module *T*. □

In [22], Herzog called a morphism $f : M \to N$ in *R*-Mod a *phantom morphism* if the induced morphism $\mathrm{Tor}_1(A,f) : \mathrm{Tor}_1(A,M) \to \mathrm{Tor}_1(A,N)$ is 0 for every (finitely presented) right *R*-module *A*. Similarly, a morphism $g : X \to Y$ in *R*-Mod is called an Ext-*phantom morphism* [23] if the induced morphism $\mathrm{Ext}^1(B,g) : \mathrm{Ext}^1(B,X) \to \mathrm{Ext}^1(B,Y)$ is 0 for every finitely presented left *R*-module *B*.

Obviously, we have the following implications:

phantom morphisms ⇒ *RD*-phantom morphisms.

Ext-phantom morphisms ⇒ *RD*-Ext-phantom morphisms.

But the inverse implications are not true in general.

Proposition 2.4. *Let $f : M \to N$ be a morphism in R-Mod, where M is an RD-flat left R-module. Then f is an RD-phantom morphism if and only if f is a phantom morphism.*

Proof. By [8, Proposition I.1], $M = \varinjlim M_i$, where each M_i is a finite direct sum of cyclically presented left R-modules. Let $g_i : M_i \to \varinjlim M_i$ be the structural morphism of this direct limit. Then $f = \varinjlim(fg_i)$.

Suppose that f is an RD-phantom morphism. Then fg_i factors through a projective left R-module by Theorem 2.2. So for any right R-module A, we have

$$\mathrm{Tor}_1(A, f) = \mathrm{Tor}_1(A, \varinjlim(fg_i)) = \varinjlim \mathrm{Tor}_1(A, fg_i) = 0.$$

Whence f is a phantom morphism. □

Recall that a left R-module M is said to be *absolutely pure* [31] or *FP-injective* [36] if any exact sequence $0 \to M \to N \to L \to 0$ of left R-modules is pure, equivalently, if $\mathrm{Ext}^1(A, M) = 0$ for any finitely presented left R-module A.

Proposition 2.5. *The following conditions are equivalent for a commutative domain R:*

1. *R is a Prüfer domain.*
2. *Every RD-phantom morphism is a phantom morphism.*
3. *Every RD-*Ext*-phantom morphism is an* Ext*-phantom morphism.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) By [37, Theorem 1], every finitely presented R-module is a direct summand of a direct sum of cyclically presented R-modules. Thus every RD-phantom morphism is a phantom morphism and every RD-Ext-phantom morphism is an Ext-phantom morphism.

(2) ⇒ (1) Let M be a torsionfree R-module. Then the identity $M \to M$ is an RD-phantom morphism and so is a phantom morphism by (2). Thus M is flat. So R is a Prüfer domain by [21, Theorem 4.4.10] or [26, Lemma 5.1].

(3) ⇒ (1) Let N be a divisible R-module. Then the identity $N \to N$ is an RD-Ext-phantom morphism and so is an Ext-phantom morphism by (3). Hence N is absolutely pure. So R is a Prüfer domain by [19, IX Proposition 3.4, p. 314]. □

Theorem 2.6. *The following conditions are equivalent for a morphism $\alpha : X \to Y$ in R-Mod:*

1. *α is an RD-*Ext*-phantom morphism.*
2. *For any left R-module C, $\mathrm{Ext}^1(C, \alpha) : \mathrm{Ext}^1(C, X) \to \mathrm{Ext}^1(C, Y)$ takes values in the subgroup consisting of RD-exact sequences.*
3. *For each morphism $\beta : Y \to Z$ with Z an RD-injective left R-module, the composition $\beta\alpha$ factors through an injective left R-module.*
4. *$\mathrm{Ext}^1(B, \alpha) = 0$ for every RD-projective left R-module B.*

Proof. (1) ⇒ (2) Let $\eta : 0 \to X \to H \to C \to 0$ be any exact sequence. Then we get the following pushout η' of η along α:

$$\begin{array}{llllllllll}
\eta : & 0 & \longrightarrow & X & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \| & & \\
\eta' : & 0 & \longrightarrow & Y & \longrightarrow & Q & \xrightarrow{h} & C & \longrightarrow & 0.
\end{array}$$

