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On Geometry of the Midlocus Associated to a Smooth Curve in Plane and Space

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Abstract. The singularities of the midpoint map associated to a smooth plane curve, which is a map from the plane to the plane, are classified. The midlocus associated to a regular space curve is introduced. The geometric conditions for the midlocus of a space curve to have a crosscap or an $S_1^{\pm}$ singularities are investigated. A more general map, the λ-point map, associated to a space curve is introduced and many known surface singularities are realized as a special cases of this construction.

1. Introduction

The midlocus of a plane curve had been introduced by Brady under the name ”smoothed local symmetry” (cf. [2]). In [6] the second author and Brassett give the condition for the midlocus of a plane curve to be a regular curve. Also, they study the behaviour of the midpoint map. In [8] the second author and Warder present a method to create the boundary plane curve using the information provided by the midlocus and the radius function. This method is summarized in creating a system of ordinary differential equations using the midlocus and the radius function — the solution of this system is the symmetry set of the boundary curve and in this case the boundary curve is created as the envelope of circles centred on the symmetry set. For more details on envelope we refer reader to [3–7]. This method had been generalized to the higher dimensions by the first author [1].
This paper is divided into seven main sections, the first section deals with the introduction and the second section will be dedicated to the classification of the midpoint map as a map from the plane to the plane, in §2 we will give the geometric conditions for the midpoint map to have cusp, fold, lips, beaks and swallowtails singularities. The third section provides some examples illustrating our result in §2 . In §4 , we will prove that the midlocus associated to a smooth regular space curve is a surface and we give the geometric conditions for this surface to have a crosscap and an $S_1^{\pm}$ singularity. In §5, we will give some examples to illustrate the results in §4. In §6 we study the singularity of the λ-point map which is more general than the midpoint map. Also, through §6 the special values of λ are introduced and the singularity of the λ-point map associated the special values of λ will be investigated. Last section is the appendix, and in this section we give a geometric interpretation of the coefficients occur in our results in §6.

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2. Singularity of the midpoint map associated to a plane curve as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$

In this section we investigate the singularity of the midpoint map of a plane curve as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Recall that the midpoint map of a smooth plane curve γ is defined by $m : I \subset \mathbb{R} \times J \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ such that $m(t_1, t_2) = \frac{1}{2}(\gamma_1(t_1) + \gamma_2(t_2))$, where γ_1 and γ_2 are two smooth parts of γ parametrized by t_1 and t_2 respectively.

Before the discussion of the singularities of the midpoint map, we review some basic concepts related to the singularity of a smooth map from the plane into the plane. The map germ $(\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ with corank one singularity (a map germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^m, 0)$ has a corank one singularity at p if the rank of the Jacobian matrix of f at p is equal to $min(n, m) - 1$) and $\mathcal{A}_e$-codimension ≤ 6 had been classified up to $\mathcal{A}$-equivalence by J. Rieger [14] using the technique of complete transversal and finite determinacy [18]. The main purpose of this section is to give the geometric conditions for the midpoint map of a plane curve to have fold, cusp, beaks, lips and swallowtail singularities. The normal forms of these singularities are (x, y^2), $(x, xy + y^3)$, $(x, y^3 - x^2y)$, $(x, y^3 + x^2y)$ and $(x, y^4 + xy)$ respectively.

The second author and S. Janeczko found the conditions for the midpoint map to have cusp, beaks, lips and swallowtail singularities. The conditions they found are related to the centre symmetry set (CSS) and the inflexion points of the boundary curve [10]. In our results we give more precise conditions related to the geometry of the boundary curve.

Definition 2.1. *Two map-germs $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ $(i = 1, 2)$ are $\mathcal{A}$-equivalent if there exist germs of C^∞-diffeomorphisms ϑ and φ such that $\varphi \circ f_1 = f_2 \circ \vartheta$ holds, where $\vartheta : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\varphi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$.*

To give the geometric conditions for the midpoint map to have the mentioned singularities we use the criteria in [16, 19]. Since the map $m : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ has a corank one singularity when $T_1(0) = \pm T_2(0)$, there exists a neighbourhood U of 0, and non-vanishing vector field η such that $dm_p(\eta) = 0$ for all $p \in S(m) \bigcap U$, where $S(m)$ is the singular set of m. The vector field η is called the null vector field. The discriminant function which plays a central role in the criteria which we are going to use is defined by

$$\Lambda(t_1, t_2) = \det\left(\frac{\partial m}{\partial t_1}, \frac{\partial m}{\partial t_2}\right).$$

The expression $\eta\Lambda$ is the directional derivative of Λ by η. For more detail on the discriminant function and the null vector field we refer reader to [12, 16]. Now we state the criteria.

