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Iterative Approximation of Solution of Split Variational Inclusion Problem

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Abstract. Following recent important results of Moudafi [Journal of Optimization Theory and Applications 150(2011), 275-283] and other related results on variational problems, we introduce a new iterative algorithm for approximating a solution of monotone variational inclusion problem involving multi-valued mapping. The sequence of the algorithm is proved to converge strongly in the setting of Hilbert spaces. As application, we solved split convex optimization problems.

1. Introduction

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. A mapping $S : C \to C$ is said to be

(i) *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\| \ \ \forall \ \ x, y \in C,$$

(ii) *μ-strictly pseudocontractive* in the sense of Browder and Petryshyn [12] if for $0 \leq \mu < 1$,

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \mu\|(I - S)x - (I - S)y\|^2 \ \ \forall \ \ x, y \in C.$$

A point $x \in C$ is called a *fixed point* of S if $Sx = x$. The set of fixed points of S is denoted by $F(S)$, and it is generally known that if $F(S) \neq \emptyset$, then $F(S)$ is closed and convex. For more information on strictly pseudocontrative mappings, see [1, 12, 32, 43] and references therein.
A mapping $M : H \to H$ is said to be
(i) *monotone*, if

$$\langle Mx - My, x - y\rangle \geq 0, \ \ \forall x, y \in H,$$

(ii) *α-inverse strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y\rangle \geq \alpha\|Mx - My\|^2, \ \ \forall x, y \in H,$$

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(iii) *firmly nonexpansive*, if

$$\langle Mx - My, x - y\rangle \geq \|Mx - My\|^2, \ \ \forall x, y \in H,$$

(iv) *Lipschitz*, if there exists a constant $L > 0$ such that

$$\|Mx - My\| \leq L\|x - y\|, \ \forall x, y \in H.$$

Remark 1.1. *[9] It is well known that M is α-inverse strongly monotone if and only if it is $\frac{1}{\alpha}$-Lipschitz continuous.*

If M is a multivalued mapping, i.e. $M : H \to 2^H$, then M is called monotone if

$$\langle x - y, u - v\rangle \geq 0 \ \forall x, y \in H, \ u \in M(x), \ v \in M(y),$$

and M is maximal monotone if the graph $G(M)$ of M defined by

$$G(M) =: \{(x, y) \in H \times H : y \in M(x)\}$$

is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$.
The resolvent operator J_λ^M associated with a mapping M and λ is the mapping $J_\lambda^M : H \to 2^H$ defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}x, \ \ x \in H, \lambda > 0. \tag{1}$$

It is known that if the mapping M is monotone, then J_λ^M is single valued and firmly nonexpansive (see [11]).
A mapping $f : C \to C$ is said to be averaged nonexpansive if $\forall x, y \in C$, $f = (1 - \beta)I + \beta S$ holds for a nonexpansive operator $S : C \to C$ and $\beta \in (0, 1)$. The term "averaged mapping" was coined by Biallon et al [8]. Recall that a mapping f is firmly nonexpansive if and only if f can be expressed as $f = \frac{1}{2}(I + S)$, where S is nonexpansive (see [34]). Thus, we make the following remark which can be easily verified.

Remark 1.2. *In a Hilbert space, f is firmly nonexpansive if and only if it is averaged with $\beta = \frac{1}{2}$.*

The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : \ y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x\rangle \leq 0, \ \ \forall z \in C$ and P_C is a firmly nonexpansive mapping. Thus, P_C is nonexpansive. For more information on metric projections, see [19, 24].
Recall that the normal cone of C at the point $z \in H$ is defined as

$$N_C z := \{d \in H : \langle d, y - z\rangle \leq 0, \ \ \forall \ y \in C\} \text{ if } z \in C \text{ and } \emptyset, \text{ otherwise.}$$

In 1994, Censor and Elfving [17] introduced the following Split Feasibility Problem (SFP): Find a point

$$x \in C \ \text{ such that } \ Ax \in Q, \tag{2}$$

where C and Q are nonempty closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and A is an $m \times n$ real matrix. The SFP is known to have wide applications in many fields such as phase retrieval, medical image reconstruction, signal processing, radiation therapy treatment planning, among others (for example, see [13, 16–18] and the references therein).
Byrne [14] applied the forward-backward method, a type of projected gradient method, thus, presenting the so-called CQ-iterative procedure for approximating a solution of (2), which he defined as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \ n \in \mathbb{N}, \tag{3}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A. Byrne [14] proved that the sequence generated by Algorithm 3 converges weakly to a solution of (2).

