Filomat 32:8 (2018), 2901–2910
https://doi.org/10.2298/FIL1808901E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Existence and Stability of Solution of Boundary Value Problem for Fractional Integro-Differential Equations with Complex Order

E. M. Elsayed[a], K. Kanagarajan[b], D. Vivek[c]

[a]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia.*
[b]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.*
[c]*Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, India.*

Abstract. In this paper, we establish sufficient conditions for the existence and stability of solutions for fractional integro-differential equations with boundary conditions involving complex order. The proofs are based upon the Banach contraction principle. An example is included to show the applicability of our results.

1. Introduction

The study of fractional differential equations (FDEs) ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. FDEs appear naturally in a number of fields such as physics, polymer rheology, regular variational in thermodynamics, biophysics, electrical circuits, electron-analytical chemistry, biology, control theory, etc. An excellent description in the study of FDEs can be found in [13, 23, 24, 28]. It is considerable that there are many works about fractional integro-differential equations (FIDEs) (see, for example, [4, 25, 30, 38]).

Ulam's stability problem [14] has been attracted by many famous researchers, for example, see Andras, Jung and Rus [2, 19, 31]. For more recent contribution on such interesting topic, see [2, 15, 22, 29, 36, 37] and references therein. Rabha W. Ibrahim studied Ulam stability for FDEs in Complex Domain in [16]. The author also considered a generalization of the admissible functions in complex Banach spaces; one can refer to [17, 18].

The topics of FDEs, which attracted a growing interest for some time, in particular, in relation to the complex order in fractional calculus, have been rapidly developed recent years. E. R. Love [21] started the research of fractional derivatives of imaginary order. The concept is usual definitions of fractional integrals and derivatives by defining derivatives of purely imaginary orders. The notion of fractional operator of complex order, introduced by Samko et al.[32]. In this direction, several notions of fractional derivative of complex order were discussed [1, 33]. For instance, Carla M.A.Pinto [10] introduced the two approximations of the complex order van der Pol oscillator. In the paper [27], the authors investigated the existence of solutions of boundary value problems(BVPs) with complex order. Most recently, Vivek

2010 *Mathematics Subject Classification*. 26A33, 93B05, 34A60
Keywords. fractional integro-differential equations, boundary condition, complex order, existence, fixed point, Ulam-Hyers stablity
Received: 20 July 2017; Accepted: 21 December 2017
Communicated by Jelena Manojlović
Email addresses: emmelsayed@yahoo.com (E. M. Elsayed), kanagarajank@gmail.com (K. Kanagarajan), peppyvivek@gmail.com (D. Vivek)

et al. studied the existence and stability results for pantograph equations[35] and integro-differential equations[34] with nonlocal conditions involving complex order.

Motivated by the works mentioned in [4, 21, 27, 33, 34], in this paper, we estabilish four types of Ulam stability, namely Ulam-Hyers(U-H) stability, generalized U-H stability, U-H-Rassias and generalized U-H-Rassias stability for the following BVPs for FIDEs with complex order

$$D^{\theta}_{0^+}x(t) = f\left(t, x(t), \int_0^t h(t,s,x(s))ds\right), \quad t \in J := [0,T], \quad \theta = m + i\alpha, \tag{1}$$

$$ax(0) + bx(T) = c, \tag{2}$$

where $D^{\theta}_{0^+}$ is Caputo fractional derivative of order $\theta \in \mathbb{C}$. Let $\alpha \in \mathbb{R}^+$, $m \in (0,1]$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : \Delta \times \mathbb{R} \to \mathbb{R}$ are continuous. Here, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$. For brevity let us take

$$Hx(t) = \int_0^t h(t,s,x(s))ds.$$

There have been many papers (see, for example, in [3, 6–8, 11, 26, 39]) dealing with BVPs of FDEs.

The paper is organized as follows. In Section 2, we recall some basic definitions from fractional calculus and establish auxiliary lemmas which play a pivotal role in the sequel. Section 3 contains existence and Ulam stability results for the problem (1)-(2).

2. Prerequisites

In this section, we recall some definitions and lemmas used further.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous function J into $\mathbb{R}$ with the norm

$$\|x\|_\infty := \sup\{|x(t)| : t \in J\}.$$

By $L^1(J)$ we denote the space of Lebesgue-integrable function $x : J \to \mathbb{R}$ with the norm $\|x\|_{L^1} = \int_0^T |x(t)|\,dt$.