For any $a \in R$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}(R/Ra, H) & \longrightarrow & \mathrm{Hom}(R/Ra, C) & \xrightarrow{\theta} & \mathrm{Ext}^1(R/Ra, X) \\
\downarrow & & \| & & \downarrow{\scriptstyle \mathrm{Ext}^1(R/Ra,\alpha)} \\
\mathrm{Hom}(R/Ra, Q) & \xrightarrow{h_*} & \mathrm{Hom}(R/Ra, C) & \xrightarrow{\xi} & \mathrm{Ext}^1(R/Ra, Y).
\end{array}$$

Since $\mathrm{Ext}^1(R/Ra, \alpha) = 0$, we have $\xi = \mathrm{Ext}^1(R/Ra, \alpha)\theta = 0$. So h_* is an epimorphism. Whence η' is an RD-exact sequence.

(2) ⇒ (3) There exists an exact sequence $\zeta : 0 \to X \to E \to L \to 0$ with E injective. Consider the following pushout ζ' of ζ along α:

$$\begin{array}{cccccccccc}
\zeta: & 0 & \longrightarrow & X & \xrightarrow{\lambda} & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \| & & \\
\zeta': & 0 & \longrightarrow & Y & \xrightarrow{\omega} & H & \longrightarrow & L & \longrightarrow & 0.
\end{array}$$

Then ζ' is an RD-exact sequence by (2). For each morphism $\beta : Y \to Z$ with Z an RD-injective left R-module, there exists $\tau : H \to Z$ such that $\tau\omega = \beta$. So $\beta\alpha = \tau\omega\alpha = (\tau\gamma)\lambda$ and we may infer that $\beta\alpha$ factors through the injective left R-module E.

(3) ⇒ (1) By [19, XIII Theorem 1.6, p. 425], there exists an RD-exact sequence $0 \to Y \xrightarrow{\iota} Z \to V \to 0$, where Z is an RD-injective left R-module. Also there exists an exact sequence $0 \to X \to E \to W \to 0$ with E injective. By (3), $\iota\alpha$ factors through an injective R-module. So one obtains the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \vdots & & \vdots{\scriptstyle \chi} & & \\
0 & \longrightarrow & Y & \xrightarrow{\iota} & Z & \xrightarrow{\rho} & V & \longrightarrow & 0.
\end{array}$$

For any $a \in R$, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}(R/Ra, E) & \longrightarrow & \mathrm{Hom}(R/Ra, W) & \xrightarrow{\psi} & \mathrm{Ext}^1(R/Ra, X) & \longrightarrow & \mathrm{Ext}^1(R/Ra, E) = 0 \\
\downarrow & & \downarrow{\scriptstyle \chi_*} & & \downarrow{\scriptstyle \mathrm{Ext}^1(R/Ra,\alpha)} & & \downarrow \\
\mathrm{Hom}(R/Ra, Z) & \xrightarrow{\rho_*} & \mathrm{Hom}(R/Ra, V) & \xrightarrow{\phi} & \mathrm{Ext}^1(R/Ra, Y) & \longrightarrow & \mathrm{Ext}^1(R/Ra, Z).
\end{array}$$

Since ρ_* is an epimorphism, $\phi = 0$. Therefore

$$\mathrm{Ext}^1(R/Ra, \alpha)\psi = \phi\chi_* = 0.$$

Note that ψ is an epimorphism, which implies $\mathrm{Ext}^1(R/Ra, \alpha) = 0$.