In 2010, Censor *et al.* [20] introduced a new class of problem called the Split Variational Inequality Problem (SVIP) by combining the Variational Inequality Problem (VIP) and the SFP. They defined the SVIP as follows: Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \ \forall x \in C, \tag{4}$$

and such that $y^* = Ax^* \in Q$ solves

$$\langle g(y^*), y - y^* \rangle \geq 0 \ \forall y \in Q, \tag{5}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given operators. If (4) and (5) are considered separately, we have that (4) is a VIP with its solution set $VIP(C, f)$ and (5) is a VIP with its solution set $VIP(Q, g)$. To solve the SVIP (4)-(5), Censor *et al.* proposed the following algorithm and obtained a weak convergence result. For $x_1 \in H_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I - \lambda f)(x_n + \gamma A^*(P_Q(I - \lambda g) - I)Ax_n), n \geq 1, \tag{6}$$

where $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A.
Based on the work of Censor *et al.* [20], Moudafi [34] recently introduced and studied a new type of split problem called Split Monotone Variational Inclusion Problem (SMVIP), which is to find

$$x^* \in H_1 \text{ such that } 0 \in f(x^*) + M_1(x^*), \tag{7}$$

and such that $y^* = Ax^* \in H_2$ solves

$$0 \in g(y^*) + M_2(y^*), \tag{8}$$

where $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ are multivalued mappings, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are single valued operators. We also note that if (7) and (8) are considered separately, we have that (7) is a Monotone Variational Inclusion Problem (MVIP) with its solution set $(M_1 + f)^{-1}(0)$ and (8) is a MVIP with its solution set $(M_2 + g)^{-1}(0)$. In [34], Moudafi proved that $x^* \in (M_1 + f)^{-1}(0)$ if and only if $x^* = J_\lambda^{M_1}(I - \lambda f)(x^*), \ \forall \lambda > 0$. It was also shown in [34] that, if f is an α-inverse strongly monotone mapping and M is a maximal monotone mapping, then $J_\lambda^{M_1}(I - \lambda f)$ is averaged with $0 < \lambda < 2\alpha$. Thus, $J_\lambda^{M_1}(I - \lambda f)$ is a nonexpansive mapping with $0 < \lambda < 2\alpha$. In addition, $(M_1 + f)^{-1}(0)$ is closed and convex.
To solve the SMVIP (7)-(8), Moudafi [34] proposed the following iterative algorithm and obtained weak convergence results: For $x_1 \in H_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_\lambda^{M_1}(I - \lambda f)(x_n + \gamma A^*(J_\lambda^{M_2}(I - \lambda g) - I)Ax_n), \ n \in \mathbb{N}, \tag{9}$$

where $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A.

Remark 1.3. *[34] As observed by Moudafi, setting $M_1 = N_C$ and $M_2 = N_Q$ in SMVIP* (7)-(8)*, where N_C and N_Q are the normal cones of C and Q respectively, we recover the SVIP* (4)-(5)*. Thus, the SMVIP can be viewed as an important generalization of the SVIP, SFP and other related problems (see also [33]).*

Moreover, MVIP is generally known to be very useful in the study of wide classes of problems. It has been an important tools for solving problems arising from mechanics, optimization, nonlinear programming, economics, finance, applied sciences, among others (see for example [2–4, 21, 33] and the references therein). Very recently, Tian and Jiang [39] proposed a class of SVIP which is to find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \ \forall x \in C, \text{ and such that } Ax^* \in F(S), \tag{10}$$

where C is a nonempty, closed and convex subset of H_1, $A : H_1 \to H_2$ is a bounded linear operator, $f : C \to H_1$ is a single valued operator and $S : H_2 \to H_2$ is a nonlinear mapping. To approximate solutions of (10), Tian and Jiang [39] proposed the following iterative algorithm by combining Algorithm (6) with the