Definition 2.1. *([28]) The Riemann-Liouville fractional integral of order $\theta \in \mathbb{C}$, $(Re(\theta) > 0)$ of a function f is*

$$I^{\theta}_{0^+}f(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}f(s)ds.$$

Definition 2.2. *([28]) For a function f given by on the interval J, the Caputo fractional-order $\theta \in \mathbb{C}$, $(Re(\theta) > 0)$ of f, is defined by*

$$(D^{\theta}_{0^+}f)(t) = \frac{1}{\Gamma(n-\theta)}\int_0^t (t-s)^{n-\theta-1}f^{(n)}(s)ds,$$

where $n = [Re(\theta)] + 1$ and $[Re(\theta)]$ denotes the integral part of the real number θ.

Definition 2.3. *([20]) The Stirling asymptotic formula of the Gamma function for $z \in \mathbb{C}$ is following*

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{\frac{z-1}{2}} e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right], \quad (\left|arg(z)\right| < \pi; |z| \to \infty), \tag{3}$$

and its results for $|\Gamma(u+iv)|$, $(u, v \in \mathbb{R})$ is

$$|\Gamma(u+iv)| = (2\pi)^{\frac{1}{2}}\,|v|^{u-\frac{1}{2}}\,e^{-u-\pi|v|/2}\left[1 + O\left(\frac{1}{v}\right)\right], \quad (v \to \infty). \tag{4}$$

Lemma 2.4. *(see Lemma 7.1.1,([14])) Let $z, w : [0, T) \to [0, \infty)$ be continuous functions where $T \leq \infty$. If w is nondecreasing and there are constants $k \geq 0$ and $0 < \nu < 1$ such that*

$$z(t) \leq w(t) + k \int_0^t (t-s)^{\nu-1} z(s)ds, \quad t \in [0, T),$$

then

$$z(t) \leq w(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(k\Gamma(\nu))^n}{\Gamma(n\nu)} (t-s)^{n\nu-1} w(s) \right) ds, \quad t \in [0, T).$$

Remark 2.5. *Under the hypothesis of Lemma 2.4, let $w(t)$ be a nondecreasing function on $[0, T)$. Then we have $z(t) \leq w(t) E_{\nu,1}(k\Gamma(\nu)t^{\nu})$.*

For the FIDE with complex order (1), we adopt the definitions from Rus [31] of the U-H stability, generalized U-H stability, U-H-Rassias and generalized U-H-Rassias stability.

Now we consider the problem (1) and the following inequalities

$$\left| D_{0^+}^{\theta} z(t) - f(t, z(t), Hz(t)) \right| \leq \epsilon, \ t \in J, \tag{5}$$

$$\left| D_{0^+}^{\theta} z(t) - f(t, z(t), Hz(t)) \right| \leq \epsilon\varphi(t), \quad t \in J, \tag{6}$$

$$\left| D_{0^+}^{\theta} z(t) - f(t, z(t), Hz(t)) \right| \leq \varphi(t), \quad t \in J, \tag{7}$$

Definition 2.6. *The equation* (1) *is U-H stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C(J, \mathbb{R})$ of the inequality* (**??**) *there exists a solution $x \in C(J, \mathbb{R})$ of equation* (1) *with*

$$|z(t) - x(t)| \leq C_f \epsilon, \ t \in J.$$

Definition 2.7. *The equation* (1) *is generalized U-H stable if there exists $\psi_f \in C([0, \infty), [0, \infty))$, $\psi_f(0) = 0$ such that for each solution $z \in C(J, \mathbb{R})$ of the inequality* (**??**) *there exists a solution $x \in C(J, \mathbb{R})$ of equation* (1) *with*

$$|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 2.8. *The equation* (1) *is U-H-Rassias stable with respect to $\varphi \in C(J, \mathbb{R})$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C(J, \mathbb{R})$ of the inequality* (**??**) *there exists a solution $x \in C(J, \mathbb{R})$ of equation* (1) *with*

$$|z(t) - x(t)| \leq C_f \epsilon\varphi(t), \quad t \in J.$$

Definition 2.9. *The equation* (1) *is generalized U-H-Rassias stable with respect to $\varphi \in C(J, \mathbb{R})$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C(J, \mathbb{R})$ of the inequality* (5) *there exists a solution $x \in C(J, \mathbb{R})$ of equation* (1) *with*

$$|z(t) - x(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.$$

Remark 2.10. *A function $z \in C(J, \mathbb{R})$ is a solution of the inequality* (**??**) *if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depend on z) such that*

1. $\left| g(t) \right| \leq \epsilon, \ \forall \ t \in J$;
2. $D_{0^+}^{\theta} z(t) = f(t, z(t), Hz(t)) + g(t), \ t \in J$.

