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On Positive Solution to Multi-point Fractional h-Sum Eigenvalue Problems for Caputo Fractional h-Difference Equations

Saowaluck Chasreechai[a], Jarunee Soontharanon[a], Thanin Sitthiwirattham[b]

[a]*Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*
[b]*Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10700, Thailand*

Abstract. In this article, we study the existence of at least one positive solution to a multi-point fractional h-sum eigenvalue problem for Caputo fractional h-difference equation, by using the Guo-Krasnoselskii's fixed point theorem. Moreover, we present some examples to display the importance of these results.

1. Introduction

Fractional calculus is an emerging field recently drawing attention from both theoretical and applied disciplines. Fractional order differential equations play a vital role in describing many phenomena related to chemistry, physics, mechanics, flow in porous media, control systems, electrical networks and mathematical biology. For a reader interested in the systematic development of the topic, we refer to the books [1]-[3]. A variety of results on initial and boundary value problems of fractional differential equations and inclusions can easily be found in the literature on the topic. For some recent results, we can refer to [4]-[11] and references cited therein.

Fractional difference equations have arttracted the attention of many mathematicians since they can be used for describing many problems in the real-world phenomena such as physics, mechanics, chemistry, control systems, electrical networks, and flow in porous media. In recent years, mathematicians have used this fractional calculus to model and solve various related problems. In particular, fractional calculus is a powerful tool for the processes which appear in nature, e.g. biology, ecology and other areas (research works can be found in [12]-[13], and the references therein). Some good papers dealing with boundary value problems for fractional difference equations have helped to build up some of the basic theory of this field (see for example the textbooks [14] and the papers [15]-[45] and references cited therein). Some recent works about the monotonicity of some new class of fractional difference operators with discrete exponential and Mittag-Leffler kernels (see [52] and [53]), Lyapunov type and Gronwalls inequalities for such operators (see[54] and [55]). The paper [56] is recent and develop the theory of fractional difference variational calculus.

Presently, there are many research presenting discrete fractional calculus on $\mathbb{Z}$, they focus on the difference operator with step size 1. Our knowledge, there is a gap in the literature about the details of this

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Email addresses: saowaluck.c@sci.kmutnb.ac.th (Saowaluck Chasreechai), jarunee.s@sci.kmutnb.ac.th (Jarunee Soontharanon), thanin_sit@dusit.ac.th (Thanin Sitthiwirattham)

operation. To make it more general and flexible in the sense that it has the freedom to choose the step size. However, the development of discrete fractional calculus on $h\mathbb{Z}$ are rare (see [57]-[62]).

The eigenvalue problem for h-difference equations has not been studied. These are the motivation for this research. In this paper, we consider a multi-point fractional h-sum eigenvalue problems for Caputo fractional h-difference equations of the form

$$\begin{aligned} &{}^{C}\Delta_h^{\alpha}u(t) + \lambda F\Big[t + (\alpha - N)\,h, u\,(t + (\alpha - N)\,h)\Big] = 0, \quad t \in (h\mathbb{N})_{0,Th} \\ &u\Big((\alpha - N - 1)\,h\Big) = {}^{C}\Delta^{\beta_i}u\Big((\alpha - \beta_i)\,h\Big) = 0, \quad i \in \mathbb{N}_{1,N-1}, \\ &u\Big((T + \alpha - N + 1)\,h\Big) = \mu\Delta_h^{-\gamma}u\Big((T + \alpha + \gamma - N + 1)\,h\Big), \end{aligned} \tag{1}$$

where $(h\mathbb{N})_{0,Th} := \{0, h, 2h, \ldots, Th\}$; $\alpha \in (N, N + 1]$, $N \in \mathbb{N}_2 := \{2, 3, ...\}$; $\beta_i \in (i, i + 1]$; $\gamma \in (0, 1]$; $0 < \mu < \frac{\Gamma(\gamma)}{h\sum_{s=\alpha-2N}^{T+\alpha-N+1} s\Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}}}$ and
$F \in C\Big((h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h} \times [0, \infty), [0, \infty)\Big)$. For example, the particular case of system (1) when $2 < \alpha < 3$, we have

$$\begin{aligned} &{}^{C}\Delta_h^{\alpha}u(t) + \lambda F\Big[t + (\alpha - 2)\,h, u\,(t + (\alpha - 2)\,h)\Big] = 0, \quad t \in (h\mathbb{N})_{0,Th} \\ &u\Big((\alpha - 3)\,h\Big) = {}^{C}\Delta^{\beta_1}u\Big((\alpha - \beta_1)\,h\Big) = 0, \\ &u\Big((T + \alpha - 1)\,h\Big) = \mu\Delta_h^{-\gamma}u\Big((T + \alpha + \gamma - 1)\,h\Big), \end{aligned} \tag{2}$$

where $N = 2$, $\beta_1 \in (1, 2]$, and the domain of F, u are $(h\mathbb{N})_{(\alpha-4)h,(T+\alpha-1)h}$. In the case of system (1) when $3 < \alpha < 4$, we have

$$\begin{aligned} &{}^{C}\Delta_h^{\alpha}u(t) + \lambda F\Big[t + (\alpha - 3)\,h, u\,(t + (\alpha - 3)\,h)\Big] = 0, \quad t \in (h\mathbb{N})_{0,Th} \\ &u\Big((\alpha - 3)\,h\Big) = {}^{C}\Delta^{\beta_1}u\Big((\alpha - \beta_1)\,h\Big) = {}^{C}\Delta^{\beta_2}u\Big((\alpha - \beta_2)\,h\Big) = 0, \\ &u\Big((T + \alpha - 2)\,h\Big) = \mu\Delta_h^{-\gamma}u\Big((T + \alpha + \gamma - 2)\,h\Big), \end{aligned} \tag{3}$$

where $N = 3$, $\beta_1 \in (1, 2]$, $\beta_2 \in (2, 3]$, and the domain of F, u are $(h\mathbb{N})_{(\alpha-6)h,(T+\alpha-2)h}$.

The aim of this paper is to give some results for the existence of at least one positive solution to (1). For the positive solution of (1), we mean that a function $u(t) : (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h} \to [0, \infty)$ and satisfies the problem (1). The plan of this paper is as follows. In Section 2 we recall some definitions and basic lemmas. Also, we derive a representation for the solution to (1) by converting the problem to an equivalent summation equation. In Section 3, we show the existence of at least one positive solution to (1) by the following well-known Guo-Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1. *[63] Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$.
(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$ or
Then, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 1.2. *[64] (Arzelá-Ascoli theorem) A set of functions in $C[a, b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a, b]$.*

Lemma 1.3. *[64] If a set is closed and relatively compact then it is compact.*

2. Preliminaries

In the following, there are notations, definitions, and lemmas which are used in the main results.