Korpelevich's extra-gradient method (see [27]) and Byrne's CQ algorithm: For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_C(x_n - \gamma_n A^*(I-S)Ax_n), \\ t_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n f(t_n)), \end{cases} \tag{11}$$

for each $n \in \mathbb{N}$, where $\{\gamma_n\} \subset [a,b]$ for some $a,b \in (0, \frac{1}{\|A\|^2})$ and $\{\lambda_n\} \subset [a,b]$ for some $c,d \in (0, \frac{1}{k})$, $S : H_2 \to H_2$ is a nonexpansive mapping and $f : C \to H_1$ is a monotone and k-Lipschitz continuous mapping. They proved that the sequence generated by Algorithm (11) converges weakly to a solution of (10). Furthermore, Tian and Jian [39] showed that Algorithm (11) can be used to solve the SVIP of Censor *et al.* [20] by setting $S = P_Q(I - \lambda g)$ in Algorithm (11), since $P_Q(I-\lambda g)$ is a nonexpansive mapping for $\lambda \in (0, 2\alpha)$. For more results on VIPs and MVIPs, see [5–7, 15, 19, 23, 26, 29, 30, 35] and the references therein.
Motivated by the works of Moudafi [34], Tian and Jiang [39], and in view of Remark 1.3, we propose an extension of the class of SVIP studied by Tian and Jiang [39] to the following class of SMVIP: Find

$$x^* \in H_1 \text{ such that } 0 \in f(x^*) + M(x^*), \text{ and such that } Ax^* \in F(S), \tag{12}$$

where $M : H_1 \to 2^{H_1}$ is a multivalued mapping, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ is a single valued operator and $S : H_2 \to H_2$ is a nonlinear mapping. Furthermore, we propose an iterative algorithm and using the algorithm, we state and prove some strong convergence results for the approximation of solutions of (12) and (7)-(8). Finally, we applied our results to study split convex minimization problems. Our results extend and improve the results of Censor *et al.* [20], Moudafi [34], Tian and Jiang [39], and a host of other important results.

2. Preliminaries

We state some useful results which will be needed in the proof of our main theorem.

Lemma 2.1. *[22] Let H be a Hilbert space, then for all $x, y \in H$ and $\alpha \in (0,1)$, the following hold:*

(i) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2$,
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

Lemma 2.2. *[40] Let H be a Hilbert space and $T : H \to H$ be a nonlinear mapping, then the following hold.*

(i) f is nonexpansive if and only if the complement $I - f$ is $\frac{1}{2}$-ism.

(ii) f is ν-ism and $\gamma > 0$, then γf is $\frac{\nu}{\gamma}$-ism.

(iii) f is averaged if and only if the complement $I - f$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\beta \in (0,1)$, f is β-averaged if and only if $I - f$ is $\frac{1}{2\beta}$-ism.

(iv) If f_1 is β_1-averaged and f_2 is β_2-averaged, where $\beta_1, \beta_2 \in (0,1)$, then the composite $f_1 f_2$ is β-averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.

(v) If f_1 and f_2 are averaged and have a common fixed point, then $F(f_1 f_2) = F(f_1) \cap F(f_2)$.

Lemma 2.3. *[37] Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq 0$, and $S : H_2 \to H_2$ be a nonexpansive mapping. Then $A^*(I-S)A$ is $\frac{1}{2\|A\|^2}$-ism.*

Lemma 2.4. *[39] Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $S : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $\gamma > 0$ and $x^* \in H_1$. Then the following are equivalent.*

(i) $x^* = P_C(I - \gamma A^*(I - S)A)x^*$;

(ii) $0 \in A^*(I - S)Ax^* + N_C x^*$;

(iii) $x^* \in C \cap A^{-1}F(S)$.

Lemma 2.5. *[41] Let H be a real Hilbert space and* $S : H \to H$ *be a nonexpansive mapping with* $F(S) \neq \emptyset$. *If* $\{x_n\}$ *is a sequence in H converging weakly to* x^* *and if* $\{(I - S)x_n\}$ *converges strongly to* y, *then* $(I - S)x^* = y$.