(1) ⇔ (4) follows from the fact that any RD-projective left R-module is a direct summand of a direct sum of cyclically presented left R-modules. □

Lemma 2.7. *A morphism $f : M \to N$ in R-Mod is an RD-phantom morphism if and only if $f^+ : N^+ \to M^+$ is an RD-*Ext*-phantom morphism in Mod-R.*

Proof. For any $a \in R$, we have the following commutative square:

$$\begin{array}{ccc}
\mathrm{Tor}_1(R/aR, N)^+ & \xrightarrow{\mathrm{Tor}_1(R/aR,f)^+} & \mathrm{Tor}_1(R/aR, M)^+ \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
\mathrm{Ext}^1(R/aR, N^+) & \xrightarrow{\mathrm{Ext}^1(R/aR,f^+)} & \mathrm{Ext}^1(R/aR, M^+),
\end{array}$$

where both α and β are the standard isomorphisms by [34, p. 360]. Consequently $\mathrm{Tor}_1(R/aR, f) = 0$ if and only if $\mathrm{Tor}_1(R/aR, f)^+ = 0$ if and only if $\mathrm{Ext}^1(R/aR, f^+) = 0$. Thus f is an RD-phantom morphism if and only if f^+ is an RD-Ext-phantom morphism. □

Next we give new characterizations of some rings using RD-phantom and RD-Ext-phantom morphisms.

Recall that R is a *left P-coherent ring* [27] if every principal left ideal of R is finitely presented. Obviously, any left PP ring is left P-coherent.

Theorem 2.8. *The following conditions are equivalent for a ring R:*

1. *R is a left P-coherent ring.*
2. *f is an RD-Ext-phantom morphism in R-Mod if and only if f^+ is an RD-phantom morphism in Mod-R.*
3. *f is an RD-Ext-phantom morphism in R-Mod if and only if f^{++} is an RD-Ext-phantom morphism in R-Mod.*
4. *f is an RD-phantom morphism in Mod-R if and only if f^{++} is an RD-phantom morphism in Mod-R.*
5. *The class of RD-phantom morphisms in Mor-R is closed under direct products.*
6. *The class of RD-Ext-phantom morphisms in R-Mor is closed under direct limits.*

Proof. (1) ⇒ (2) Let $f : M \to N$ be a morphism in R-Mod. For any $a \in R$, consider the following commutative square:

$$\begin{array}{ccc} \mathrm{Tor}_1(N^+, R/Ra) & \xrightarrow{\mathrm{Tor}_1(f^+, R/Ra)} & \mathrm{Tor}_1(M^+, R/Ra) \\ \varphi \downarrow & & \downarrow \psi \\ \mathrm{Ext}^1(R/Ra, N)^+ & \xrightarrow{\mathrm{Ext}^1(R/Ra, f)^+} & \mathrm{Ext}^1(R/Ra, M)^+, \end{array}$$

where both φ and ψ are isomorphisms by [7, Lemma 2.7 (2)]. Then $\mathrm{Ext}^1(R/Ra, f) = 0$ if and only if $\mathrm{Ext}^1(R/Ra, f)^+ = 0$ if and only if $\mathrm{Tor}_1(f^+, R/Ra) = 0$. Therefore f is an RD-Ext-phantom morphism if and only if f^+ is an RD-phantom morphism.

(2) ⇒ (3) f is an RD-Ext-phantom morphism in R-Mod if and only if f^+ is an RD-phantom morphism in Mod-R if and only if f^{++} is an RD-Ext-phantom morphism in R-Mod by Lemma 2.7.

(3) ⇒ (4) f is an RD-phantom morphism in Mod-R if and only if f^+ is an RD-Ext-phantom morphism in R-Mod if and only if f^{+++} is an RD-Ext-phantom morphism in R-Mod if and only if f^{++} is an RD-phantom morphism in Mod-R by Lemma 2.7.

(4) ⇒ (1) By (4), a right R-module A is torsionfree if and only if $A \xrightarrow{1_A} A$ is an RD-phantom morphism in Mod-R if and only if $A^{++} \xrightarrow{1_{A^{++}}} A^{++}$ is an RD-phantom morphism if and only if A^{++} is a torsionfree right R-module.