Definition 2.1. *[57] For any $t, \alpha \in \mathbb{R}$ and $h > 0$, the h-falling function is defined by*

$$t_h^{\underline{\alpha}} := h^{\alpha} \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\alpha\right)} = h^{\alpha}\left(\frac{t}{h}\right)^{\underline{\alpha}},$$

where $\frac{t}{h}+1 \notin \mathbb{Z}^- \cup \{0\}$, and we use the convention that division at a pole yields zero. If $h=1$, then $t_h^{\underline{\alpha}} = t^{\underline{\alpha}}$.

Definition 2.2. *[57] For $\alpha, h > 0$ and f defined on $(h\mathbb{N})_a := \{a, a+h, a+2h, \ldots\}$, the α-order fractional h-sum of f is defined by*

$$\Delta_h^{-\alpha} f(t) := \frac{h}{\Gamma(\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\alpha} \left(t - \sigma_h(hs)\right)_h^{\underline{\alpha-1}} f(hs),$$

where $t \in (h\mathbb{N})_{a+\alpha h} := \{a+\alpha h, a+(\alpha+1)h, a+(\alpha+2)h, ...\}$ and $\sigma_h(hs) = (s+1)h$. If $h=1$, then $\Delta_h^{-\alpha} f(t) = \Delta^{-\alpha} f(t)$.

Definition 2.3. *[60] For $\alpha > 0$ and f defined on $(h\mathbb{N})_a$, the α-order Caputo fractional h-difference of f is defined by*

$$^{C}\Delta_h^{\alpha} f(t) := \Delta_h^{-(N-\alpha)} \Delta_h^N f(t) = \frac{h}{\Gamma(N-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(N-\alpha)} \left(t - \sigma_h(hs)\right)_h^{\underline{N-\alpha-1}} \Delta_h^N f(sh),$$

where $t \in (h\mathbb{N})_{a+(N-\alpha)h}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha < N$. If $\alpha = N$ then $^{C}\Delta_h^{\alpha} f(t) = \Delta_h^N f(t)$, and if $h=1$ then $^{C}\Delta_h^{\alpha} f(t) = \Delta_C^{\alpha} f(t)$.

To define the solution of the boundary value problem (1) we need the following lemma that deals with a linear variant of the boundary value problem (1) and gives a representation of the solution.

Lemma 2.4. *Let $\alpha \in (N, N+1]$, $N \in \mathbb{N}_2 := \{2,3,...\}$; $\beta_i \in (i, i+1], i \in \mathbb{N}_{1,N-1}$; $\gamma \in (0,1]$; $0 < \mu < \frac{(T+2)\Gamma(\gamma)}{h\sum_{s=\alpha-2N}^{T+\alpha-N+1}[s-\alpha+N+1]\left((T+\alpha+\gamma-N+1)h-\sigma_h(s)\right)_h^{\underline{\gamma-1}}}$ and $f \in C\Big((h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}, [0,\infty)\Big)$ be given. Then, the problem*

$$^{C}\Delta_h^{\alpha} u(t) + f\Big[t + (\alpha-N)h\Big] = 0, \quad t \in (h\mathbb{N})_{0,Th}, \tag{4}$$

$$\begin{cases} u\Big((\alpha-N-1)h\Big) = {}^{C}\Delta_h^{\beta_i} u\Big((\alpha-N-\beta_i)h\Big) = 0, \\ u\Big((T+\alpha-N+1)h\Big) = \mu\Delta_h^{-\gamma} u\Big((T+\alpha+\gamma-N+1)h\Big), \end{cases} \tag{5}$$

has the unique solution

$$u(t) = -\frac{h}{\Gamma(\alpha)} \sum_{s=0}^{\frac{t}{h}-\alpha} \left((t-\sigma_h(s)\right)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) + \left[\frac{\frac{t}{h}-(\alpha-N-1)}{T+2}\right] \times \left[\frac{h}{\Gamma(\alpha)} \sum_{s=0}^{T-N+1} \left((T+\alpha-N+1)h - \sigma_h(s)\right)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) + \mu\mathcal{A}(u)\right], \tag{6}$$

where the functional $\mathcal{A}(u)$ is defined by

$$\mathcal{A}(u) := \frac{h^2}{\Lambda\Gamma(\alpha)\Gamma(\gamma)} \sum_{s=\alpha-2N}^{T+\alpha-N+1} \Big((T+\alpha+\gamma-N+1)h - \sigma_h(s)\Big)_h^{\underline{\gamma-1}} \times$$
$$\Bigg[\Big(\frac{s-\alpha+N+1}{T+2}\Big) \sum_{x=0}^{T-N+1} \Big((T+\alpha-N+1)h - \sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big)$$
$$- \sum_{x=0}^{s-\alpha} \Big(sh - \sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big)\Bigg], \tag{7}$$

and

$$\Lambda = 1 - \frac{\mu h}{\Gamma(\gamma)} \sum_{s=\alpha-2N}^{T+\alpha-N+1} \Big[\frac{s-\alpha+N+1}{T+2}\Big] \Big((T+\alpha+\gamma-N+1)h - \sigma_h(s)\Big)_h^{\underline{\gamma-1}}. \tag{8}$$

Proof. Using the fractional h-sum of order α for (4), we obtain

$$u(t) = -C_0 - \sum_{k=1}^{N} C_k t_h^{\underline{k}} - \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{\frac{t}{h}-\alpha} \Big(t - \sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big), \tag{9}$$

for $t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}$.
By substituting $t = (\alpha - N - 1)h$ into (9) and employing the condition of (5): $u\Big((\alpha-N-1)h\Big) = 0$, we have

$$C_0 + \sum_{k=1}^{N} C_k\Big((\alpha-N-1)h\Big)_h^{\underline{k}} = 0. \tag{10}$$

Using the Caputo fractional h-difference of order β_i, $i \in \mathbb{N}_{1,N-1}$ for (9), we have

$${}^C\Delta_h^{\beta_i} u(t) = \frac{h}{\Gamma(i+1-\beta_i)} \sum_{s=\alpha-2N}^{\frac{t}{h}-(i+1-\beta_i)} \Big(t - \sigma_h(s)\Big)_h^{\underline{i-\beta_i}} {}_s\Delta_h^{i+1} u(sh), \tag{11}$$

for $t \in (h\mathbb{N})_{(\alpha-2N+i+1-\beta_i)h,(T+\alpha-N+i+2-\beta_i)h}$.
By substituting $t = (\alpha - N - \beta_i)h$ into (11) and using (9), we have

$$\frac{h}{\Gamma(i+1-\beta_i)} \sum_{s=\alpha-2N}^{\alpha-i-N-1} \Big((\alpha-N-\beta_i)h - \sigma_h(s)\Big)_h^{\underline{i-\beta_i}} {}_s\Delta_h^{i+1} u(sh)$$
$$= -\frac{h}{\Gamma(i+1-\beta_i)} \sum_{s=\alpha-2N}^{\alpha-i-N-1} \Big((\alpha-N-\beta_i)h - \sigma_h(s)\Big)_h^{\underline{i-\beta_i}} \times$$
$${}_s\Delta_h^{i+1} \Bigg[\sum_{k=1}^{N} C_k (sh)_h^{\underline{k}} + \frac{h}{\Gamma(\alpha)} \sum_{x=0}^{s-\alpha} \Big(sh - \sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big)\Bigg]$$
$$= -\frac{h}{\Gamma(i+1-\beta_i)} \sum_{s=\alpha-2N}^{\alpha-i-N-1} \Big((\alpha-N-\beta_i)h - \sigma_h(s)\Big)_h^{\underline{i-\beta_i}} {}_s\Delta_h^{i+1} \Bigg[\sum_{k=1}^{N} C_k (sh)_h^{\underline{k}}\Bigg]. \tag{12}$$