One can have similar remarks for the inequality (**??**) and (5).

Remark 2.11. *Let $\theta = m + i\alpha$, $m \in (0,1]$ and $\alpha \in \mathbb{R}^+$, if $z \in C(J,\mathbb{R})$ is a solution of the inequality (**??**), then z is a solution of the following integral inequality*

$$\left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right| \leq \frac{\epsilon}{|\Gamma(\theta)|} \frac{T^m}{m} \left(1 + \frac{|b|}{|a+b|} \right).$$

Indeed, by Remark 2.10, we have that

$$D_{0^+}^{\theta} z(t) = f(t, z(t), Hz(t)) + g(t), \quad t \in J.$$

Then

$$\begin{aligned} z(t) &= A_z + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s)) ds + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} g(s) ds \\ &\quad - \left(\frac{b}{a+b} \right) \frac{1}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} g(s) ds, \ t \in J. \end{aligned}$$

with

$$A_z = \frac{1}{a+b} \left[c - \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, z(s) Hz(s)) ds \right].$$

From this it follows that

$$\begin{aligned} &\left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right| \\ &= \left| \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} g(s) ds - \left(\frac{b}{a+b} \right) \frac{1}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} g(s) ds \right| \\ &\leq \frac{1}{|\Gamma(\theta)|} \int_0^t (t-s)^{m-1} \left| g(s) \right| ds + \left(\frac{|b|}{|a+b|} \right) \frac{1}{|\Gamma(\theta)|} \int_0^T (T-s)^{m-1} \left| g(s) \right| ds \\ &\leq \frac{\epsilon}{|\Gamma(\theta)|} \frac{T^m}{m} \left(1 + \frac{|b|}{|a+b|} \right). \end{aligned}$$

Remark 2.12. *Clearly,*

1. *Definition 2.6$\Rightarrow$ Definition 2.7.*
2. *Definition 2.8$\Rightarrow$ Definition 2.9.*

Remark 2.13. *A solution of the FIDEs with complex order inequality (**??**) is called an fractional ϵ-solution of the problem* (1)-(2)*.*

3. Existence and U-H stability results

Lemma 3.1. *Let $\theta = m + i\alpha$, $0 < m \leq 1$, $\alpha \in \mathbb{R}^+$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : \Delta \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Then the FIDEs with complex order*

$$D_{0^+}^{\theta} x(t) = f(t, x(t), Hx(t)), \quad t \in J, \tag{8}$$

$$ax(0) + bx(T) = c \tag{9}$$

has a unique solution which is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s)) ds \\ &\quad - \frac{1}{a+b} \left[\frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s)) ds - c \right]. \end{aligned} \tag{10}$$

Proof. By integration of eqn. (**??**), we obtain

$$x(t) = x_0 + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s))ds. \tag{11}$$

We use condition (8) to compute the constant x_0, so we have

$$ax(0) = ax_0 \quad \text{and} \quad bx(T) = bx_0 + \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s))ds,$$

then $ax(0) + bx(T) = c$, since

$$x_0 = -\frac{1}{(a+b)} \left[\frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s))ds - c \right].$$

Substituting in Eqn.(11) leads to formula (10). □

First, we give the following result based on Banach contraction principle.

Lemma 3.2. *Assume hypotheses*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists a constant* $L_1 > 0$ *such that*

$$\left| f(t, x_1, y_1) - f(t, x_2, y_2) \right| \leq L_1 \left[|x_1 - x_2| + \left| y_1 - y_2 \right| \right], \forall\, x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

(H2) The function $h : \Delta \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists a constant* $H_1 > 0$ *such that*

$$|h(t, s, x_1) - h(t, s, x_2)| \leq H_1 |x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

If

$$\left(\frac{L_1(1 + H_1)}{m\, |\Gamma(\theta)|} T^m \left[1 + \frac{|b|}{|a+b|} \right] \right) < 1, \tag{12}$$

the problem (1)-(2) *has a unique solution.*

Proof. Transform the problem (1)-(2) into a fixed point problem.
Consider the operator $P : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by

$$\begin{aligned} (Px)(t) = {} & \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s))ds \\ & - \frac{1}{a+b} \left[\frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s))ds - c \right]. \end{aligned} \tag{13}$$

Clearly, the fixed points of the operator P are solution of the problem (1)-(2). We shall use the Banach contraction principle to prove that P defined by (13) has a fixed point. We shall show that P is a contraction.