Lemma 2.6. *[28] Let* $M : H \to 2^H$ *be a maximal monotone mapping and* $f : H \to H$ *be a Lipschitz continuous mapping. Then, the mapping* $(M + f) : H \to 2^H$ *is maximal monotone.*

Lemma 2.7. *[42] Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. *[43] Let H be a real Hilbert space and* $S : H \to H$ *be* μ*-strictly pseudocontractive mapping with* $\mu \in [0, 1)$. *Let* $T_\gamma := \gamma I + (1 - \gamma)S$, *where* $\gamma \in [\mu, 1)$, *then*

(i) $F(T) = F(T_\gamma)$,

(ii) T_γ *is a nonexpansive mapping.*

Lemma 2.9. *[31] Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}_{j\geq 0}$ *of* $\{\Gamma_n\}$*such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \forall j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ *defined by*

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\Gamma_n\}_{n\geq n_0}$ *is a nondecreasing sequence such that* $\tau(n) \to \infty$, *as* $n \to 0$, *and for all* $n \geq n_0$, *the following two estimates hold:*

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

3. Main Results

Proposition 3.1. *Let H be a real Hilbert space. Let* $M : H \to 2^H$ *be a maximal monotone mapping and* $f : H \to H$ *be an* α*-inverse strongly monotone mapping. Let* $z = J_\lambda^M(I - \lambda f)x$, *then*

$$\|y - z\|^2 + \|x - z\|^2 \leq \|y - x\|^2, \ \forall x \in H, y \in F(J_\lambda^M(I - \lambda f)), \text{ and } \lambda \in (0, 2\alpha).$$

Proof. Let $\lambda \in (0, 2\alpha)$, since $J_\lambda^M(I - \lambda f)$ is averaged, then it follows from Remark 1.2 that $J_\lambda^M(I - \lambda f)$ is a firmly nonexpansive mapping. Thus, for any $x \in H$ and $y \in F(J_\lambda^M(I - \lambda f))$, we have from Lemma 2.1 that

$$\begin{aligned} \|z - y\|^2 &= \|J_\lambda^M(I - \lambda f)x - y\|^2 \\ &\leq \langle z - y, x - y\rangle \\ &= \frac{1}{2}\left[\|z - y\|^2 + \|x - y\|^2 - \|z - x\|^2\right], \end{aligned}$$

which implies

$$\|y - z\|^2 + \|x - z\|^2 \leq \|y - x\|^2.$$

□

Theorem 3.2. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $M : H_1 \to 2^{H_1}$ be multivalued maximal monotone mapping and $f : H_1 \to H_1$ be an α-inverse strongly monotone mapping. Let $S : H_2 \to H_2$ be μ-strictly pseudocontractive mapping. Assume that $\Gamma = \{z \in (M+f)^{-1}(0) : Az \in F(S)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - T_\gamma)Au_n), \\ x_{n+1} = J_\lambda^M (I - \lambda f)y_n, \ n \geq 1, \end{cases} \tag{13}$$

where $T_\gamma := \gamma I + (1-\gamma)S$ with $\gamma \in [\mu, 1)$, $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $\lambda \in (0, 2\alpha)$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Γ.

Proof. From Lemma 2.8, Lemma 2.2 (ii), (iii), (iv) and Lemma 2.3, we obtain that $P_C(I - \gamma_n A^*(I - T_\gamma)A)$ is $\frac{1+\gamma_n\|A\|^2}{2}$-averaged. That is, $P_C(I - \gamma_n A^*(I - T_\gamma)A) = (1-\alpha_n)I + \alpha_n T_n$, where $\alpha_n = \frac{1+\gamma_n\|A\|^2}{2}$ and T_n is a nonexpansive mapping for each $n \geq 1$. Thus, we can rewrite y_n as