Employing the condition of (5): ${}^C\Delta_h^{\beta_i} u\Big((\alpha-N-\beta_i)h\Big) = 0$ for $i = N-1, ..., 2, 1$, we have the system of $N-1$

equations:

$$(E_1)\quad -\frac{hN!\,C_N}{\Gamma(N-\beta_{N-1})}\Big((N-1-\beta_{N-1})h\Big)_h^{\underline{N-1-\beta_{N-1}}} = 0, \qquad \text{so } C_N = 0,$$

$$\dots$$

$$(E_{N-2})\quad -\frac{h3!\,C_3}{\Gamma(3-\beta_2)}\sum_{s=\alpha-2N}^{\alpha-N-3}\Big((\alpha-N-\beta_2)h-\sigma_h(s)\Big)_h^{\underline{2-\beta_2}} = 0, \quad \text{so } C_3 = 0,$$

$$(E_{N-1})\quad -\frac{h2!\,C_2}{\Gamma(2-\beta_1)}\sum_{s=\alpha-2N}^{\alpha-N-2}\Big((\alpha-N-\beta_1)h-\sigma_h(s)\Big)_h^{\underline{1-\beta_1}} = 0, \quad \text{so } C_2 = 0.$$

Substituting the constants $C_i,\ i = 2, 3, ..., N$ into (10), we have

$$C_0 + (\alpha - N - 1)h\,C_1 = 0. \tag{13}$$

Next, taking the fractional h-sum of order γ for (9), weget

$$\Delta_h^{-\gamma}u(t) = \frac{h}{\Gamma(\gamma)}\sum_{s=\alpha-2N}^{\frac{t}{h}-\gamma}\Big(t-\sigma_h(s)\Big)_h^{\underline{\gamma-1}}u(sh), \tag{14}$$

for $t \in (h\mathbb{N})_{(\alpha-2N+\gamma)h,(T+\alpha-N+\gamma+1)h}$.

Employing the last condition of (5), we obtain

$$\begin{aligned} &C_0 + C_1[T+\alpha-N+1]h \\ =& -\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) \\ &-\frac{\mu h}{\Gamma(\gamma)}\sum_{s=\alpha-2N}^{T+\alpha-N+1}\Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}} u\Big(sh\Big). \end{aligned} \tag{15}$$

The constants C_0 and C_1 can be obtained by solving the system of equations (13) and (15) as given by

$$\begin{aligned} C_0 =& \left(\frac{\alpha-N-1}{T+2}\right)\Bigg[\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) \\ &+\frac{\mu h}{\Gamma(\gamma)}\sum_{s=\alpha-2N}^{T+\alpha-N+1}\Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}} u\Big(sh\Big)\Bigg], \end{aligned} \tag{16}$$

$$\begin{aligned} C_1 =& -\frac{1}{(T+2)h}\Bigg[\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) \\ &+\frac{\mu h}{\Gamma(\gamma)}\sum_{s=\alpha-2N}^{T+\alpha-N+1}\Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}} u\Big(sh\Big)\Bigg]. \end{aligned} \tag{17}$$

Substituting all constants $C_i,\ i = 0, 1, 2, ..., N$ into (10), we get

$$u(t) = -\frac{h}{\Gamma(\alpha)} \sum_{s=0}^{\frac{t}{h}-\alpha} \Big((t-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) + \left(\frac{\frac{t}{h}-(\alpha-N-1)}{T+2}\right)$$
$$\left[\frac{h}{\Gamma(\alpha)} \sum_{s=0}^{T-N+1} \Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big)\right.$$
$$\left.+\frac{\mu h}{\Gamma(\gamma)} \sum_{s=\alpha-2N}^{T+\alpha-N+1} \Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}} u\Big(sh\Big)\right]. \tag{18}$$

Letting

$$\mathcal{A}(u) = \frac{h}{\Gamma(\gamma)} \sum_{s=\alpha-2N}^{T+\alpha-N+1} \Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}} u\Big(sh\Big)$$

then, from (17), we deduce that

$$\mathcal{A}(u) = \frac{h}{\Gamma(\gamma)} \sum_{s=\alpha-2N}^{T+\alpha-N+1} \Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}} \times \tag{19}$$
$$\left\{-\frac{h}{\Gamma(\alpha)} \sum_{x=0}^{s-\alpha} \Big(sh-\sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big) + \left(\frac{s-\alpha+N+1}{T+2}\right)\times\right.$$
$$\left.\left[\frac{h}{\Gamma(\alpha)} \sum_{x=0}^{T-N+1} \Big((T+\alpha-N+1)h-\sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big) + \mu\mathcal{A}(u)\right]\right\}$$

which implies (7). Substituting this value into (17), we obtain (6). This completes the proof. □

Lemma 2.5. *Problem* (4) *has the unique solution in the from*

$$u(t) = \sum_{s=0}^{T-N+1} G\left(\frac{t}{h}-\alpha, s\right) f\Big((s+\alpha-N)h\Big) \tag{20}$$

for $t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}$ *,*
where

$$G\left(\frac{t}{h}-\alpha, s\right) := \frac{h}{\Gamma(\alpha)} \begin{cases} -(t-\sigma_h(s))_h^{\underline{\alpha-1}} + \left[\frac{(\frac{t}{h}-\alpha)+N+1}{(T+2)\Lambda}\right]\mathcal{K}(s), & s \in \mathbb{N}_{0,\frac{t}{h}-\alpha} \\ \left[\frac{(\frac{t}{h}-\alpha)+N+1}{(T+2)\Lambda}\right]\mathcal{K}(s), & s \in \mathbb{N}_{\frac{t}{h}-\alpha+1,T-N+1}, \end{cases} \tag{21}$$

with

$$\mathcal{K}(s) := \Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} - (1-\Lambda), \tag{22}$$

and Λ *is defined by (8).*

Proof. Unique solution of problem (4) can be written as

$$\begin{aligned}
u(t) = & -\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{\frac{t}{h}-\alpha}\Big((t-\sigma_h(s))\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) + \left[\frac{\frac{t}{h}-(\alpha-N-1)}{T+2}\right]\times \\
& \left\{\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big)\right. \\
& + \frac{\mu h\Phi}{\Lambda\Gamma(\alpha)}\sum_{x=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big) \\
& - \frac{\mu h^2}{\Lambda\Gamma(\alpha)\Gamma(\gamma)}\sum_{s=\alpha-2N}^{T+\alpha-N+1}\sum_{x=0}^{s-\alpha}\Big((T+\alpha+\gamma-N+1)h-\sigma_h(s)\Big)_h^{\underline{\gamma-1}}\times \\
& \left.\Big(sh-\sigma_h(x)\Big)_h^{\underline{\alpha-1}} f\Big((x+\alpha-N)h\Big)\right\},
\end{aligned} \tag{23}$$

where the constant Φ is defined as

$$\begin{aligned}
\Phi := & \frac{h}{\Gamma(\gamma)}\sum_{x=s}^{T-N+1}\Big((T+\gamma-N+1)h-\sigma_h(x)\Big)_h^{\underline{\gamma-1}}\Big((x+\alpha)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} \\
= & \frac{1-\Lambda}{\mu} > 0.
\end{aligned} \tag{24}$$

