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On the Reciprocal Sums of Products of Fibonacci and Lucas Numbers

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Abstract. In this paper, we study the reciprocal sums of products of Fibonacci and Lucas numbers. Some identities are obtained related to the numbers $\sum_{k=n}^{\infty} 1/F_kL_{k+m}$ and $\sum_{k=n}^{\infty} 1/L_kF_{k+m}$, $m \geq 0$.

1. Introduction

As is well known, the Fibonacci numbers $(F_n)_{n=0}^{\infty}$ and the Lucas numbers $(L_n)_{n=0}^{\infty}$ are respectively generated from the recurrence relations $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0$, $F_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$ with $L_0 = 2$, $L_1 = 1$. Over the decades, much attention has been given to the properties of these two classical numbers, and numerous results have been reported in the literature [7].

Recently, Ohtsuka and Nakamura [8] found new properties of the Fibonacci numbers and proved Theorem 1.1 below, where $\lfloor \cdot \rfloor$ indicates the floor function and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

Theorem 1.1. *For the Fibonacci numbers, the following identities hold:*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{2}$$

Following the work of Ohtsuka and Nakamura [8], diverse results in the same direction have appeared in the literature [1], [3–5], [9–12]. In particular, according to Holliday and Komatsu [5], the infinite sums of reciprocal Lucas numbers satisfy the identities given in Theorem 1.2 below.

Theorem 1.2. *For the Lucas numbers, the following identities hold:*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k} \right)^{-1} \right\rfloor = \begin{cases} L_{n-2} - 1, & \text{if } n \geq 4 \text{ and } n \in \mathbb{N}_e; \\ L_{n-2}, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{3}$$

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$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k^2} \right)^{-1} \right\rfloor = \begin{cases} L_{n-1}L_n + 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ L_{n-1}L_n - 2, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (4)$$

In this paper we study the reciprocal sums of products of Fibonacci and Lucas numbers. Some identities are obtained related to the following numbers:

$$\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}}, \quad \sum_{k=n}^{\infty} \frac{1}{L_k F_{k+m}}, \quad m = 0, 1, 2, \cdots.$$

2. Main results

2.1. Reciprocal sums of $F_k L_{k+m}$

Firstly we consider the reciprocal sums of $F_k L_{k+m}$. Lemma 2.1 below will play a key role in obtaining the results.

Lemma 2.1. *For $m \geq 0$, we have*

(a) $F_n L_{n+m+1} - F_{n+m+1} L_n = 2(-1)^{n+1} F_{m+1}$.
(b) $F_{n+1} L_{n+m} - F_{n+m+1} L_n = (-1)^{n+1} F_m$.
(c) $F_{n+m+1} L_n - F_{n+m-1} L_{n+2} = (-1)^n (3F_m - F_{m+1})$.
(d) $F_{n+m+1} L_{n-1} - F_{n+m} L_n = (-1)^n (F_{m+1} - 2F_{m+2})$.
(e) $F_{n+m-1} L_n - F_{n+m-2} L_{n+1} = (-1)^n (3F_{m+1} - 4F_m)$.
(f) $F_{n+m} L_n - F_n L_{n+m} = 2(-1)^n F_m$.
(g) $F_{n+m} L_{n+1} - F_n L_{n+m+1} = (-1)^n F_m$.
(h) $F_{n+m+1} L_{n+2} - F_{n+m-1} L_n = F_n L_{n+m} + F_{n+1} L_{n+m+1}$.

Proof. (a)-(g) are special cases of [2, Theorem 2.1]. From (f), we have

$$F_{n+m} L_n + F_{n+m+1} L_{n+1} = F_n L_{n+m} + F_{n+1} L_{n+m+1}.$$

On the other hand

$$F_{n+m+1}(L_{n+2} - L_{n+1}) = (F_{n+m} + F_{n+m-1}) L_n,$$

or

$$F_{n+m+1} L_{n+2} - F_{n+m-1} L_n = F_{n+m} L_n + F_{n+m+1} L_{n+1},$$

and (h) also holds. □

Theorem 2.2. *For $m \geq 0$, (a) and (b) below hold:*

(a) If

$$\frac{2F_m - 3F_{m+1}}{3} \notin \mathbb{Z},$$

then there exist positive integers n_0 and n_1 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} F_{n+m-1} L_n + g_m, & \text{if } n \geq n_0 \text{ and } n \in \mathbb{N}_e; \\ F_{n+m-1} L_n - g_m - 1, & \text{if } n \geq n_1 \text{ and } n \in \mathbb{N}_o, \end{cases} \quad (5)$$