$$y_n = (1-\alpha_n)u_n + \alpha_n T_n u_n. \tag{14}$$

Let $p \in \Gamma$, then from (13), (14) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|J_\lambda^M (I - \lambda f)y_n - p\|^2 \\ &\leq \|y_n - p\|^2 \\ &= \|(1-\alpha_n)(u_n - p) + \alpha_n(T_n u_n - p)\|^2 \\ &= (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|T_n u_n - p\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|u_n - T_n u_n\|^2 \\ &\leq \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - T_n u_n\|^2 \\ &\leq \|(1-\beta_n)(x_n - p) + \beta_n(u - p)\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|u - p\|^2 \\ &\leq \max\{\|x_n - p\|^2, \|u - p\|^2\} \\ &\ \ \vdots \\ &\leq \max\{\|x_1 - p\|^2, \|u - p\|^2\}. \end{aligned} \tag{15}$$

Therefore, $\{\|x_n - p\|^2\}$ is bounded. Consequently, $\{x_n\}$, $\{y_n\}$ $\{u_n\}$ and $\{Au_n\}$ are all bounded.
From (13), we obtain

$$\lim_{n\to\infty} \|u_n - x_n\|^2 = \lim_{n\to\infty} \beta_n\|u - x_n\|^2 = 0. \tag{16}$$

We now consider two cases:
Case 1: Suppose that $\{\|x_n - p\|^2\}$ is monotone decreasing, then $\{\|x_n - p\|^2\}$ is convergent. Thus,

$$\lim_{n\to\infty} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) = 0. \tag{17}$$

From (15), we obtain

$$\begin{aligned} \alpha_n(1-\alpha_n)\|u_n - T_n u_n\|^2 &\leq \|u_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|u - p\|^2 \\ &\quad -\|x_{n+1} - p\|^2 \to 0, \text{ as } n \to \infty. \end{aligned} \tag{18}$$

Since $\alpha_n = \frac{1+\gamma_n\|A\|^2}{2}$, then by the condition on γ_n, we obtain

$$\lim_{n\to\infty} \|u_n - T_n u_n\|^2 = 0. \tag{19}$$

Also, from (14) and (19), we have

$$\lim_{n\to\infty} \|y_n - u_n\|^2 = \lim_{n\to\infty} \alpha_n \|T_n u_n - u_n\|^2 = 0. \tag{20}$$

We obtain from (16) and (20) that

$$\lim_{n\to\infty} \|y_n - x_n\|^2 = 0. \tag{21}$$

It follows from (13), (15) and Proposition 3.1 that

$$\begin{aligned} \|x_{n+1} - y_n\|^2 &\leq \|y_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \|u_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|u - p\|^2 - \|x_{n+1} - p\|^2 \\ &= \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) \\ &\quad +\beta_n(\|u-p\|^2 - \|x_n - p\|^2) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{22}$$

From (20) and (22), we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\|^2 = 0. \tag{23}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to z. Without loss of generality, the subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ converges to a point $\bar{\gamma} \in \left(0, \frac{1}{\|A\|^2}\right)$. By Lemma 2.3, $A^*(I - T_\gamma)A$ is inverse strongly monotone, thus $\{A^*(I-T_\gamma)Au_{n_k}\}$ is bounded. It then follows from the firmly nonexpansivity of P_C that

$$\|P_C(I - \gamma_{n_k}A^*(I - T_\gamma)A)u_{n_k} - P_C(I - \bar{\gamma}A^*(I - T_\gamma)A)u_{n_k}\| \leq |\gamma_{n_k} - \bar{\gamma}|\|A^*(I - T_\gamma)Au_{n_k}\| \to 0, \text{ as } k \to \infty.$$

That is,

$$\lim_{k\to\infty} \|y_{n_k} - P_C(I - \bar{\gamma}A^*(I - T_\gamma)A)u_{n_k}\| = 0,$$

which implies from (20) that

$$\lim_{k\to\infty} \|u_{n_k} - P_C(I - \bar{\gamma}A^*(I - T_\gamma)A)u_{n_k}\| = 0. \tag{24}$$