By the properties of summation, we obtain

$$\begin{aligned}
u(t) = & -\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{\frac{t}{h}-\alpha}\Big((t-\sigma_h(s))\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) + \left[\frac{\frac{t}{h}-(\alpha-N-1)}{T+2}\right]\times \\
& \left\{\frac{h}{\Gamma(\alpha)}\sum_{s=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big)\right. \\
& + \frac{\mu h\Phi}{\Lambda\Gamma(\alpha)}\sum_{s=0}^{T-N+1}\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big) \\
& - \frac{\mu h^2}{\Lambda\Gamma(\alpha)\Gamma(\gamma)}\sum_{s=0}^{T-N+1}\sum_{x=s}^{T-N+1}\Big((T+\gamma-N+1)h-\sigma_h(x)\Big)_h^{\underline{\gamma-1}}\times \\
& \left.\Big((x+\alpha)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} f\Big((s+\alpha-N)h\Big)\right\} \\
= & \frac{h}{\Gamma(\alpha)}\sum_{s=0}^{\frac{t}{h}-\alpha}\left\{-\Big((t-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} + \left(\frac{\frac{t}{h}-(\alpha-N-1)}{(T+2)\Lambda}\right)\times\right. \\
& \left[\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}} - \frac{\mu h}{\Gamma(\gamma)}\sum_{x=s}^{T-N+1}\Big((x+\alpha)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}}\times\right. \\
& \left.\left.\Big((T+\gamma-N+1)h-\sigma_h(x)\Big)_h^{\underline{\gamma-1}}\right]\right\} f\Big((s+\alpha-N)h\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{h}{\Gamma(\alpha)}\sum_{s=\frac{t}{h}-\alpha+1}^{T-N+1}\Bigg\{\left(\frac{\frac{t}{h}-(\alpha-N-1)}{(T+2)\Lambda}\right)\Bigg[\Big((T+\alpha-N+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}}\\
&-\frac{\mu h}{\Gamma(\gamma)}\sum_{x=s}^{T-N+1}\Big((T+\gamma-N+1)h-\sigma_h(x)\Big)_h^{\underline{\gamma-1}}\Big((x+\alpha)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}}\Bigg]\Bigg\}\times\\
&f\Big((s+\alpha-N)h\Big)\\
=&\sum_{s=0}^{T-N+1}G\left(\frac{t}{h}-\alpha,s\right)f\Big((s+\alpha-N)h\Big).
\end{aligned}
$$

This completes the proof. □

Lemma 2.6. *Let G be the Green's function related to problem (4)-(5) given by* (2.18). *For $0<\mu<\frac{1}{\Phi}$ where Φ is defined on (24), the following property holds:*

$$
\left[\frac{\left(\frac{t}{h}-\alpha\right)+N+1}{T+2}\right]G(T-N+1,s)\ \le\ G\left(\frac{t}{h}-\alpha,s\right)\ \le\ \frac{1}{\Theta}G(T-N+1,s),
$$

where

$$
\Theta:=\frac{\mu\Phi\left[\big((\alpha-1)h\big)_h^{\underline{\alpha-1}}-1\right]}{\big((T-N+\alpha)h\big)_h^{\underline{\alpha-1}}}. \tag{25}
$$

Proof. Assume that $0\le\frac{t}{h}-\alpha\le s<T-N+1$. In such a case:

$$
\begin{aligned}
\mathcal{H}\left(\frac{t}{h}-\alpha,s\right)=\frac{G\left(\frac{t}{h}-\alpha\right)}{G(T-N+1,s)}\ &=\ \frac{\left[\frac{\left(\frac{t}{h}-\alpha\right)+N+1}{(T+2)\Lambda}\right]\mathcal{K}(s)}{\frac{\mathcal{K}(s)}{\Lambda}-\Big((T-N+\alpha+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}}}\\
&=\ \frac{\left[\frac{\left(\frac{t}{h}-\alpha\right)+N+1}{T+2}\right]\mathcal{K}(s)}{\mu\Phi\left[\Big((T-N+\alpha+1)h-\sigma_h(s)\Big)_h^{\underline{\alpha-1}}-1\right]},
\end{aligned}
$$

for all $0<\frac{t}{h}-\alpha\le s<T-N+1$.

Now, it is immediate to verify the following inequalities:

$$
\frac{\left(\frac{t}{h}-\alpha\right)+N+1}{T+2}\ <\ \frac{1}{\mu\Phi}\left[\frac{\left(\frac{t}{h}-\alpha\right)+N+1}{T+2}\right]\ \le\ \mathcal{H}\left(\frac{t}{h}-\alpha,s\right),
$$

and

$$
\begin{aligned}
\mathcal{H}\left(\frac{t}{h}-\alpha,s\right)\ &\le\ \frac{\mathcal{K}(0)}{\mu\Phi\left[\Big((T-N+\alpha+1)h-\sigma_h(T-N+1)\Big)_h^{\underline{\alpha-1}}-1\right]}\\
&<\ \frac{\big((T-N+\alpha)h\big)_h^{\underline{\alpha-1}}}{\mu\Phi\left[\big((\alpha-1)h\big)_h^{\underline{\alpha-1}}-1\right]}\ =\ \frac{1}{\Theta},
\end{aligned}
$$

for all $0<\frac{t}{h}-\alpha\le s<T-N+1$.

On the contary, if $0 \leq s \leq \frac{t}{h} - \alpha \leq T - N + 1$ we obtain

$$\mathcal{H}\left(\frac{t}{h} - \alpha, s\right) = \frac{\left[\frac{\left(\frac{t}{h}-\alpha\right)+N+1}{T+2}\right]\mathcal{K}(s) - \Lambda\Big(t - \sigma_h(s)\Big)_h^{\underline{\alpha-1}}}{\mathcal{K}(s) - \Lambda\Big((T-N+\alpha+1)h - \sigma_h(s)\Big)_h^{\underline{\alpha-1}}},$$

for all $0 < s \leq \frac{t}{h} - \alpha < T - N + 1$.