where

$$g_m = \left\lfloor \frac{2F_m - 3F_{m+1}}{3} \right\rfloor.$$

(b) If

$$\frac{2F_m - 3F_{m+1}}{3} \in \mathbb{Z},$$

then there exist positive integers n_2 and n_3 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} F_{n+m-1}L_n + \hat{g}_m - 1, & \text{if } n \geq n_2 \text{ and } n \in \mathbb{N}_e; \\ F_{n+m-1}L_n - \hat{g}_m - 1, & \text{if } n \geq n_3 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{6}$$

where

$$\hat{g}_m = \frac{2F_m - 3F_{m+1}}{3}.$$

Proof. (a) Firstly, consider

$$\begin{aligned} X_1 &= \frac{1}{F_{n+m-1}L_n + (-1)^n g_m} - \frac{1}{F_{n+m+1}L_{n+2} + (-1)^n g_m} - \frac{1}{F_n L_{n+m}} - \frac{1}{F_{n+1}L_{n+m+1}} \\ &= \frac{\hat{X}_1}{\{F_{n+m-1}L_n + (-1)^n g_m\}\{F_{n+m+1}L_{n+2} + (-1)^n g_m\}F_n L_{n+m} F_{n+1} L_{n+m+1}}, \end{aligned}$$

where, by Lemma 2.1(h)

$$\hat{X}_1 = (F_n L_{n+m} + F_{n+1}L_{n+m+1})\tilde{X}_1,$$

with

$$\tilde{X}_1 = F_n F_{n+1} L_{n+m} L_{n+m+1} - F_{n+m-1} F_{n+m+1} L_n L_{n+2} - (-1)^n g_m (F_{n+m-1}L_n + F_{n+m+1}L_{n+2}) - g_m^2.$$

By Lemma 2.1(a)-(d), we have

$$\begin{aligned} & F_n F_{n+1} L_{n+m} L_{n+m+1} - F_{n+m-1} F_{n+m+1} L_n L_{n+2} \\ = & \left\{F_{n+m+1}L_n + 2(-1)^{n+1}F_{m+1}\right\}\left\{F_{n+m+1}L_n + (-1)^{n+1}F_m\right\} - F_{n+m+1}L_n\left\{F_{n+m+1}L_n + (-1)^n(F_{m+1} - 3F_m)\right\} \\ = & (-1)^{n+1}F_{n+m+1}L_n(3F_{m+1} - F_m) + 2F_{m+1}F_m, \end{aligned}$$

and

$$\begin{aligned} F_{n+m-1}L_n + F_{n+m+1}L_{n+2} &= 3F_{n+m+1}L_n + F_{n+m+1}L_{n-1} - F_{n+m}L_n \\ &= 3F_{n+m+1}L_n + (-1)^n(F_{m+1} - 2F_{m+2}). \end{aligned}$$

Then

$$\tilde{X}_1 = (-1)^n F_{n+m+1}L_n(2F_m - 3F_{m+1} - 3g_m) + 2F_{m+1}F_m - g_m(F_{m+1} - 2F_{m+2}) - g_m^2.$$

Assume that $n \in \mathbb{N}_e$. Since $2F_m - 3F_{m+1} - 3g_m > 0$, then there exists a positive integer l_0 such that, for $n \geq l_0$, $X_1 > 0$ and

$$\frac{1}{F_n L_{n+m}} + \frac{1}{F_{n+1}L_{n+m+1}} < \frac{1}{F_{n+m-1}L_n + g_m} - \frac{1}{F_{n+m+1}L_{n+2} + g_m}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} < \frac{1}{F_{n+m-1}L_n + g_m}, \text{ if } n \geq l_0 \text{ and } n \in \mathbb{N}_e. \tag{7}$$

Similarly there exists a positive integer l_1 such that

$$\frac{1}{F_{n+m-1}L_n - g_m} < \sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}}, \text{ if } n \geq l_1 \text{ and } n \in \mathbb{N}_o. \tag{8}$$