It then follows from Lemma 2.5 that $z \in F(P_C(I - \bar{\gamma}A^*(I - T_\gamma)A))$. Thus, from Lemma 2.4, we obtain that

$$z \in C \cap A^{-1}F(T_\gamma).$$

Thus,

$$Az \in F(T_\gamma) = F(S).$$

Next we show that $z \in (M + f)^{-1}(0)$. Since f is α-inverse strongly monotone, f is $\frac{1}{\alpha}$−Lipschitz continuous and monotone. It then follows from Lemma 2.6 that $M + f$ is maximal monotone. Let $(v, w) \in G(M + f)$, then $w - fv \in M(v)$. From $x_{n_k+1} = J_\lambda^M(I - \lambda f)y_{n_k}$, we obtain

$$(I - \lambda f)y_{n_k} \in (I + \lambda M)x_{n_k+1}.$$

That is,

$$\frac{1}{\lambda}(y_{n_k} - \lambda f y_{n_k} - x_{n_k+1}) \in M(x_{n_k+1}).$$

Since $M + f$ is maximal monotone, it is monotone. Thus, we have

$$\langle v - x_{n_k+1}, w - fv - \frac{1}{\lambda}(y_{n_k} - \lambda f y_{n_k} - x_{n_k+1})\rangle \geq 0, \tag{25}$$

which implies

$$\begin{aligned}\langle v - x_{n_k+1}, w\rangle &\geq \langle v - x_{n_k+1}, fv + \frac{1}{\lambda}(y_{n_k} - x_{n_k+1}) - f y_{n_k}\rangle \\ &= \langle v - x_{n_k+1}, fv - f(x_{n_k+1})\rangle + \langle v - x_{n_k+1}, f(x_{n_k+1}) - f(y_{n_k})\rangle \\ &\quad + \langle v - x_{n_k+1}, \frac{1}{\lambda}(y_{n_k} - x_{n_k+1})\rangle \\ &\geq \langle v - x_{n_k+1}, f(x_{n_k+1}) - f(y_{n_k})\rangle + \langle v - x_{n_k+1}, \frac{1}{\lambda}(y_{n_k} - x_{n_k+1})\rangle. \end{aligned} \tag{26}$$

From (22), we have

$$\|f(x_{n_k+1}) - f(y_{n_k})\| \leq \frac{1}{\alpha}\|x_{n_k+1} - y_{n_k}\| \to 0, \text{ as } n \to \infty. \tag{27}$$

Also, from (23), we have that $\{x_{n_k+1}\}$ converges weakly to z. Thus, we obtain from (26) that

$$\langle v - z, w\rangle \geq 0.$$

By the maximal monotonicity of $M + f$, we have that $0 \in (M + f)z$. That is, $z \in (M + f)^{-1}(0)$. Therefore, $z \in \Gamma$. We now show that $\{x_n\}$ converges strongly to z. From (15), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|u_n - z\|^2 \\ &= \|(1 - \beta_n)(x_n - z) + \beta_n(u - z)\|^2 \\ &= (1 - \beta)^2\|x_n - z\|^2 + \beta_n^2\|u - z\|^2 + 2\beta_n(1 - \beta_n)\langle x_n - z, u - z\rangle \\ &\leq (1 - \beta_n)\|x_n - z\|^2 + \beta_n\left[\beta_n\|u - z\|^2 + 2(1 - \beta_n)\langle x_n - z, u - z\rangle\right]. \end{aligned} \tag{28}$$

Applying Lemma 2.7 to (28), we conclude that $\{x_n\}$ converges strongly to z.

Case 2. Assume that $\{\|x_n - x^*\|^2\}$ is not monotone decreasing. Set $\Gamma_n = \|x_n - x^*\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, by Lemma 2.9, we have that $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

From (18), we have

$$\begin{aligned}\alpha_{\tau(n)}(1 - \alpha_{\tau(n)})\|u_{\tau(n)} - T_{\tau(n)}u_{\tau(n)}\|^2 &\leq \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 + \beta_{\tau(n)}\|u - p\|^2 - \beta_{\tau(n)}\|x_{\tau(n)} - p\|^2 \\ &\leq \beta_{\tau(n)}(\|u - p\|^2 - \|x_{\tau(n)} - p\|^2) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{29}$$