Let $x, y \in C(J, \mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned}
&\left|(Px)(t) - (Py)(t)\right| \\
&\leq \frac{1}{|\Gamma(\theta)|} \int_0^t \left|(t-s)^{\theta-1}\right| \left|f(s, x(s), Hx(s)) - f(s, y(s), Hy(s))\right| ds \\
&\quad + \frac{|b|}{|\Gamma(\theta)|\,|a+b|} \int_0^T \left|(T-s)^{\theta-1}\right| \left|f(s, x(s), Hx(s)) - f(s, y(s), Hy(s))\right| ds \\
&\leq \frac{L_1(1+H_1)}{|\Gamma(\theta)|} \int_0^t \left|(t-s)^{\theta-1}\right| \left|x(s) - y(s)\right| ds + \frac{L_1(1+H_1)\,|b|}{|\Gamma(\theta)|\,|a+b|} \int_0^T \left|(T-s)^{\theta-1}\right| \left|x(s) - y(s)\right| ds \\
&\leq \frac{L_1(1+H_1)\left\|x-y\right\|_\infty}{|\Gamma(\theta)|} \int_0^t (t-s)^{m-1} ds + \frac{L_1(1+H_1)\,|b|\left\|x-y\right\|_\infty}{|\Gamma(\theta)|\,|a+b|} \int_0^T (T-s)^{m-1} ds \\
&\leq \left(\frac{L_1(1+H_1)}{m\,|\Gamma(\theta)|} T^m \left[1 + \frac{|b|}{|a+b|}\right]\right) \left\|x-y\right\|_\infty .
\end{aligned}$$

Thus

$$\left\|Px - Py\right\|_\infty \leq \left(\frac{L_1(1+H_1)}{m\,|\Gamma(\theta)|} T^m \left[1 + \frac{|b|}{|a+b|}\right]\right) \left\|x-y\right\|_\infty ,$$

From (12), it follows that P has a unique fixed point which is solution of the problem (1)-(2). □

Theorem 3.3. *In the conditions (H1), (H2) and* (12), *the problem* (1)-(2) *is U-H stable.*

Proof. Let $z \in C(J, \mathbb{R})$ be a solution of the inequality (**??**). Denote by $x \in C(J, \mathbb{R})$ the unique solution of the following problem

$$\begin{cases} D^{\theta}_{0^+} x(t) = f(t, x(t), Hx(t)); \quad t \in J, \quad \theta = m + i\alpha, \\ x(0) = z(0), \quad x(T) = z(T), \end{cases}$$

where $m \in (0, 1]$, $\alpha \in \mathbb{R}^+$.
Using Lemma 3.1, we have that

$$x(t) = A_x + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s)) ds,$$

with

$$A_x = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s)) ds\right].$$

On the other hand, if $x(T) = z(T)$ and $x(0) = z(0)$, then $A_x = A_z$.
Indeed,

$$\begin{aligned}
|A_x - A_z| &\leq \frac{|b|}{|a+b|\,|\Gamma(\theta)|} \int_0^T \left|(T-s)^{\theta-1}\right| \left|f(s, x(s), Hx(s)) - f(s, z(s), Hz(s))\right| ds \\
&\leq \frac{L_1(1+H_1)\,|b|}{|a+b|} I^{\theta}_{0^+} \left|x(T) - z(T)\right| \\
&= 0.
\end{aligned}$$