Let $(M_i)_{i\in I}$ be a family of torsionfree right R-modules. Then $\oplus_{i\in I}M_i$ is torsionfree and so $(\oplus_{i\in I}M_i)^{++}$ is torsionfree. Since $\oplus_{i\in I}M_i^+$ is a pure submodule of $\prod_{i\in I} M_i^+$ by [6, Lemma 1 (1)], $(\prod_{i\in I} M_i^+)^+ \to (\oplus_{i\in I}M_i^+)^+ \to 0$ is split. Note that $(\prod_{i\in I} M_i^+)^+ \cong (\oplus_{i\in I}M_i)^{++}$ is torsionfree, so $\prod_{i\in I} M_i^{++} \cong (\oplus_{i\in I}M_i^+)^+$ is torsionfree. Since $\prod_{i\in I} M_i$ is a pure submodule of $\prod_{i\in I} M_i^{++}$ by [6, Lemma 1 (2)], $\prod_{i\in I} M_i$ is torsionfree by [27, Lemma 2.6]. Whence R is a left P-coherent ring by [27, Theorem 2.7].

(1) ⇒ (5) Let $(f_i : M_i \to N_i)_{i\in I}$ be a family of RD-phantom morphisms in Mor-R and $\prod_{i\in I} f_i : \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ be the induced morphism. For every $a \in R$, by [7, Lemma 2.10], we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Tor}_1(\prod_{i\in I} M_i, R/Ra) & \xrightarrow{\mathrm{Tor}_1(\prod_{i\in I} f_i, R/Ra)} & \mathrm{Tor}_1(\prod_{i\in I} N_i, R/Ra) \\ \cong \downarrow & & \downarrow \cong \\ \prod_{i\in I} \mathrm{Tor}_1(M_i, R/Ra) & \xrightarrow{\prod_{i\in I} \mathrm{Tor}_1(f_i, R/Ra)} & \prod_{i\in I} \mathrm{Tor}_1(N_i, R/Ra). \end{array}$$

Since $\prod_{i\in I} \mathrm{Tor}_1(f_i, R/Ra) = 0, \mathrm{Tor}_1(\prod_{i\in I} f_i, R/Ra) = 0$. So $\prod_{i\in I} f_i : \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ is an RD-phantom morphism.

(5) ⇒ (1) It is clear that $1_R : R_R \to R_R$ is an RD-phantom morphism in Mod-R. By (5), $\prod_{i\in I} 1_R : \prod_{i\in I} R_R \to \prod_{i\in I} R_R$ is an RD-phantom morphism. Thus $\mathrm{Tor}_1(\prod_{i\in I} R_R, R/Ra) = 0$ for every $a \in R$. Hence $\prod_{i\in I} R_R$ is torsionfree. We conclude that R is a left P-coherent ring by [27, Theorem 2.7].

(1) ⇒ (6) Let $(\tau_i : M_i \to N_i)_{i\in I}$ be a morphism between two direct systems of morphisms $\{f_{ij} : M_i \to M_j\}_{i\leq j\in I}$ and $\{g_{ij} : N_i \to N_j\}_{i\leq j\in I}$ in R-Mod such that each $\tau_i : M_i \to N_i$ is an RD-Ext-phantom morphism in R-Mod. Let $\varinjlim \tau_i : \varinjlim M_i \to \varinjlim N_i$ be the induced morphism. By [7, Lemma 2.9], we obtain the following

commutative diagram for any $a \in R$:

$$\begin{array}{ccc} \varinjlim \mathrm{Ext}^1(R/Ra, M_i) & \xrightarrow{\varinjlim \mathrm{Ext}^1(R/Ra,\tau_i)} & \varinjlim \mathrm{Ext}^1(R/Ra, N_i) \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Ext}^1(R/Ra, \varinjlim M_i) & \xrightarrow{\mathrm{Ext}^1(R/Ra,\varinjlim \tau_i)} & \mathrm{Ext}^1(R/Ra, \varinjlim N_i). \end{array}$$

Since $\varinjlim \mathrm{Ext}^1(R/Ra, \tau_i) = 0$, we have $\mathrm{Ext}^1(R/Ra, \varinjlim \tau_i) = 0$. Consequently, $\varinjlim \tau_i : \varinjlim M_i \to \varinjlim N_i$ is an *RD*-Ext-phantom morphism in *R*-Mod.