We next consider

$$_t\Delta_h^2 G\left(\frac{t}{h} - \alpha, s\right) = -\Lambda(\alpha-2)(\alpha-3)\Big(t - \sigma_h(s)\Big)_h^{\underline{\alpha-3}} < 0, \tag{26}$$

for all $0 < s \leq \frac{t}{h} - \alpha < T - N + 1$.

Since $\mathcal{H}(T - N + 1, s) = 1$, furthermore

$$\mathcal{H}(s,s) = \frac{\left[\frac{s+N+1}{T+2}\right]\mathcal{K}(s)}{\mathcal{K}(s) - \Lambda\Big((T-N+\alpha+1)h - \sigma_h(s)\Big)_h^{\underline{\alpha-1}}} > \frac{s+N+1}{T+2}, \tag{27}$$

for all $0 < s < T - N + 1$.

From the fact that

$$_t\Delta_h^2 G\left(\frac{t-1}{h} - \alpha, s\right) = \frac{1}{h^2}\left[G\left(\frac{t+1}{h} - \alpha, s\right) - 2G\left(\frac{t}{h} - \alpha, s\right) + G\left(\frac{t-1}{h} - \alpha, s\right)\right],$$

together with (26)-(27), allow us to conclude that

$$G\left(\frac{t}{h} - \alpha, s\right) \geq \frac{1}{2}\left[G\left(\frac{t+1}{h} - \alpha, s\right) + G\left(\frac{t-1}{h} - \alpha, s\right)\right], \tag{28}$$

so

$$\frac{G\left(\frac{t+1}{h} - \alpha, s\right)}{\left(\frac{t+1}{h} - \alpha\right) + N + 1} < \frac{G\left(\frac{t-1}{h} - \alpha, s\right)}{\left(\frac{t-1}{h} - \alpha\right) + N + 1}, \tag{29}$$

for all $0 < s < \frac{t}{h} - \alpha < T - N + 1$,
it implies that

$$\mathcal{H}\left(\frac{t}{h} - \alpha, s\right) = \frac{G\left(\frac{t}{h} - \alpha\right)}{G(T-N+1, s)} > \frac{\left(\frac{t}{h} - \alpha\right) + N + 1}{T+2}, \tag{30}$$

for all $0 < s < \frac{t}{h} - \alpha < T - N + 1$.

Finally, it is easy to verify that

$$\mathcal{H}\left(\frac{t}{h} - \alpha, s\right) = \frac{G\left(\frac{t}{h} - \alpha\right)}{G(T-N+1, s)} < \frac{1}{\Theta}, \tag{31}$$

for all $0 < s < \frac{t}{h} - \alpha < T - N + 1$.

This completes the proof. □

3. Existence and Nonexistence of Positive Solution

In this section, we wish to establish the existence of at least one positive solution to (1). To accomplish this, we denote $C = C\Big((h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}, \mathbb{R}\Big)$. The Banach space of all function u with the norm is defined by $\|u\| = \max_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} |u(t)|$. For this purpose, we consider the cone

$$P = \left\{ u \in C : u \geqslant \Theta \left[\frac{\left(\frac{t}{h} - \alpha\right) + N + 1}{T+2} \right] \|u\| \right\},$$

where Θ is defined as (25).

Suppose that u is a solution of problem (1). It is clear from Lemma 2.4 that

$$u(t) = \lambda \sum_{s=0}^{T-N+1} G\left(\frac{t}{h} - \alpha, s\right) F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big],$$

for all $t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}$.

Next, define the operator $\mathcal{S}_\lambda : \mathcal{P} \to C$ as follow:

$$(\mathcal{S}_\lambda u)(t) = \lambda \sum_{s=0}^{T-N+1} G\left(\frac{t}{h} - \alpha, s\right) F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big], \tag{32}$$

for all $t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}$.

Lemma 3.1. *The operator $\mathcal{S}_\lambda$ is completely continuous.*

Proof. Since $0 < \mu < \frac{1}{\Phi}$ or $0 < 1 - \Lambda < 1$, it is clearly that $G\left(\frac{t}{h} - \alpha, s\right) \geq 0$.
So, we have

$$\begin{aligned} &\|(\mathcal{S}_\lambda u)\| \\ =\ & \lambda \max_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \sum_{s=0}^{T-N+1} G\left(\frac{t}{h} - \alpha, s\right) F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big] \\ \leq\ & \lambda \sum_{s=0}^{T-N+1} \frac{1}{\Theta} G(T-N+2, s)\, F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big], \end{aligned} \tag{33}$$

and

$$\begin{aligned} &(\mathcal{S}_\lambda u) \\ \geq\ & \lambda\Theta \left[\frac{\left(\frac{t}{h} - \alpha\right) + N + 1}{T+2} \right] \sum_{s=0}^{T-N+1} \frac{G(T-N+1, s)}{\Theta} F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big] \\ \leq\ & \Theta \left[\frac{\left(\frac{t}{h} - \alpha\right) + N + 1}{T+2} \right] \|(\mathcal{S}_\lambda u)\|. \end{aligned} \tag{34}$$

Hence, $\mathcal{S}_\lambda(\mathcal{P}) \subset \mathcal{P}$.

Obviously, $\mathcal{S}_\lambda : \mathcal{P} \to \mathcal{P}$ is continuous. Letting $\Omega \subset C$ be bounded, there exists a constant $R > 0$ such that $\|u\| \le R$ for all $u \in \Omega$. Define

$$\mathbf{L} := 1 + \max_{(t,u)\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}\times[0,R]} |F(t,u)|.$$

Thus, for all $u \in \Omega$, it is satisfies that

$$\begin{aligned} |\mathcal{S}_\lambda u(t)| &\le \lambda \sum_{s=0}^{T-N+1} G\Big(\frac{t}{h}-\alpha, s\Big) F\Big[(s+\alpha-N)h, u\Big((s+\alpha-N)h\Big)\Big] \\ &\le \mathbf{L}\lambda \sum_{s=0}^{T-N+1} G\Big(\frac{t}{h}-\alpha, s\Big), \end{aligned} \tag{35}$$

for all $t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}$, which implies $\mathcal{S}_\lambda(\Omega)$ is bounded in C.