Next, consider

$$\begin{aligned} X_2 &= \frac{1}{F_{n+m-1}L_n + (-1)^n g_m - 1} - \frac{1}{F_{n+m}L_{n+1} + (-1)^{n+1} g_m - 1} - \frac{1}{F_n L_{n+m}} \\ &= \frac{\hat{X}_2}{\{F_{n+m-1}L_n + (-1)^n g_m - 1\}\{F_{n+m}L_{n+1} + (-1)^{n+1} g_m - 1\} F_n L_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} \hat{X}_2 &= F_n F_{n+m} L_{n+1} L_{n+m} - F_{n+m} F_{n+m-1} L_n L_{n+1} - F_n F_{n+m-1} L_n L_{n+m} \\ &\quad -(-1)^n g_m (2F_n L_{n+m} - F_{n+m-1} L_n + F_{n+m} L_{n+1}) + F_{n+m-1} L_n + F_{n+m} L_{n+1} + g_m^2 - 1. \end{aligned}$$

Using Lemma 2.1(e),(f),(g), we have

$$\begin{aligned} &F_n F_{n+m} L_{n+1} L_{n+m} - F_{n+m} F_{n+m-1} L_n L_{n+1} - F_n F_{n+m-1} L_n L_{n+m} \\ = \;& F_n F_{n+m} L_{n+1} L_{n+m} - F_{n+m-1} L_{n+1} \Big\{ F_n L_{n+m} + 2(-1)^n F_m \Big\} - F_n F_{n+m-1} L_n L_{n+m} \\ = \;& F_n L_{n+m} (F_{n+m-2} L_{n+1} - F_{n+m-1} L_n) - 2(-1)^n F_{n+m-1} L_{n+1} F_m \\ = \;& (-1)^n F_n L_{n+m} (4F_m - 3F_{m+1}) - 2(-1)^n \Big\{ F_n L_{n+m} + (-1)^n F_{m-1} \Big\} F_m \\ = \;& (-1)^n F_n L_{n+m} (2F_m - 3F_{m+1}) - 2F_{m-1} F_m, \end{aligned}$$

and

$$\begin{aligned} &2F_n L_{n+m} + F_{n+m} L_{n+1} - F_{n+m-1} L_n \\ = \;& 2F_n L_{n+m} + F_{n+m} L_n + F_{n+m} L_{n-1} - F_{n+m-1} L_n \\ = \;& 2F_n L_{n+m} + \Big\{ F_n L_{n+m} + 2(-1)^n F_m \Big\} - (-1)^n (3F_{m+3} - 4F_{m+2}) \\ = \;& 3F_n L_{n+m} + 2(-1)^n F_m - (-1)^n (2F_{m+3} - 4F_{m+2}). \end{aligned}$$

Hence

$$\begin{aligned} \hat{X}_2 &= (-1)^n F_n L_{n+m} (2F_m - 3F_{m+1} - 3g_m) + F_{n+m-1} L_n + F_{n+m} L_{n+1} - 2F_{m-1} F_m - 2g_m F_m \\ &\quad + g_m (3F_{m+3} - 4F_{m+2}) + g_m^2 - 1. \end{aligned}$$

Suppose that $n \in \mathbb{N}_o$. From Lemma 2.1(f), we obtain

$$F_{n+m-1} L_n + F_{n+m} L_{n+1} = F_n L_{n+m-1} + F_{n+1} L_{n+m}.$$

Since

$$1 \leq 2F_m - 3F_{m+1} - 3g_m \leq 2,$$

then

$$\begin{aligned} &-F_n L_{n+m} (2F_m - 3F_{m+1} - 3g_m) + (F_{n+m-1} L_n + F_{n+m} L_{n+1}) \\ \geq \;& -2F_n L_{n+m} + F_n L_{n+m-1} + F_{n+1} L_{n+m} \\ = \;& -F_n L_{n+m} + F_n L_{n+m-1} + F_{n-1} L_{n+m} \\ = \;& (F_{n-1} - F_n)(L_{n+m-1} + L_{n-m-2}) + F_n L_{n+m-1} \\ = \;& F_{n-1} L_{n+m-1} - F_{n-2} L_{n+m-2}. \end{aligned}$$

Hence there exists a positive integer l_2 such that, for $n \geq l_2$, $X_2 > 0$ and

$$\frac{1}{F_n L_{n+m}} < \frac{1}{F_{n+m-1}L_n + (-1)^n g_m - 1} - \frac{1}{F_{n+m}L_{n+1} + (-1)^{n+1} g_m - 1}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} < \frac{1}{F_{n+m-1}L_n - g_m - 1}, \text{ if } n \geq l_2 \text{ and } n \in \mathbb{N}_o. \tag{9}$$