Thus

$$A_x = A_z.$$

Then, we have

$$x(t) = A_z + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s)) ds.$$

By integration of the inequality (**??**) and using Remark 2.11, we have

$$\left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right| \leq \frac{\epsilon T^m}{m\,|\Gamma(\theta)|} \left(1 + \frac{|b|}{|a+b|}\right).$$

for all $t \in J$. From above it follows:

$$\begin{aligned}
&|z(t) - x(t)| \\
&= \left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right| \\
&\quad + \frac{1}{|\Gamma(\theta)|} \int_0^t \left|(t-s)^{\theta-1}\right| \left| f(s, z(s), Hz(s)) - f(s, x(s), Hx(s)) \right| ds \\
&\leq \frac{\epsilon T^m}{m\,|\Gamma(\theta)|} \left(1 + \frac{|b|}{|a+b|}\right) + \frac{L_1(1+H_1)}{|\Gamma(\theta)|} \int_0^t (t-s)^{m-1} |z(s) - x(s)|\, ds.
\end{aligned}$$

By Lemma 2.4(Gronwall inequality) and Remark 2.5,for all $t \in J$, we have that

$$|z(t) - x(t)| \leq \frac{\epsilon T^m}{m\,|\Gamma(\theta)|} \left(1 + \frac{|b|}{|a+b|}\right) E_{m,1} \left(\frac{L_1(1+H_1)}{|\Gamma(\theta)|} \cdot \Gamma(m) T^m \right).$$

Thus, the problem (1)-(2) is U-H stable. □

We have the following generalized U-H -Rassias stability results.

Theorem 3.4. *In the conditions (H1), (H2),* (12) *and*

(H3) There exists an increasing function $\varphi \in C(J, \mathbb{R})$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$

$$I_{0^+}^{\theta} \varphi(t) \leq \lambda_\varphi \varphi(t),$$

the problem (1)-(2) *is generalized U-H-Rassias stable.*

Proof. Let $z \in C(J, \mathbb{R})$ be solution of the following inequality

$$\left| D_{0^+}^{\theta} z(t) - f(t, z(t), Hz(t)) \right| \leq \epsilon \varphi(t), \quad t \in J, \quad \epsilon > 0, \tag{14}$$

and let $x \in C(J, \mathbb{R})$ be the unique solution of the following problem

$$\begin{cases} D_{0^+}^{\theta} x(t) = f(t, x(t), Hx(t)); \quad t \in J, \quad \theta = m + i\alpha, \\ x(0) = z(0), \quad x(T) = z(T), \end{cases}$$

where $m \in (0, 1]$, $\alpha \in \mathbb{R}^+$.
By Lemma 3.1,

$$x(t) = A_z + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s)) ds,$$

with

$$A_z = \frac{1}{a+b} \left[c - \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s)) ds \right].$$

By integration of the inequality (14), we obtain

$$\left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s))ds \right| \leq \epsilon \lambda_\varphi \varphi(t) \left(1 + \frac{|b|}{|a+b|}\right).$$

We have for any $t \in J$

$$\begin{aligned}
&|z(t) - x(t)| \\
&\leq \left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s))ds \right| \\
&\quad + \frac{1}{|\Gamma(\theta)|} \int_0^t \left|(t-s)^{\theta-1}\right| \left| f(s, z(s), Hz(s)) - f(s, x(s), Hx(s)) \right| ds \\
&\leq \epsilon \lambda_\varphi \varphi(t) \left(1 + \frac{|b|}{|a+b|}\right) + \frac{L_1(1+H_1)}{|\Gamma(\theta)|} \int_0^t (t-s)^{m-1} |z(s) - x(s)| \, ds.
\end{aligned}$$

Using Lemma 2.4(Gronwall inequality) and Remark 2.5, we obtain

$$|z(t) - x(t)| \leq \epsilon \lambda_\varphi \varphi(t) \left(1 + \frac{|b|}{|a+b|}\right) E_{m,1} \left(\frac{L_1(1+H_1)}{|\Gamma(\theta)|} \cdot \Gamma(m) T^m\right), \quad t \in J.$$

Thus, the problem (1)-(2) is generalized U-H-Rassias stable. □

Remark 3.5. *([9]) The boundary value problem* (1)-(2) *are appropriate for the following problems:*

1. *Initial value problem:* $a = 1, b = 0, c = 0$.
2. *Terminal value problem:* $a = 0, b = 1$, *c arbitrary.*
3. *Anti-periodic problem:* $a = 1, b = 1, c = 0$.

However, they are not for the periodic problem, i.e., for $a = 1, b = -1, c = 0$.