By condition on $\{\alpha_{\tau(n)}\}$, we obtain

$$\lim_{n\to\infty} \|u_{\tau(n)} - T_{\tau(n)}u_{\tau(n)}\|^2 = 0. \tag{30}$$

Also, from (22), we have

$$\begin{aligned}\|x_{\tau(n)+1} - y_{\tau(n)}\|^2 &\leq \left(\|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2\right) + \beta_{\tau(n)}(\|u - p\|^2 - \|x_{\tau(n)} - p\|^2) \\ &\leq \beta_{\tau(n)}(\|u - p\|^2 - \|x_{\tau(n)} - p\|^2) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{31}$$

Following the same line of argument as in Case 1, we can show that $\{x_{\tau(n)}\}$ converges weakly to $z \in \Gamma$.

Now for all $n \geq n_0$, we have from (28) that

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - z\|^2 - [\|x_{\tau(n)} - z\|^2 \\ &\leq (1-\beta_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \beta_{\tau(n)}[\beta_{\tau(n)}\|u - z\|^2 + 2(1-\beta_{\tau(n)})\langle x_{\tau(n)} - z, u - z\rangle] - \|x_{\tau(n)} - z\|^2, \end{aligned}$$

which implies

$$\|x_{\tau(n)} - z\|^2 \leq \beta_{\tau(n)}\|u - z\|^2 + 2(1-\beta_{\tau(n)})\langle x_{\tau(n)} - z, u - z\rangle \to 0, \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is clear that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.
Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty} \Gamma_n = 0$. That is $\{x_n\}$ converges strongly to z. □

If S is a nonexpansive mapping defined on H_2, then we obtain the following result.

Corollary 3.3. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $M : H_1 \to 2^{H_1}$ be multivalued maximal monotone mapping and $f : H_1 \to H_1$ be an α-inverse strongly monotone mapping. Let $S : H_2 \to H_2$ be a nonexpansive mapping. Assume that $\Gamma = \{z \in (M + f)^{-1}(0) : Az \in F(S)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - S)Au_n), \\ x_{n+1} = J_\lambda^M(I - \lambda f)y_n, \ n \geq 1, \end{cases} \tag{32}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $\lambda \in (0, 2\alpha)$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim\limits_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Γ.

In view of Remark 1.3, we obtain the following result.

Corollary 3.4. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $f : C \to H_1$ be an α-inverse strongly monotone mapping and $S : H_2 \to H_2$ be μ-strictly pseudocontractive mapping. Assume that $\Gamma = \{z \in VIP(C, f) : Az \in F(S)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in C$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - S)Au_n), \\ x_{n+1} = P_C(I - \lambda f)y_n, \ n \geq 1, \end{cases} \tag{33}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $\lambda \in (0, 2\alpha)$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim\limits_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Γ.

Proof. From Theorem 3 of [36], we have that $(f + N_C)^{-1}(0) = VIP(C, f)$, where N_C is the normal cone of C. Thus, by setting $M = N_C$ in Theorem 3.2, we obtain the desired result. □

In the following Theorem, we study the class of SMVIP introduced by Moudafi [34].

Theorem 3.5. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ be multivalued maximal monotone mappings. Let $f : H_1 \to H_1$ be α-inverse strongly monotone mapping and $g : H_2 \to H_2$ be β-inverse strongly monotone mapping. Assume that $\Gamma = \{z \in (M_1 + f)^{-1}(0) : \ Az \in (M_2 + g)^{-1}(0)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1 - \beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - J_\lambda^{M_2}(I - \lambda g))Au_n), \\ x_{n+1} = J_\lambda^{M_1}(I - \lambda f)y_n, \ n \geq 1, \end{cases} \tag{34}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 < \lambda < 2\alpha, 2\beta$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Γ.

Proof. We know that, for any $\lambda > 0$, $F(J_\lambda^{M_2}(I - \lambda g)) = (M_2 + g)^{-1}(0)$ and for $\lambda \in (0, 2\beta)$, $J_\lambda^{M_2}(I - \lambda g)$ is nonexpansive. Thus, setting $S = J_\lambda^{M_2}(I - \lambda g)$ in Corollary 3.3, we obtain the desired result. □

By setting $M_1 = N_C$ and $M_2 = N_Q$ in Theorem 3.5, where N_C and N_Q are the normal cones of C and Q respectively, we obtain the following result.