Finally, consider

$$\begin{aligned} X_3 &= \frac{1}{F_{n+m-1}L_n + (-1)^n g_m + 1} - \frac{1}{F_{n+m}L_{n+1} + (-1)^{n+1} g_m + 1} - \frac{1}{F_n L_{n+m}} \\ &= \frac{\hat{X}_3}{\{F_{n+m-1}L_n + (-1)^n g_m + 1\}\{F_{n+m-1}L_{n+1} + (-1)^{n+1} g_m + 1\} F_n L_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} \hat{X}_3 &= \hat{X}_2 - 2(F_{n+m-1}L_n + F_{n+m}L_{n+1}) \\ &= (-1)^n F_n L_{n+m}(2F_m - 3F_{m+1} - 3g_m) - F_{n+m-1}L_n - F_{n+m}L_{n+1} - 2F_{m-1}F_m - 2g_m F_m \\ &\quad + g_m(3F_{m+3} - 4F_{m+2}) + g_m^2 - 1. \end{aligned}$$

Suppose that $n \in \mathbb{N}_e$. Since

$$F_n L_{n+m}(2F_m - 3F_{m+1} - 3g_m) - F_{n+m-1}L_n - F_{n+m}L_{n+1} \leq F_{n-2}L_{n+m-2} - F_{n-1}L_{n+m-1},$$

then there exists a positive integer l_3 such that, for $n \geq l_3$, $X_3 < 0$ and

$$\frac{1}{F_{n+m-1}L_n + (-1)^n g_m + 1} - \frac{1}{F_{n+m}L_{n+1} + (-1)^{n+1} g_m + 1} < \frac{1}{F_n L_{n+m}},$$

from which we have

$$\frac{1}{F_{n+m-1}L_n + g_m + 1} < \sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}}, \text{ if } n \geq l_3 \text{ and } n \in \mathbb{N}_e. \tag{10}$$

Then, (5) follows from (7), (8), (9) and (10).

(b) Suppose that

$$\frac{2F_m - 3F_{m+1}}{3} \in \mathbb{Z}.$$

We recall the proof of (a). Replacing g_m by $\hat{g}_m$, we have

$$\tilde{X}_1 = 2F_{m+1}F_m - \hat{g}_m(F_{m+1} - 2F_{m+2}) - \hat{g}_m^2.$$

We can show that $\tilde{X}_1 < 0$ whenever $2F_m - 3F_{m+1} \equiv 0 \pmod 3$. Then there exist positive integers l_4 and l_5 such that $X_1 > 0$ if $n \geq l_4$ and $n \in \mathbb{N}_e$ or if and $n \geq l_5$ and $n \in \mathbb{N}_o$. Hence

$$\frac{1}{F_{n+m-1}L_n + (-1)^n \hat{g}_m} < \sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}}, \text{ if } n \geq l_4 \ (n \in \mathbb{N}_e) \text{ or if } n \geq l_5 \ (n \in \mathbb{N}_o). \tag{11}$$

Similarly there exist positive integers l_6 and l_7 such that $X_2 > 0$ if $n \geq l_6$ and $n \in \mathbb{N}_e$, or if $n \geq l_7$ and $n \in \mathbb{N}_o$, from which we have

$$\sum_{k=n}^{\infty} \frac{1}{F_k L_{k+m}} < \frac{1}{F_{n+m-1}L_n + (-1)^n \hat{g}_m - 1}, \text{ if } n \geq l_6 \ (n \in \mathbb{N}_e) \text{ or if } n \geq l_7 \ (n \in \mathbb{N}_o). \tag{12}$$

Then, (6) follows from (11) and (12). □

Remark 2.3. *From Theorem 2.2, we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{F_kL_k}\right)^{-1}\right\rfloor = \begin{cases} F_{n-1}L_n-2, & \text{if } n\ge 2 \text{ and } n\in\mathbb{N}_e;\\ F_{n-1}L_n, & \text{if } n\ge 3 \text{ and } n\in\mathbb{N}_o,\end{cases}$$

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{F_kL_{k+1}}\right)^{-1}\right\rfloor = \begin{cases} F_nL_n-1, & \text{if } n\ge 2 \text{ and } n\in\mathbb{N}_e;\\ F_nL_n, & \text{if } n\ge 1 \text{ and } n\in\mathbb{N}_o,\end{cases}$$

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{F_kL_{k+4}}\right)^{-1}\right\rfloor = \begin{cases} F_{n+3}L_n-4, & \text{if } n\ge 2 \text{ and } n\in\mathbb{N}_e;\\ F_{n+3}L_n+2, & \text{if } n\ge 1 \text{ and } n\in\mathbb{N}_o,\end{cases}$$

etc.