(6) ⇒ (1) Let $(M_i)_{i\in I}$ be a family of divisible left *R*-modules, where *I* is a directed set. It is obvious that $1_{M_i} : M_i \to M_i$ is an *RD*-Ext-phantom morphism. So $\varinjlim 1_{M_i} : \varinjlim M_i \to \varinjlim M_i$ is an *RD*-Ext-phantom morphism by (6). Thus $\mathrm{Ext}^1(R/Ra, \varinjlim M_i) = 0$ for any $a \in R$. Hence $\varinjlim M_i$ is a divisible left *R*-module. So *R* is a left *P*-coherent ring by [27, Theorem 2.7]. □

Proposition 2.9. *The following conditions are equivalent for a ring R:*

1. *R is a left PP ring.*
2. *R is a left P-coherent ring and every submorphism of a projective morphism in R-Mor is an RD-phantom morphism.*
3. *R is a left P-coherent ring and every submorphism of a projective morphism in Mor-R is an RD-phantom morphism.*
4. *Every quotient of an injective morphism in R-Mor is an RD-*Ext-*phantom morphism.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) Let $M_1 \xrightarrow{f} M_2$ be a submorphism of a projective morphism $P_1 \xrightarrow{p} P_2$ in *R*-Mor (or Mor-*R*). Then we get the following exact sequence in *R*-Mor (or Mor-*R*):

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \to & P_1 & \to & H_1 & \to & 0 \\ & & f\downarrow & & p\downarrow & & h\downarrow & & \\ 0 & \to & M_2 & \to & P_2 & \to & H_2 & \to & 0. \end{array}$$

Since *R* is a left *PP* ring, M_1 and M_2 are torsionfree by [11, Corollary 3.3 and Proposition 3.6]. Whence *f* is an *RD*-phantom morphism.

(2) ⇒ (1) Let $a \in R$. Then $Ra \xrightarrow{1} Ra$ is a submorphism of the projective morphism ${}_RR \xrightarrow{1} {}_RR$ and so is an *RD*-phantom morphism by (2). Hence *Ra* is torsionfree. So *Ra* is flat by [35, 5(a), p. 2047]. Thus *Ra* is projective since *Ra* is finitely presented.

(3) ⇒ (1) Let $a \in R$. Then *aR* is torsionfree by (3). So *aR* is flat by [35, 5(a), p. 2047]. Hence *Ra* is also flat by [24, Theorem 2.2]. Thus *R* is a left *PP* ring.

(1) ⇒ (4) Let $L^1 \xrightarrow{l} L^2$ be a quotient of an injective morphism $E^1 \xrightarrow{e} E^2$ in *R*-Mor. Then one obtains the following exact sequence in *R*-Mor:

$$\begin{array}{ccccccccc} 0 & \to & K^1 & \to & E^1 & \to & L^1 & \to & 0 \\ & & k\downarrow & & e\downarrow & & l\downarrow & & \\ 0 & \to & K^2 & \to & E^2 & \to & L^2 & \to & 0. \end{array}$$

Since *R* is a left *PP* ring, L^1 and L^2 are divisible by [27, Theorem 5.3]. So *l* is an *RD*-Ext-phantom morphism.

(4) ⇒ (1) Let *X* be any quotient of an injective left *R*-module *E*. Then the identity $X \xrightarrow{1_X} X$ is a quotient of the injective morphism $E \xrightarrow{1_E} E$. So $X \xrightarrow{1_X} X$ is an *RD*-Ext-phantom morphism by (4). Thus *X* is divisible. It follows that *R* is a left *PP* ring by [27, Theorem 5.3]. □

Proposition 2.10. *The following conditions are equivalent for a ring R:*

1. *R is a von Neumann regular ring.*
2. *Every morphism in R-Mod is an RD-phantom morphism.*
3. *Every morphism in R-Mod is an RD-*Ext-*phantom morphism.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are obvious since every left *R*-module is torsionfree and divisible.