4. An example

Consider the following fractional integro-differential equation with complex order

$$D^\theta x_p(t) = \frac{t\left|x_p\right|}{10\left(1 + \left|x_p\right|\right)} + \frac{1}{2p} \int_0^t e^{-(s-t)} x_p(s)ds, \ t \in J := [0, 1], \tag{15}$$

$$x_p(0) = 0, \quad x_p(1) = 0, \tag{16}$$

where $\theta = \alpha + im$, $m = \frac{1}{2}$ and $\alpha = 1$.
Set

$$\begin{aligned}
f_p &= \frac{t\left|x_p\right|}{10\left(1 + \left|x_p\right|\right)} + \frac{1}{2p} \int_0^t e^{-(s-t)} x_p(s)ds, \\
H_p x(t) &= \frac{1}{2p} \int_0^t e^{-(s-t)} x_p(s)ds, \\
k(t,s) &= e^{-(s-t)}.
\end{aligned}$$

Let $x_p, y_p \in \mathbb{R}$ and $t \in J$. Then, we have

$$\begin{aligned}
\left| f(t, x_p(t), (Hx_p)(t)) - f(t, y_p(t), (Hy_p)(t)) \right| &\leq L_1 \left(\left|x_p - y_p\right| + \left|Hx_p - Hy_p\right| \right), \\
&\leq L_1(1 + H_1) \left|x_p - y_p\right|,
\end{aligned}$$

where $L_1 = \frac{t}{10}$, $H_1 = \frac{1}{2}$.
Hence the conditions (H1), (H2) hold with $L_1 = \frac{t}{10}$ and $H_1 = \frac{1}{2}$. We shall check that condition (12) is satisfied for suitable values of $\alpha = 1$, $m = \frac{1}{2}$ with $a = b = T = 1$. Indeed,

$$\left(\frac{L_1(1 + H_1)}{m\,|\Gamma(\theta)|} T^m \left[1 + \frac{|b|}{|a + b|} \right] \right) < 1.$$

It follows from Lemma 3.2 that the problem (15)-(16) has a unique solution on J. In addition, Theorem (3.3) implies that the problem (15)-(16) is Ulam-Hyers stable.

Acknowledgements

The authors are grateful to the refrees for their careful reading of the manuscript and valuable comments. The authors thank the help from editor too.

Author contributions

All of the authors equally contributed to the conception and development of this manuscript.

References

[1] R. Andriambololona, R. Tokiniaina, H. Rakotoson, Definitions of complex order integrals and complex order derivatives using operator approach, *International Journal of Latest Research in Science and Technology*, **1(4)** (2012) 317-323.
[2] S. Andras, J. J. Kolumban, On the Ulam-Hyers stability of first order differential systems with nonlocal initial conditions, *Nonlinear Anal. Theory Methods Appl.*, **82**(2013) 1-11.
[3] A.Arara, M.Benchohra, N.Hamidi, J.J.Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal. Theory Methods Appl.*, **72 (2)** (2010) 580-586.
[4] K. Balachandran, K. Uchiyama, Existence of local solutions of quasilinear integrodifferential equations in Banach spaces, *Appl. Anal.*, **76** (1998) 1-8.
[5] Bertram Ross, Francis H. Northover, A use for a derivative of complex order in the fractional calculus, **9(4)** (1977) 400-406.
[6] Z. Bai, H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* , **311 (2)** (2005) 495-505.
[7] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal. Theory Methods Appl.*, **72 (2)** (2010) 916-924.
[8] Z. Bai, H. Lu, Positive solutions for a boundary value problem of nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **311**(2005) 495-505.
[9] M. Benchohra, S. Bouriah, Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order, *Moroccan J. Pure and Appl. Anal*, **1(1)** (2015) 22-37.
[10] Carla M. A. Pinto, J. A. Tenreiromachado, Complex order van der Pol oscillator, *Nonlinear Dyn.*, **65 (3)** (2010) 247-254.
[11] C.S. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* , **23** (2010) 1050-1055.
[12] A. Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.
[13] R.Hilfer, Application of fractional Calculus in Physics, World Scientific, Singapore, 1999.
[14] DH. Hyers, G. Isac, TM. Rassias, Stability of functional equation in several variables, Vol. 34, Progress in nonlinea differential equations their applications, Boston (MA): Birkhauser; 1998.
[15] Rabha W. Ibrahim, Generalized Ulam-Hyers stability for fractional differential equations, *Int. J. Math.*,**23** (2012), 9 pages, doi:10.1142/S0129167X12500565.
[16] Rabha W. Ibrahim, Ulam Stability for Fractional Differential Equationin Complex Domain, *Abstr. Appl. Anal.*, Volume 2012, Article ID 649517, 8 pages, doi:10.1155/2012/649517.
[17] Rabha W. Ibrahim, On Generalized Hyers-Ulam Stability of Admissible Functions, *Abstr. Appl. Anal.*, Volume 2012, Article ID 749084, 10 pages, doi:10.1155/2012/749084.
[18] Rabha W. Ibrahim, Ulam-Hyers Stability for Cauchy Fractional Differential Equation in the Unit Disk, *Abstr. Appl. Anal.*, Volume 2012, Article ID 613270, 10 pages, doi:10.1155/2012/613270.
[19] S. M. Jung, Hyers-Ulam stability of linear differential equations of first order, *Appl. Math. Lett.*, **17**(2004) 1135-1140.
[20] A. A. Kilbas, H. M. Srivasta, J. J. Trujillo, Theory and application of fractional differential equations, Elsevier B. V, Netherlands, 2016.
[21] E.R. Love, Fractional derivatives of imaginary order, *J. London Math. Soc.*, **2(2-3)** (1971) 241-259 .
[22] P. Muniyappan, S. Rajan, Hyers-Ulam-Rassias stability of fractional differential equation, *Int. J. Pure Appl. Math.*, **102** (2015)631-642.
[23] R.L. Magin, Fractional Calculus in Bioengineering, Begell House, 2006.
[24] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[25] S. Manimaran, T. Gunasekar, G. V. Subramaniyan, F. Paul Samuel, existence of solutions for neutral functional integrodifferential evolution equations with non local conditions, *Indian J. Sci. Technol.*, **8(4)** (2015) 358-363.