2.2. *Reciprocal sums of L_kF_{k+m}*

Next we consider the reciprocal sums of L_kF_{k+m}. To this end, we need Lemma 2.4 below.

Lemma 2.4. *For $m \ge 0$, we have*

(a) $L_nF_{n+m+1} - L_{n+m+1}F_n = 2(-1)^nF_{m+1}$.
(b) $L_{n+1}F_{n+m} - L_{n+m+1}F_n = (-1)^nF_m$.
(c) $L_{n+m+1}F_n - L_{n+m-1}F_{n+2} = (-1)^n(L_m - L_{m+1})$.
(d) $L_{n+m+1}F_{n-1} - L_{n+m}F_n = (-1)^nL_{m+1}$.
(e) $L_{n+m-1}F_n - L_{n+m-2}F_{n+1} = (-1)^n(L_{m+1} - 2L_m)$.
(f) $L_{n+m}F_n - L_nF_{n+m} = 2(-1)^{n+1}F_m$.
(g) $L_{n+m+1}F_{n+2} - L_{n+m-1}F_n = L_nF_{n+m} + L_{n+1}F_{n+m+1}$.

Proof. (a)-(f) are also special cases of [2, Theorem 2.1]. From (f), we have

$$L_{n+m}F_n + L_{n+m+1}F_{n+1} = L_nF_{n+m} + L_{n+1}F_{n+m+1}.$$

On the other hand

$$L_{n+m+1}(F_{n+2} - F_{n+1}) = (L_{n+m} + L_{n+m-1})F_n,$$

or

$$L_{n+m+1}F_{n+2} - L_{n+m-1}F_n = L_{n+m}F_n + L_{n+m+1}F_{n+1},$$

and (g) also holds. □

Theorem 2.5 below can be proved similarly to Theorem 2.2.

Theorem 2.5. *For $m \ge 0$, (a) and (b) below hold:*
(a) If

$$\frac{2F_m + 2L_m - L_{m+1}}{3} \notin \mathbb{Z},$$

then there exist positive integers n_4 and n_5 such that

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{L_kF_{k+m}}\right)^{-1}\right\rfloor = \begin{cases} L_{n+m-1}F_n + h_m, & \text{if } n\ge n_4 \text{ and } n\in\mathbb{N}_e;\\ L_{n+m-1}F_n - h_m - 1, & \text{if } n\ge n_5 \text{ and } n\in\mathbb{N}_o,\end{cases} \tag{13}$$

where

$$h_m = \left\lfloor\frac{2F_m + 2L_m - L_{m+1}}{3}\right\rfloor + 1.$$

(b) If

$$\frac{2F_m + 2L_m - L_{m+1}}{3} \in \mathbb{Z},$$

then there exist positive integers n_6 and n_7 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k F_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} L_{n+m-1}F_n + \hat{h}_m - 1, & \text{if } n \geq n_6 \text{ and } n \in \mathbb{N}_e; \\ L_{n+m-1}F_n - \hat{h}_m - 1, & \text{if } n \geq n_7 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{14}$$

where

$$\hat{h}_m = \frac{2F_m + 2L_m - L_{m+1}}{3}.$$

Proof. (a) Consider

$$\begin{aligned} Y_1 &= \frac{1}{L_{n+m-1}F_n + (-1)^n h_m} - \frac{1}{L_{n+m+1}F_{n+2} + (-1)^n h_m} - \frac{1}{L_n F_{n+m}} - \frac{1}{L_{n+1}F_{n+m+1}} \\ &= \frac{\hat{Y}_1}{\{L_{n+m-1}F_n + (-1)^n h_m\}\{L_{n+m+1}F_{n+2} + (-1)^n h_m\} L_n F_{n+m} L_{n+1} F_{n+m+1}}, \end{aligned}$$

where, by Lemma 2.4(g)

$$\hat{Y}_1 = (L_n F_{n+m} + L_{n+1}F_{n+m+1})\tilde{Y}_1,$$

with

$$\tilde{Y}_1 = L_n L_{n+1} F_{n+m} F_{n+m+1} - L_{n+m-1} L_{n+m+1} F_n F_{n+2} - (-1)^n h_m (L_{n+m-1}F_n + L_{n+m+1}F_{n+2}) - h_m^2.$$