(2) ⇒ (1) For any left *R*-module M, the identity $M \to M$ is an *RD*-phantom morphism by (2). Thus M is torsionfree. So R is a von Neumann regular ring by [11, Theorem 2.2].

(3) ⇒ (1) For any left *R*-module M, the identity $M \to M$ is an *RD*-Ext-phantom morphism. Thus M is divisible. So R is a von Neumann regular ring by [25, Proposition 3.18]. □

We need the following lemma in order to investigate *RD*-phantom precovers and *RD*-Ext-phantom preenvelopes.

Lemma 2.11. *Consider the following commutative diagram with exact rows in R-Mod:*

$$\begin{array}{rccccccccc} \eta_1: & 0 & \longrightarrow & K_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & L_1 & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\gamma} & & \\ \eta_2: & 0 & \longrightarrow & K_2 & \xrightarrow{\alpha_2} & M_2 & \xrightarrow{\beta_2} & L_2 & \longrightarrow & 0. \end{array}$$

1. *If η_1 is an RD-exact sequence and φ is an RD-phantom morphism, then γ is an RD-phantom morphism.*
2. *If η_2 is an RD-exact sequence and φ is an RD-*Ext-*phantom morphism, then ψ is an RD-*Ext-*phantom morphism.*

Proof. (1) For any $a \in R$, the *RD*-exact sequence η_1 induces the exact sequence

$$\mathrm{Tor}_1(R/aR, M_1) \overset{\mathrm{Tor}_1(R/aR,\beta_1)}{\to} \mathrm{Tor}_1(R/aR, L_1) \to (R/aR) \otimes K_1 \overset{(R/aR)\otimes\alpha_1}{\to} (R/aR) \otimes M_1.$$

Since $(R/aR) \otimes \alpha_1$ is a monomorphism, $\mathrm{Tor}_1(R/aR, \beta_1)$ is an epimorphism. Because

$$\mathrm{Tor}_1(R/aR, \gamma)\mathrm{Tor}_1(R/aR, \beta_1) = \mathrm{Tor}_1(R/aR, \beta_2)\mathrm{Tor}_1(R/aR, \varphi) = 0,$$

we have $\mathrm{Tor}_1(R/aR, \gamma) = 0$.

(2) For any $a \in R$, the *RD*-exact sequence η_2 induces the exact sequence

$$\mathrm{Hom}(R/Ra, M_2) \overset{(\beta_2)_*}{\to} \mathrm{Hom}(R/Ra, L_2) \to \mathrm{Ext}^1(R/Ra, K_2) \overset{\mathrm{Ext}^1(R/Ra,\alpha_2)}{\to} \mathrm{Ext}^1(R/Ra, M_2).$$

Since $(\beta_2)_*$ is an epimorphism, $\mathrm{Ext}^1(R/Ra, \alpha_2)$ is a monomorphism. Note that

$$\mathrm{Ext}^1(R/Ra, \alpha_2)\mathrm{Ext}^1(R/Ra, \psi) = \mathrm{Ext}^1(R/Ra, \varphi)\mathrm{Ext}^1(R/Ra, \alpha_1) = 0.$$

Whence $\mathrm{Ext}^1(R/Ra, \psi) = 0$. □

Theorem 2.12. *Let R be a ring.*

1. *Every left R-module morphism has an epic RD-phantom cover in R-Mor.*
2. *Every left R-module morphism has a monic RD-*Ext-*phantom preenvelope in R-Mor.*

Proof. (1) Note that the class of *RD*-phantom morphisms in *R*-Mor is closed under pure quotients by [29, Lemma 2.12] and Lemma 2.11(1), and closed under direct limits. So every left *R*-module morphism has an *RD*-phantom cover in *R*-Mor by [10, Theorem 2.6]. The *RD*-phantom cover is an epimorphism because any projective generator of *R*-Mor is an *RD*-phantom morphism.