[26] Moustafa El-Shahed, Positive solutions for boundary value problem of nonlinear fractional differential equation, *Abstr. Appl. Anal.*, 2007 (2007) Article ID 10368, 8 pages.

[27] A. Neamaty, M. Yadollahzadeh, R. Darzi, On fractional differential equation with complex order, *Progress in fractional differential equations and Apllications*, **1(3)** (2015) 223-227.

[28] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[29] Rabha W. Ibrahim, Ulam stability of boundary value problem, *Kragujevac Journal of Mathematics*, **37(2)** (2013) 287-297.

[30] Ravi P Agarwa, Sotiris K Ntouyas, Bashir Ahmad, Mohammed S Alhothuali, Existence of solutions for integro-differential equations of fractional order with nonlocal three-point fractional boundary conditions, *Adv. Differential Equations*, 128 (2013) 1-9.

[31] I. A. Rus, Ualm stabilities of ordinary differential equations in a Banach space, *Carpathian Journal Mathematics*, **26** (2010) 103-107.

[32] S. G. Samko, A.A. Kilbas O. I. Marichev, Fractional Integrals and Derivatives-Theory and Applications, Gordon and Breach Science Publishers, Amsterdam , 1993.

[33] Teodor M. Atanackovi, Sanja Konjik, Stevan Pilipovic, Dusan Zorica, Complex order fractional derivatives in viscoelasticity, *Mech Time-Depend Mater*, **1** (2016) 1-21.

[34] D.Vivek, K. Kanagarajan, S. Harikrishnan, Dynamics and stability results for integro-differential equations with complex order, *Discontinuity, Nonlinearity and Complexity*,(2007),(Accepted manuscript, ID:DNC-D-2017-0007).

[35] D.Vivek, K. Kanagarajan, S. Harikrishnan, Dynamics and stability results for pantograph equations with complex order, *Journal of Applied Nonlinear Dynamics*,(2007),(Accepted manuscript, ID:JAND-D-2017-0011).

[36] J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative,*Electron J. Qual. Theory Differ. Equ.*,**63**(2011) 1-10.

[37] J. Wang, Y. Zhou, New concepts and results in stability of fractional differential equations,*Commun. Nonlinear Sci. Numer. Simul.*,**17** (2012) 2530-2538.

[38] Young Chel ahn, Hae Won KIM, Dong Gun Park, the existence and controllability of solutions for the neutral functional integro-differential equations with delay terms, *J. Appl. Math. And Informatics*, **27** (2009), No. 5 - 6, 1145 - 1156.

[39] S. Zhang, Existence of solution for a boundary value problem of fractional order, *Acta Math. Sci. Series B. English Edition* , **26 (2)** (2006) 220-228.