By Lemma 2.4(a)-(d) and the identity $F_m + L_m = 2F_{m+1}$ [7], we have

$$\begin{aligned} & L_n L_{n+1} F_{n+m} F_{n+m+1} - L_{n+m-1} L_{n+m+1} F_n F_{n+2} \\ =\; & \left\{ L_{n+m+1}F_n + 2(-1)^n F_{m+1} \right\} \left\{ L_{n+m+1}F_n + (-1)^n F_m \right\} - L_{n+m+1}F_n \left\{ L_{n+m+1}F_n + (-1)^n (L_{m+1} - L_m) \right\} \\ =\; & (-1)^n L_{n+m+1}F_n (2F_{m+1} + F_m + L_m - L_{m+1}) + 2F_{m+1}F_m \\ =\; & (-1)^n L_{n+m+1}F_n (2F_m + 2L_m - L_{m+1}) + 2F_{m+1}F_m, \end{aligned}$$

and

$$\begin{aligned} L_{n+m-1}F_n + L_{n+m+1}F_{n+2} &= 3L_{n+m+1}F_n + L_{n+m+1}F_{n-1} - L_{n+m}F_n \\ &= 3L_{n+m+1}F_n + (-1)^n L_{m+1}. \end{aligned}$$

Then

$$\tilde{Y}_1 = (-1)^n L_{n+m+1}F_n (2F_m + 2L_m - L_{m+1} - 3h_m) + 2F_{m+1}F_m - h_m L_{m+1} - h_m^2.$$

Assume that $n \in \mathbb{N}_e$. Since $2F_m + 2L_m - L_{m+1} - 3h_m < 0$, then there exists a positive integer m_0 such that, for $n \geq m_0$, $Y_1 < 0$ and

$$\frac{1}{L_{n+m-1}F_n + h_m} - \frac{1}{L_{n+m+1}F_{n+2} + h_m} < \frac{1}{L_n F_{n+m}} + \frac{1}{L_{n+1}F_{n+m+1}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{L_{n+m-1}F_n + h_m} < \sum_{k=n}^{\infty} \frac{1}{L_k F_{k+m}}, \text{ if } n \geq m_0 \text{ and } n \in \mathbb{N}_e. \tag{15}$$

Similarly we can find a positive integer m_1 such that

$$\sum_{k=n}^{\infty} \frac{1}{L_k F_{k+m}} < \frac{1}{L_{n+m-1}F_n - h_m}, \text{ if } n \geq m_1 \text{ and } n \in \mathbb{N}_o. \tag{16}$$

Next, consider

$$\begin{aligned} Y_2 &= \frac{1}{L_{n+m-1}F_n + (-1)^n h_m - 1} - \frac{1}{L_{n+m}F_{n+1} + (-1)^{n+1} h_m - 1} - \frac{1}{L_n F_{n+m}} \\ &= \frac{\hat{Y}_2}{\{L_{n+m-1}F_n + (-1)^n h_m - 1\}\{L_{n+m}F_{n+1} + (-1)^{n+1} h_m - 1\} L_n F_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} \hat{Y}_2 &= L_n L_{n+m} F_{n+1} F_{n+m} - L_{n+m} L_{n+m-1} F_n F_{n+1} - L_n L_{n+m-1} F_n F_{n+m} \\ &\quad -(-1)^n h_m (2L_n F_{n+m} - L_{n+m-1} F_n + L_{n+m} F_{n+1}) + L_{n+m-1} F_n + L_{n+m} F_{n+1} + h_m^2 - 1. \end{aligned}$$

Using Lemma 2.1(b) and Lemma 2.4(e),(f), we have

$$\begin{aligned} & L_n L_{n+m} F_{n+1} F_{n+m} - L_{n+m} L_{n+m-1} F_n F_{n+1} - L_n L_{n+m-1} F_n F_{n+m} \\ =\ & L_n L_{n+m} F_{n+1} F_{n+m} - L_{n+m-1} F_{n+1} \left\{ L_n F_{n+m} + 2(-1)^{n+1} F_m \right\} - L_n L_{n+m-1} F_n F_{n+m} \\ =\ & L_n F_{n+m} (L_{n+m-2} F_{n+1} - L_{n+m-1} F_n) - 2(-1)^{n+1} L_{n+m-1} F_{n+1} F_m \\ =\ & (-1)^n L_n F_{n+m} (2L_m - L_{m+1}) - 2(-1)^{n+1} \left\{ L_n F_{n+m} + (-1)^{n+1} F_{m-1} \right\} F_m \\ =\ & (-1)^n L_n F_{n+m} (2F_m + 2L_m - L_{m+1}) - 2F_{m-1} F_m, \end{aligned}$$

and

$$\begin{aligned} & 2L_n F_{n+m} - L_{n+m-1} F_n + L_{n+m} F_{n+1} \\ =\ & 2L_n F_{n+m} + L_{n+m} F_n + L_{n+m} F_{n-1} - L_{n+m-1} F_n \\ =\ & 2L_n F_{n+m} + \left\{ L_n F_{n+m} + 2(-1)^{n+1} F_m \right\} - (-1)^n (L_{m+3} - 2L_{m+2}) \\ =\ & 3L_n F_{n+m} - 2(-1)^n F_m - (-1)^n (L_{m+3} - 2L_{m+2}). \end{aligned}$$