Corollary 3.6. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $f : H_1 \to H_1$ be α-inverse strongly monotone mapping and $g : Q \to H_2$ be β-inverse strongly monotone mapping. Assume that $\Gamma = \{z \in VIP(C, f) : \ Az \in VIP(Q, g)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in C$ by*

$$\begin{cases} u_n = (1 - \beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - P_Q(I - \lambda g))Au_n), \\ x_{n+1} = P_C(I - \lambda f)y_n, \ n \geq 1, \end{cases} \tag{35}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 < \lambda < 2\alpha, 2\beta$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Γ.

4. Application to Split convex minimization problems

Let $F : H \to \mathbb{R}$ be a convex and differentiable function, and $M : H \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. We know that if ∇F is $\frac{1}{\alpha}$-Lipschitz continuous, then it is α-inverse strongly monotone, where ∇F is the gradient of F (see Remark 1.1). It is also known that the subdifferential ∂M of M is maximal monotone (see [36]). Moreover,

$$F(x^*) + M(x^*) = \min_{x\in H} [F(x) + M(x)] \Leftrightarrow 0 \in \nabla F(x^*) + \partial M(x^*).$$

Now, consider the following class of Split Convex Minimization Problem (SCMP): Find

$$x^* \in H_1 \text{ such that } F(x^*) + M(x^*) = \min_{x\in H_1} [F(x) + M(x)], \text{ and such that } Ax^* \in F(S), \tag{36}$$

where $A : H_1 \to H_2$ is a bounded linear operator, F and M is as defined above, $S : H_2 \to H_2$ is a strictly pseudocontractive mapping. Suppose the solution set of problem (36) is Ω, then setting $M = \partial M$ and $f = \nabla F$ in Theorem 3.2, we obtain the following result.

Theorem 4.1. *Let H_1 and H_2 be real Hilbert spaces, and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $F : H_1 \to \mathbb{R}$ be a convex and differentiable function such that ∇F is $\frac{1}{\alpha}$-Lipschitz continuous, and $M : H_1 \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous* *function. Let $S : H_2 \to H_2$ be μ-strictly pseudocontractive mapping. Suppose $\Omega \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I-S)Au_n), \\ x_{n+1} = J_\lambda^{\partial M}(I - \lambda \nabla F)y_n, \ n \geq 1, \end{cases} \tag{37}$$

where $\{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 < \lambda < 2\alpha, 2\beta$ and $\{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Ω.

Next, we consider the following class of SCMP: Find

$$x^* \in H_1 \text{ such that } F(x^*) + M_1(x^*) = \min_{x \in H_1} [F(x) + M_1(x)], \tag{38}$$

and such that $y^* = Ax^* \in H_2$, solves

$$G(x^*) + M_2(x^*) = \min_{x \in H_2} [G(x) + M_2(x)]. \tag{39}$$

Suppose the solution set of problem (38)-(38) is Ω, then setting $M_1 = \partial M_1$, $M_2 = \partial M_2$, $f = \nabla F$ and $g = \nabla G$ in Theorem 3.5, we obtain the following result.

Theorem 4.2. *Let H_1 and H_2 be real Hilbert spaces, and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $M_1 : H_1 \to (-\infty, +\infty]$ and $M_2 : H_2 \to (-\infty, +\infty]$ be proper convex and lower semi-continuous functions. Let $F : H_1 \to H_1$ be convex and differentiable function such that ∇F is $\frac{1}{\alpha}$-Lipschitz continuous and $G : H_2 \to H_2$ be convex and differentiable function such that ∇G is $\frac{1}{\beta}$-Lipschitz continuous. Assume that $\Omega \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - J_\lambda^{\partial M_2}(I - \lambda \nabla G))Au_n), \\ x_{n+1} = J_\lambda^{\partial M_1}(I - \lambda \nabla F)y_n, \ n \geq 1, \end{cases} \tag{40}$$

where $\{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 < \lambda < 2\alpha, 2\beta$ and $\{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to an element of Ω.

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