(2) Since the class of *RD*-Ext-phantom morphisms in *R*-Mor is closed under pure submorphisms by [29, Lemma 2.12] and Lemma 2.11(2), and closed under direct products, every left *R*-module morphism has an *RD*-Ext-phantom preenvelope in *R*-Mor by [10, Theorem 4.1]. These *RD*-Ext-phantom preenvelopes are monomorphisms because any injective object of *R*-Mor is an *RD*-Ext-phantom morphism. □

Recall that an additive subbifunctor of the bifunctor $\mathrm{Hom}_R(-,-) : R\text{-Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}$ is called an *ideal* $\mathcal{I}$ of R-Mod. This means that, for every pair of left R-modules M and N, the morphisms $M \to N$ in $\mathcal{I}$ form a subgroup of the Abelian group $\mathrm{Hom}_R(M, N)$, and given any three left R-module morphisms f, g, h for which fgh is defined and $g \in \mathcal{I}$, we have $fgh \in \mathcal{I}$.

Clearly, both RD-phantom and RD-Ext-phantom morphisms constitute ideals.

In ideal approximation theory, the concepts of classical covers and envelopes for classes of objects were generalized to ideals of morphisms. Let $\mathcal{I}$ be an ideal of R-Mod. Recall that a morphism $\phi : M \to N$ in $\mathcal{I}$ is an *$\mathcal{I}$-precover of N* [17] if for any morphism $\psi : C \to N$ in $\mathcal{I}$, there is a morphism $\theta : C \to M$ such that $\phi\theta = \psi$. An $\mathcal{I}$-precover $\phi : M \to N$ is called an *$\mathcal{I}$-cover* if every endomorphism h of M such that $\phi h = \phi$ is an isomorphism. An $\mathcal{I}$-preenvelope and an $\mathcal{I}$-envelope are defined dually.

Just as a phantom precover is called special [22, p. 230] when its kernel is pure-injective, we will call an RD-phantom precover $f : M \to N$ *special* if the kernel of f is RD-injective. Similarly, we will call an RD-Ext-phantom preenvelope $g : A \to B$ *special* if the cokernel of g is RD-projective.

Theorem 2.13. *Let R be a ring.*

1. *Every left R-module has an epic RD-phantom cover which is special.*
2. *Every left R-module has a monic RD-*Ext-*phantom preenvelope which is special.*

Proof. (1) Given a left R-module M, Theorem 2.12(1) implies that $M \stackrel{1_M}{\to} M$ has an epic RD-phantom cover $(N \stackrel{g}{\to} G) \stackrel{(f,h)}{\to} (M \stackrel{1_M}{\to} M)$, so $f = hg$. Note that $f : N \to M$ is an RD-phantom morphism.

Since $(N \stackrel{f}{\to} M) \stackrel{(f,1_M)}{\to} (M \stackrel{1_M}{\to} M)$ is a morphism in R-Mor, there exists $(N \stackrel{f}{\to} M) \stackrel{(\mu,\nu)}{\to} (N \stackrel{g}{\to} G)$ such that $g\mu = \nu f$ and

$$(f, h)(\mu, \nu) = (f, 1_M).$$

So $f\mu = f$ and $h\nu = 1_M$. Because $(N \stackrel{g}{\to} G) \stackrel{(\mu,\nu h)}{\to} (N \stackrel{g}{\to} G)$ is a morphism in R-Mor such that

$$(f, h)(\mu, \nu h) = (f, h),$$

we have $(\mu, \nu h)$ is an isomorphism. It is easy to see that h is an isomorphism.

By [28, Lemma 2.7], $f : N \to M$ is an epic RD-phantom precover of M. Next we prove that $f : N \to M$ is also an epic RD-phantom cover of M.

Let $\delta : N \to N$ such that $f\delta = f$. Notice that $g\delta = h^{-1}f\delta = h^{-1}f = g$. Therefore $(N \stackrel{g}{\to} G) \stackrel{(\delta,1_G)}{\to} (N \stackrel{g}{\to} G)$ is a morphism in R-Mor such that

$$(f, h)(\delta, 1_G) = (f, h).$$

Hence we have $(\delta, 1_G)$ is an isomorphism. So δ is an isomorphism, i.e., f is an epic RD-phantom cover of M.