Hence

$$\begin{aligned} \hat{Y}_2 &= (-1)^n L_n F_{n+m} (2F_m + 2L_m - L_{m+1} - 3h_m) + L_{n+m-1} F_n + L_{n+m} F_{n+1} - 2F_{m-1} F_m + 2h_m F_m \\ &\quad + h_m (L_{m+3} - 2L_{m+2}) + h_m^2 - 1. \end{aligned}$$

Suppose that $n \in \mathbb{N}_e$. From Lemma 2.4(f), we obtain

$$L_{n+m-1} F_n + L_{n+m} F_{n+1} = L_n F_{n+m-1} + L_{n+1} F_{n+m}.$$

Since

$$-2 \leq 2F_m + 2L_m - L_{m+1} - 3h_m \leq -1,$$

and

$$\begin{aligned} & L_n F_{n+m} (2F_m + 2L_m - L_{m+1} - 3h_m) + (L_{n+m-1} F_n + L_{n+m} F_{n+1}) \\ \geq\ & -2L_n F_{n+m} + L_n F_{n+m-1} + L_{n+1} F_{n+m} \\ =\ & -L_n F_{n+m} + L_n F_{n+m-1} + L_{n-1} F_{n+m} \\ =\ & (L_{n-1} - L_n)(F_{n+m-1} + F_{n-m-2}) + L_n F_{n+m-1} \\ =\ & L_{n-1} F_{n+m-1} - L_{n-2} F_{n+m-2}, \end{aligned}$$

then there exists a positive integer m_2 such that, for $n \geq m_2$, $Y_2 > 0$ and

$$\frac{1}{L_nF_{n+m}} < \frac{1}{L_{n+m-1}F_n + (-1)^n h_m - 1} - \frac{1}{L_{n+m}F_{n+1} + (-1)^{n+1}h_m - 1},$$

from which we obtain

$$\sum_{k=n}^{\infty} \frac{1}{L_kF_{k+m}} < \frac{1}{L_{n+m-1}F_n + h_m - 1}, \text{ if } n \geq m_2 \text{ and } n \in \mathbb{N}_e. \tag{17}$$

Finally, consider

$$\begin{aligned} Y_3 &= \frac{1}{L_{n+m-1}F_n+(-1)^n h_m+1} - \frac{1}{L_{n+m}F_{n+1}+(-1)^{n+1}h_m+1} - \frac{1}{L_nF_{n+m}} \\ &= \frac{\hat{Y}_3}{\{L_{n+m-1}F_n+(-1)^n h_m+1\}\{L_{n+m-1}F_{n+1}+(-1)^{n+1}h_m+1\}L_nF_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} \hat{Y}_3 &= \hat{Y}_2 - 2L_{n+m-1}F_n - 2L_{n+m}F_{n+1}) \\ &= (-1)^n L_nF_{n+m}(2F_m + 2L_m - L_{m+1} - 3h_m) - L_{n+m-1}F_n - L_{n+m}F_{n+1} - 2F_{m-1}F_m + 2h_mF_m \\ &\quad + h_m(L_{m+3} - 2L_{m+2}) + h_m^2 - 1. \end{aligned}$$

Assume that $n \in \mathbb{N}_o$. Since

$$-L_nF_{n+m}(2F_m + 2L_m - L_{m+1} - 3h_m) - L_{n+m-1}F_n - L_{n+m}F_{n+1} \leq L_{n-1}F_{n+m-1} - L_{n-2}F_{n+m-2},$$

then there exists a positive integer m_3 such that, for $n \geq m_3$, $Y_3 < 0$ and

$$\frac{1}{L_{n+m-1}F_n + (-1)^n h_m + 1} - \frac{1}{L_{n+m}F_{n+1} + (-1)^{n+1}h_m + 1} < \frac{1}{F_nL_{n+m}},$$

from which we have

$$\frac{1}{L_{n+m-1}F_n - h_m + 1} < \sum_{k=n}^{\infty} \frac{1}{L_kF_{k+m}}, \text{ if } n \geq m_3 \text{ and } n \in \mathbb{N}_o. \tag{18}$$