On the other hand, for each $u \in \Omega$ we have

$$\begin{aligned} & |{}_t\Delta_h (\mathcal{S}_\lambda u)(t)| \\ \le & \left| -\frac{\lambda h}{\Gamma(\alpha)} \sum_{s=0}^{T-N+1} (\alpha-1)(t-\sigma_h(s))_h^{\underline{\alpha-2}} F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big] \right. \\ & \left. + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{T-N+1} \frac{\mathcal{K}(s)}{T+2} F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big] \right| \\ \le & \frac{\lambda h}{\Gamma(\alpha-1)} \sum_{s=0}^{T-N+1} (t-\sigma_h(s))_h^{\underline{\alpha-2}} \left| F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big] \right| \\ & + \frac{\lambda}{(T+2)\Gamma(\alpha)} \sum_{s=0}^{T-N+1} \mathcal{K}(s) \left| F\Big[\Big(s+\alpha-N\Big)h, u\Big((s+\alpha-N)h\Big)\Big] \right| \\ \le & \lambda \mathbf{L}\left[\frac{h}{\Gamma(\alpha-1)} \sum_{s=0}^{T-N+1} ((T+\alpha-N+1)h - \sigma_h(s))_h^{\underline{\alpha-2}} + \frac{1}{(T+2)\Gamma(\alpha)} \sum_{s=0}^{T-N+1} \mathcal{K}(s) \right] \\ \le & \frac{\mathbf{L}\lambda}{\Gamma(\alpha)} \left[((T+\alpha-N+1)h)_h^{\underline{\alpha-1}} - \Gamma(\alpha)h^{\alpha-1} + \frac{((T+\alpha-N+1)h)_h^{\underline{\alpha}}}{\alpha h(T+2)} \right. \\ & \left. + [1-\Lambda]\Big(\frac{T-N+2}{T+2}\Big) \right] := \mathbf{M}. \end{aligned} \tag{36}$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta = \frac{h}{\mathbf{M}}\epsilon \quad \text{for all } \ t_1, t_2 \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}.$$

Hence, for each $u \in \Omega$ and $t_1, t_2 \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}$ with $t_1 < t_2$, we have

$$\Big| (\mathcal{S}_\lambda u)(t_2) - (\mathcal{S}_\lambda u)(t_1) \Big| \le \sum_{s=\frac{t_1}{h}-\alpha+1}^{\frac{t_2}{h}-\alpha} |{}_t\Delta_h (\mathcal{S}_\lambda u)(t)| \le \frac{\mathbf{M}}{h}|t_2 - t_1| < \epsilon. \tag{37}$$

So, $\mathcal{S}_\lambda$ is equicontinuous. Form the Arzela-Ascoli theorem, it implies that $\mathcal{S}_\lambda : \mathcal{P} \to: \mathcal{P}$ is completely continuous. □

We next establish some sufficient conditions for the existence and nonexistence of positive solution for problem (1). For convenience, we set the notation:

$$
\begin{aligned}
F_0 &= \lim_{u\to 0^+}\left\{\max_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \frac{F\big(t,u(t)\big)}{u}\right\},\\
F_\infty &= \lim_{u\to \infty}\left\{\max_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \frac{F\big(t,u(t)\big)}{u}\right\},\\
f_0 &= \lim_{u\to 0^+}\left\{\min_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \frac{F\big(t,u(t)\big)}{u}\right\},\\
f_\infty &= \lim_{u\to \infty}\left\{\min_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \frac{F\big(t,u(t)\big)}{u}\right\}.
\end{aligned}\tag{38}
$$

Theorem 3.2. *Let $\tau\in(0,1)$ be a constant. Then for each*

$$
\lambda\in\left(\left[\tau\Theta f_\infty\sum_{s=0}^{T-N+1} s\,G(T-N+1,s)\right]^{-1},\ \left[\frac{F_0}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s)\right]^{-1}\right),\tag{39}
$$

problem (1) *has at least one positive solution.*

Proof. First, for any $\epsilon>0$, from (39) we obtain

$$
\left[\tau\Theta(f_\infty-\epsilon)\sum_{s=0}^{T-N+1} s\,G(T-N+1,s)\right]^{-1}\ \le\ \lambda\ \le\left[\frac{(F_0+\epsilon)}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s)\right]^{-1}.\tag{40}
$$

By the definition of F_0, there exists a constant $\rho_1>0$ such that, for $0<u\leqslant\rho_1$, we have

$$F(t,u)\le(F_0+\epsilon)u.$$

Let $\Omega_{\rho_1}=\{u\in C:\|u\|<\rho_1\}$, then for $u\in\mathcal{P}\cap\partial\Omega_{\rho_1}$ we get

$$
\begin{aligned}
\|\mathcal{S}_\lambda u\| &= \max_{t\in t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \lambda\sum_{s=0}^{T-N+1} G\left(\frac{t}{h}-\alpha,s\right)(F_0+\epsilon)\ u(s)\\
&\le \frac{\lambda}{\Theta}(F_0+\epsilon)\|u\|\sum_{s=0}^{T-N+1} G(T-N+1,s)\\
&\le \|u\|.
\end{aligned}\tag{41}
$$

On the other hand, by the definition of f_∞, there exists $\rho_2>\rho_1$ such that, for any $u\geqslant\rho_2$, we have

$$F(t,u)\geqslant(f_\infty-\epsilon)u.$$

Let $\Omega_{\rho_2}=\{u\in C:\|u\|<\rho_2\}$. Then, for $u\in\mathcal{P}\cap\partial\Omega_{\rho_2}$ we get

$$
\begin{aligned}
\|\mathcal{S}_\lambda u\| &\ge \tau\,(\mathcal{S}_\lambda u)\ge\lambda\sum_{s=0}^{T-N+1}\tau\,G(T-N+1,s)\,(f_\infty-\epsilon)\ u(s)\\
&\ge \tau\lambda\Theta f_\infty\|u\|\sum_{s=0}^{T-N+1} s\,G(T-N+1,s)\\
&\ge \|u\|.
\end{aligned}\tag{42}
$$

According to (41),(42) and the first part of Theorem 1.1, imply that $\mathcal{S}_\lambda$ has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_{\rho_2} \backslash \Omega_{\rho_1})$, such that $\rho_1 \leqslant \|u\| \leqslant \rho_2$. Therefore, the problem (1) has at least one positive solution. □

Corollary 3.3. *If $F_0 = 0$ and $f_\infty = \infty$, then problem* (1) *has at least one positive solution.*

Proof. Since $F_0 = 0$ and $f_\infty = \infty$, we can get

$$\frac{F_0}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1, s) = 0$$

and
$$\tau \Theta f_\infty \sum_{s=0}^{T-N+1} s\, G(T-N+1, s) = +\infty.$$

By Theorem 3.2 implies that, for $\lambda \in (0, \infty)$, problem (1) has at least one positive solution. □

Theorem 3.4. *Let $\tau \in (0, 1)$ be a constant. Then for each*

$$\left[\tau\Theta(f_0 - \epsilon) \sum_{s=0}^{T-N+1} s\, G(T-N+1, s)\right]^{-1} \leq \lambda \leq \left[\frac{(F_\infty + \epsilon)}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1, s)\right]^{-1}, \tag{43}$$

problem (1) *has at least one positive solution.*

Proof. First, for any $\epsilon > 0$, from (43) we obtain

$$\left[\tau\Theta f_0 \sum_{s=0}^{T-N+1} s\, G(T-N+1, s)\right]^{-1} \leq \lambda \leq \left[\frac{F_\infty}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1, s)\right]^{-1}. \tag{44}$$