Let $K = \ker(f)$. By [19, XIII Theorem 1.6, p. 425], there exists an RD-exact sequence $0 \to K \to B \to L \to 0$, where B is an RD-injective left R-module. Consider the following pushout:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \stackrel{\lambda}{\longrightarrow} & N & \stackrel{f}{\longrightarrow} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & {\scriptstyle\gamma}\downarrow & & \| & & \\
0 & \longrightarrow & B & \stackrel{\iota}{\longrightarrow} & H & \stackrel{\pi}{\longrightarrow} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

For any $a \in R$, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Tor}_1(R/aR,N) & \xrightarrow{\mathrm{Tor}_1(R/aR,f)} & \mathrm{Tor}_1(R/aR,M) & \xrightarrow{\theta} & (R/aR)\otimes K \\ {\scriptstyle \mathrm{Tor}_1(R/aR,\gamma)}\downarrow & & \| & & \downarrow{\scriptstyle (R/aR)\otimes\varphi} \\ \mathrm{Tor}_1(R/aR,H) & \xrightarrow{\mathrm{Tor}_1(R/aR,\pi)} & \mathrm{Tor}_1(R/aR,M) & \xrightarrow{\xi} & (R/aR)\otimes B. \end{array}$$

Since $\mathrm{Tor}_1(R/aR, f) = 0$, θ is a monomorphism. Also $(R/aR) \otimes \varphi$ is a monomorphism, so $\xi = ((R/aR) \otimes \varphi)\theta$ is a monomorphism. Thus $\mathrm{Tor}_1(R/aR, \pi) = 0$, i.e., $\pi : H \to M$ is an *RD*-phantom morphism. Hence there exist $\rho : H \to N$ and $\omega : B \to K$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\lambda} & N & \xrightarrow{f} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\iota} & H & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \vdots{\scriptstyle\omega} & & \vdots{\scriptstyle\rho} & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{\lambda} & N & \xrightarrow{f} & M & \longrightarrow & 0. \end{array}$$

Since f is a cover, $\rho\gamma$ is an isomorphism. Hence $\omega\varphi$ is an isomorphism. Thus K is isomorphic to a direct summand of B and so is *RD*-injective. Hence f is special.

(2) By Theorem 2.12(2) and [28, Lemma 2.6], every left R-module X has a monic *RD*-Ext-phantom preenvelope $\alpha : X \to Y$. Let $G = \mathrm{coker}(\alpha)$. By [19, VI Lemma 12.1, p. 240], there exists an *RD*-exact sequence $0 \to V \to U \xrightarrow{\psi} G \to 0$, where U is an *RD*-projective left R-module.

Consider the following pullback:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & V & = & V & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow{\phi} & W & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \longrightarrow & G & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

For any $a \in R$, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Hom}(R/Ra,U) & \xrightarrow{\tau} & \mathrm{Ext}^1(R/Ra,X) & \xrightarrow{\mathrm{Ext}^1(R/Ra,\phi)} & \mathrm{Ext}^1(R/Ra,W) \\ {\scriptstyle\psi_*}\downarrow & & \| & & \downarrow \\ \mathrm{Hom}(R/Ra,G) & \xrightarrow{\sigma} & \mathrm{Ext}^1(R/Ra,X) & \xrightarrow{\mathrm{Ext}^1(R/Ra,\alpha)} & \mathrm{Ext}^1(R/Ra,Y). \end{array}$$

Since $\mathrm{Ext}^1(R/Ra, \alpha) = 0$, σ is an epimorphism. Because ψ_* is an epimorphism, $\tau = \sigma\psi_*$ is an epimorphism. So $\mathrm{Ext}^1(R/Ra, \phi) = 0$. Thus $\phi : X \to W$ is an *RD*-Ext-phantom morphism. Since α is an *RD*-Ext-phantom preenvelope, it is easy to see that $\phi : X \to W$ is a special *RD*-Ext-phantom preenvelope. □

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