Then, (13) follows from (15), (16), (17) and (18).

b) Suppose that

$$\frac{2F_m + 2L_m - L_{m+1}}{3} \in \mathbb{Z}.$$

Again we recall the proof of (a). Replacing h_m by $\hat{h}_m$, we have

$$\tilde{Y}_1 = 2F_{m+1}F_m - \hat{h}_mL_{m+1} - \hat{h}_m^2.$$

We can show that $\tilde{Y}_1 < 0$ whenever $2F_m + 2L_m - L_{m+1} \equiv 0 \pmod 3$. Then there exist positive integers m_4 and m_5 such that $Y_1 < 0$ if $n \geq m_4$ and $n \in \mathbb{N}_e$, or if $n \geq m_5$ and $n \in \mathbb{N}_o$. Hence

$$\frac{1}{L_{n+m-1}F_n+(-1)^n\hat{h}_m} < \sum_{k=n}^{\infty} \frac{1}{L_kF_{k+m}}, \text{ if } n \geq m_4 \ (n \in \mathbb{N}_e) \text{ or if } n \geq m_5 \ (n \in \mathbb{N}_o). \tag{19}$$

Similarly there exist positive integers m_6 and m_7 such that $Y_2 > 0$ if $n \geq m_6$ and $n \in \mathbb{N}_e$ or if $n \geq m_7$ and $n \in \mathbb{N}_o$, from which we have

$$\sum_{k=n}^{\infty} \frac{1}{L_kF_{k+m}} < \frac{1}{L_{n+m-1}F_n + (-1)^n\hat{h}_m - 1}, \text{ if } n \geq m_6 \ (n \in \mathbb{N}_e) \text{ or if } n \geq m_7 \ (n \in \mathbb{N}_o). \tag{20}$$

Then, (14) follows from (19) and (20). □

Remark 2.6. *From Theorem 2.5, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k F_k} \right)^{-1} \right\rfloor = \begin{cases} L_{n-1}F_n, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ L_{n-1}F_n - 2, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o, \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k F_{k+1}} \right)^{-1} \right\rfloor = \begin{cases} L_n F_n, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ L_n F_n - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k F_{k+4}} \right)^{-1} \right\rfloor = \begin{cases} L_{n+3}F_n + 2, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ L_{n+3}F_n - 4, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases}$$

etc.

References

[1] M. Basbük, Y. Yazlik, On the sum of reciprocal of generalized bi-periodic Fibonacci numbers, Miskolc Mathematical Notes 17 (2016) 35–41.
[2] Y. Choo, On the generalizations of Fibonacci identities, Results in Mathematics 71 (2017) 347–356.
[3] Y. Choo, On the reciprocal sums of generalized Fibonacci numbers, International Journal of Mathematical Analysis 10 (2016) 1365–1373.
[4] Y. Choo, On the finite sums of reciprocal Lucas numbers, International Journal of Mathematical Analysis 11 (2017) 519–529.
[5] S. Holliday, T. Komatsu, On the sum of reciprocal generalized Fibonacci numbers, Integers 11A (2011) Article 11.
[6] E. Kilic, T. Arican, More on the infinite sum of reciprocal Fibonacci, Pell and higher order recurrences, Applied Mathematics and Computation 219 (2013) 7783–7788.
[7] T. Koshy, Fibonacci and Lucas numbers with applications, Wiley, New York, 2001.
[8] H. Ohtsuka, S. Nakamura, On the sum of reciprocal Fibonacci numbers, Fibonacci Quarterly 46/47 (2008/2009) 153–159.
[9] A. Wang, P. Wen, On the partial finite sums of the reciprocals of the Fibonacci numbers, Journal of Inequalities and Applications (2015) Article 73.
[10] A. Wang, T. Yuan, Alternating sums of the reciprocal Fibonacci numbers, Journal of Integer Sequences 20 (2017) Article 17.1.4.
[11] A. Wang, F. Zhang, The reciprocal sums of even and odd terms in the Fibonacci sequence, Journal of Inequalities and Applications 2015 (2015) Article 376.
[12] A. Wang, F. Zhang, The reciprocal sums of the Fibonacci 3-subsequences, Advances in Difference Equations 2016 (2016) Article 27.