By the definition of f_0, there exists a constant $\rho_1 > 0$ such that, for $0 < u \leqslant \rho_1$, we have

$$F(t, u) \geqslant (f_0 - \epsilon) u. \tag{45}$$

Let $\Omega_{\rho_1} = \{u \in C : \|u\| < \rho_1\}$. Then, for $u \in \mathcal{P} \cap \partial\Omega_{\rho_1}$ we get $\|u\| = \rho_1$. Similary to the proof in Theorem 3.2, it holds from (44) and ((45)) that

$$\|\mathcal{S}_\lambda u\| \geq \tau(\mathcal{S}_\lambda u) \geq \tau\lambda\Theta f_0 \|u\| \sum_{s=0}^{T-N+1} s\, G(T-N+1, s) \geq \|u\|. \tag{46}$$

On the other hand, by the definition of F_∞, there exists $\widehat{\rho_2} > \rho_1$ such that

$$F(t, u) \leqslant (F_\infty + \epsilon) u, \quad \text{for all } u \geqslant \widehat{\rho_2}.$$

We consider F on two cases:

Case I. Suppose F is bounded. There exists $K > 0$, such that

$$F(t, u) \leqslant K, \quad \text{for all } u \geqslant \widehat{\rho_2}.$$

Choose $\rho_3 = \max\left\{\widehat{\rho_2}, \frac{\lambda K}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1, s)\right\}$.

Let $\Omega_{\rho_3} = \{u \in \mathcal{C} : \|u\| \leq \rho_3\}$. Then, for $u \in \mathcal{P} \cap \partial\Omega_{\rho_3}$ we get

$$\begin{aligned} \|\mathcal{S}_\lambda u\| &\leq \frac{\lambda}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s)\, F\Big[\big(s+\alpha-N\big)h, u\big((s+\alpha-N)h\big)\Big] \\ &\leq \frac{\lambda K}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s) \\ &\leq \rho_3 = \|u\|. \end{aligned} \tag{47}$$

Case II. Suppose F is unbounded. There exists $\rho_4 > \widehat{\rho_2}$ such that

$$F(t,u) \leqslant u, \quad \text{for all} \ \ u \geqslant \rho_4.$$

Let $\Omega_{\rho_4} = \{u \in \mathcal{C} : \|u\| \leq \rho_4\}$. Then, for $u \in \mathcal{P} \cap \partial\Omega_{\rho_4}$ we get

$$\begin{aligned} \|\mathcal{S}_\lambda u\| &\leq \frac{\lambda}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s)\, F\Big[\big(s+\alpha-N\big)h, u\big((s+\alpha-N)h\big)\Big] \\ &\leq \frac{\lambda}{\Theta}\|u\|\sum_{s=0}^{T-N+1} G(T-N+1,s) \\ &\leq \|u\|. \end{aligned} \tag{48}$$

Combining (47) and (48) and letting

$$\Omega_{\rho_2} = \{u \in \mathcal{C} : \|u\| \leq \rho_2\} \ \text{ where } \ \rho_2 = \max\{\rho_3, \rho_4\},$$

for $u \in \mathcal{P} \cap \partial\Omega_{\rho_2}$ we have

$$\|\mathcal{S}_\lambda u\| \leq \|u\|. \tag{49}$$

Hence, from (46) and (49) together with the second part of Theorem 1.1, it implies that $\mathcal{S}_\lambda$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Therefore, the problem (1) has at least one positive solution. □

Corollary 3.5. *If $f_0 = \infty$ and $F_\infty = 0$, then problem* (1) *has at least one positive solution.*

Proof. Since $f_0 = \infty$ and $F_\infty = 0$, we can get

$$\tau\Theta f_0 \sum_{s=0}^{T-N+1} s\,G(T-N+1,s) = +\infty$$

and

$$\frac{F_\infty}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s) = 0.$$

By Theorem 3.4, it implies that, for $\lambda \in (0,\infty)$, problem (1) has at least one positive solution. □

Theorem 3.6. *Assume $F_0 < +\infty$ and $F_\infty < +\infty$. Then problem* (1) *has no positive solution when the following condition is provided*

$$\lambda < \left[\frac{\omega}{\Theta}\sum_{s=0}^{T-N+1} G(T-N+1,s)\right]^{-1}, \tag{50}$$

where ω is a constant defined by (51).

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, together with the definitions of F_0 and F_∞, there exist positive constants $\omega_1, \omega_2, \rho_1, \rho_2$ satisfying $\rho_1 < \rho_2$ such that, for $0 < u \leqslant \rho_1$, we have

$$\begin{aligned} F(t,u) &\leq \omega_1 u, \quad \text{for all } u \in [0,\rho_1], \\ F(t,u) &\leq \omega_2 u, \quad \text{for all } u \in [\rho_2,\infty). \end{aligned}$$

Let

$$\omega := \max\left\{\omega_1, \omega_2, \max_{(t,u)\in t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}\times(\omega_1,\omega_2)} \frac{F(t,u)}{u}\right\}. \tag{51}$$

It follows that $F(t,u) \leq \omega u$ for any $u \in (0,\infty)$. Suppose that $x(t)$ is a positive solution of problem (1). That is,

$$(\mathcal{S}_\lambda x)(t) = x(t), \quad \text{for all } t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}.$$

In sequence,

$$\begin{aligned} \|x\| &= \|\mathcal{S}_\lambda x\| \\ &= \max_{t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}} \lambda \sum_{s=0}^{T-N+1} G\left(\frac{t}{h}-\alpha, s\right) F\Big[\big(s+\alpha-N\big)h, x\big((s+\alpha-N)h\big)\Big] \\ &\leq \frac{\lambda}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1,s) F\Big[\big(s+\alpha-N\big)h, x\big((s+\alpha-N)h\big)\Big] \\ &\leq \frac{\lambda\omega}{\Theta}\|x\| \sum_{s=0}^{T-N+1} G(T-N+1,s) < \|x\|, \end{aligned}$$

which is a contradiction. Hence, problem (1) has at least one positive solution. □

Theorem 3.7. *Assume* $f_0 > 0$ *and* $f_\infty > 0$*. Then problem* (1) *has no positive solution when the following condition is provided*

$$\lambda > \left[\ell\Theta \sum_{s=0}^{T-N+1} s\, G(T-N+1,s)\right]^{-1}, \tag{52}$$

where ℓ *is a constant defined by (53).*

Proof. Since $f_0 > 0$ and $f_\infty > 0$, together with the definitions of f_0 and f_∞, there exist positive constants $\ell_1, \ell_2, \rho_1, \rho_2$ satisfying $\rho_1 < \rho_2$ such that, for $0 < u \leqslant \rho_1$, we have

$$\begin{aligned} F(t,u) &\geq \ell_1 u, \quad \text{for all } u \in [0,\rho_1], \\ F(t,u) &\geq \ell_2 u, \quad \text{for all } u \in [\rho_2,\infty). \end{aligned}$$

Let

$$\ell := \min\left\{\ell_1, \ell_2, \min_{(t,u)\in t\in(h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}\times(\kappa_1,\kappa_2)} \frac{F(t,u)}{u}\right\}. \tag{53}$$

It follows that $F(t,u) \geq \ell u$ for any $u \in (0,\infty)$. Suppose that $x(t)$ is a positive solution of problem (1). That is,

$$(\mathcal{S}_\lambda x)(t) = x(t), \quad \text{for all } t \in (h\mathbb{N})_{(\alpha-2N)h,(T+\alpha-N+1)h}.$$

In sequence,

$$\begin{aligned} \|x\| = \|\mathcal{S}_\lambda x\| &\geq \lambda \sum_{s=0}^{T-N+1} s\, G\,(T-N+1,s)\, F\Big[\big(s+\alpha-N\big)h, x\big((s+\alpha-N)h\big)\Big] \\ &\geq \lambda \ell \Theta \|x\| \sum_{s=0}^{T-N+1} s\, G\,(T-N+1,s) \;>\; \|x\|, \end{aligned}$$

which is a contradiction. Hence, problem (1) has at least one positive solution. □

4. Some examples

In this section, in order to illustrate our results, we consider the problem

$$ {}^{C}\Delta_2^{\frac{10}{3}} u(t) + \lambda\, F\left[t-\frac{4}{3}, u\left(t-\frac{4}{3}\right)\right] = 0, \quad t \in (2\mathbb{N})_{0,30}\,, \tag{54}$$

$$\begin{cases} u\left(-\frac{10}{3}\right) = {}^{C}\Delta_2^{\frac{3}{2}} u\left(-\frac{13}{6}\right) = {}^{C}\Delta_2^{\frac{7}{3}} u\left(-6\right) = {}^{C}\Delta_2^{\frac{15}{4}} u\left(-\frac{53}{6}\right) = 0, \\ u\left(\frac{92}{3}\right) = e^{-8} \Delta_2^{-\frac{2}{5}} u\left(\frac{472}{15}\right). \end{cases} \tag{55}$$

Setting $\alpha = \frac{10}{3},\ N = 4,\ T = 15,\ \beta_1 = \frac{3}{2},\ \beta_2 = \frac{7}{3},\ \beta_3 = \frac{15}{4},\ \gamma = \frac{2}{5},\ \mu = e^{-8}$, we get that

$$\mu < \frac{\Gamma(\gamma)}{h\sum_{s=\alpha-2N}^{T+\alpha-N+1} s\left((T+\alpha+\gamma-N+1)h-\sigma_h(s)\right)_h^{\underline{\gamma-1}}} = 0.00079,$$

$$\Phi = \frac{h}{\Gamma(\gamma)} \sum_{s=\alpha-2N}^{T+\alpha-N+1} \left[\frac{s-\alpha+N+1}{T+2}\right] \left((T+\alpha+\gamma-N+1)h-\sigma_h(s)\right)_h^{\underline{\gamma-1}} = 1265.823,$$

$$\Theta = \frac{\mu\Phi\left[\left((\alpha-1)h\right)_h^{\underline{\alpha-1}}-1\right]}{\left((T-N+\alpha)h\right)_h^{\underline{\alpha-1}}} = 0.0058,$$

$$\sum_{s=0}^{T-N+1} G(T-N+1,s) = \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{12} \left[\frac{\mathcal{K}(s)}{\Lambda} - \left(\frac{92}{3}-\sigma_h(s)\right)_2^{\underline{\frac{7}{3}}}\right] = 31153.39,$$

$$\sum_{s=0}^{T-N+1} G(T-N+1,s) = \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{12} s\left[\frac{\mathcal{K}(s)}{\Lambda} - \left(\frac{92}{3}-\sigma_h(s)\right)_2^{\underline{\frac{7}{3}}}\right] = 8719.62.$$

(i) If $F(t,u(t)) = \dfrac{(200u^2+u)(5+t^2)}{u+10}$ for $t \in (2\mathbb{N})_{-\frac{28}{3},\frac{92}{3}}$, then we have

$$F_0 = \lim_{u\to 0^+} \left\{ \max_{t\in -\frac{28}{3},\frac{92}{3}} \frac{F\big(t,u(t)\big)}{u} \right\} = 94.544,$$

$$f_\infty = \lim_{u\to \infty} \left\{ \min_{t\in -\frac{28}{3},\frac{92}{3}} \frac{F\big(t,u(t)\big)}{u} \right\} = 1088.889.$$

Choosing $\tau = \frac{1}{300}$, we obtain

$$\left[\tau\Theta f_\infty \sum_{s=0}^{T-N+1} s\,G(T-N+1,s)\right]^{-1} \leq 0.00544$$

$$\left[\frac{F_0}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1,s)\right]^{-1} \geq 1.945.$$

By Theorem 3.2, we can conclude that the problem (54)-(55) has at least one positive solution for $\lambda \in (0.00544, 1.945)$. □

(ii) If $F(t,u(t)) = u^2(10+t)$ for $t \in (2\mathbb{N})_{-\frac{28}{3},\frac{92}{3}}$, then we have

$$F_0 = 0 \quad \text{and} \quad f_\infty = +\infty.$$

By Corollary 3.3, we can conclude that the problem (54)-(55) has at least one positive solution for $\lambda \in (0,\infty)$. □

(iii) If $F(t,u(t)) = \dfrac{(\pi u^2 + eu)(u+t^2)}{e^{20}u^2 + 2\pi}$ for $t \in (2\mathbb{N})_{-\frac{28}{3},\frac{92}{3}}$, then we have

$$f_0 = \lim_{u\to 0^+}\left\{\min_{t\in -\frac{28}{3},\frac{92}{3}} \frac{F\big(t,u(t)\big)}{u}\right\} = 0.192,$$

$$F_\infty = \lim_{u\to\infty}\left\{\max_{t\in -\frac{28}{3},\frac{92}{3}} \frac{F\big(t,u(t)\big)}{u}\right\} = \pi e^{-20}.$$

Choosing $\tau = \frac{1}{20}$, we obtain

$$\left[\tau\Theta f_0 \sum_{s=0}^{T-N+1} s\,G(T-N+1,s)\right]^{-1} \leq 2.059,$$

$$\left[\frac{F_\infty}{\Theta} \sum_{s=0}^{T-N+1} G(T-N+1,s)\right]^{-1} \geq 284.191.$$

By Theorem 3.4, we can conclude that the problem (54)-(55) has at least one positive solution for $\lambda \in (2.059, 284.191)$. □

(iv) If $F(t,u(t)) = \dfrac{(\pi \sin u + 2(\pi+t)\cos u)}{u^2}$ for $t \in (2\mathbb{N})_{-\frac{28}{3},\frac{92}{3}}$, then we have

$$f_0 = +\infty \quad \text{and} \quad F_\infty = 0.$$

By Corollary 3.5, we can conclude that the problem (54)-(55) has at least one positive solution for $\lambda \in (0,\infty)$. □

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