Criteria 2.2. *[16, 19] For a map germ $f : (U \subset \mathbb{R}^2, p) \to (\mathbb{R}^2, 0)$, the following hold.*

1. *f is $\mathcal{A}$-equivalent to fold if and only if $\eta\Lambda(p) \neq 0$.*
2. *f is $\mathcal{A}$-equivalent to cusp if and only if p is non-degenerate, $\eta\Lambda(p) = 0$ and $\eta\eta\Lambda(p) \neq 0$.*
3. *f is $\mathcal{A}$-equivalent to lips if and only if p is of corank one, $d\Lambda(p) = 0$ and Λ has a Morse type critical point of index 0 or 2 at p, namely $det(Hess\Lambda(p)) > 0$.*
4. *f is $\mathcal{A}$-equivalent to beaks if and only if p is of corank one $d\Lambda(p) = 0$ and Λ has a Morse type critical point of index 1 at p, namely $det(Hess\Lambda(p)) < 0$ and $\eta\eta\Lambda(p) \neq 0$.*
5. *f is $\mathcal{A}$-equivalent to swallowtail if and only if $d\Lambda(p) \neq 0$, $\eta\Lambda(p) = \eta\eta\Lambda(p) = 0$ and $\eta\eta\eta\Lambda(p) \neq 0$.*

Remark 2.3. *It is easy to observe that $\eta\eta\Lambda(p) \neq 0$ is automatically satisfied in part 3 of Criteria 2.2 and this is because of the inequality* $\det(Hess\Lambda(p)) > 0$ *and the symmetry of $Hess\Lambda$.*

From the definition of the midpoint map m associated to a smooth plane curve γ it is easy to check that the midpoint map is singular at $m(t_1, t_2)$ if and only if γ has parallel tangents at $\gamma(t_1)$ and $\gamma(t_2)$. Let γ_1 and γ_2 be two segments of γ around $\gamma(t_1)$ and $\gamma(t_1)$ respectively. We parameterize γ_1 and γ_2 by their arc-lengths s_1 and s_2 respectively such that $t_1 = t_2 = 0$ in the new coordinates. The unit tangents of γ_1 and γ_2 are denoted by T_1 and T_2 respectively. We state the main theorem of this section.

Theorem 2.4. *Let m be the midpoint map of a smooth plane curve. Suppose that the tangents to the two boundary segments are parallel, i.e. $T_1(0) = \pm T_2(0)$. Then at $(0, 0)$*

1. *m is $\mathcal{A}$-equivalent to fold if and only if $\kappa_1(0) \neq \mp\kappa_2(0)$.*
2. *m is $\mathcal{A}$-equivalent to cusp if and only if $\kappa_1(0) = \mp\kappa_2(0) \neq 0$ and $\kappa_1'(0) \neq \kappa_2'(0)$.*
3. *m is $\mathcal{A}$-equivalent to lips if and only if $\kappa_1(0) = \kappa_2(0) = 0$ and $\kappa_1'(0)\kappa_2'(0) < 0$.*
4. *m is $\mathcal{A}$-equivalent to beaks if and only if $\kappa_1(0) = \kappa_2(0) = 0$, $\kappa_1'(0)\kappa_2'(0) > 0$ and $\kappa_1'(0) \neq \kappa_2'(0)$.*
5. *m is $\mathcal{A}$-equivalent to swallowtail if and only if $\kappa_1(0) = \mp\kappa_2(0) \neq 0$, $\kappa_1'(0) = \kappa_2'(0)$ and $\kappa_1''(0) \neq \mp\kappa_2''(0)$.*

Proof. From definition we have $m(s_1, s_2) = \frac{1}{2}(\gamma_1(s_1) + \gamma_2(s_2))$, where s_1 and s_2 are the arc-length of γ_1 and γ_2 respectively. This map is singular at $(0, 0)$ if and only if $T_1(0) = \pm T_2(0)$ see [6]. Now we will use Criteria 2.2 to prove this theorem. Let $T_1(0) = -T_2(0)$, we choose η such that $dm_{(0,0)}(\eta) = 0$, thus we take $\eta = \frac{\partial}{\partial s_1} + \frac{\partial}{\partial s_2}$. Calculations show that $\Lambda(s_1, s_2) = -T_1(s_1).N_2(s_2)$. For the purpose of calculations we omit s_1 and s_2, hence $\Lambda = -T_1.N_2$. Parts 1 and 2 in Theorem 2.4 were proved by the second author in [6], but here we present a new version of their proof using the Criteria 2.2. Calculations show that

$$\Lambda_s = -\kappa_1 N_1 \cdot N_2, \qquad \Lambda_t = \kappa_2 T_1 \cdot T_2,$$

$$\eta\Lambda = (\kappa_2 - \kappa_1)T_1 \cdot T_2, \quad \eta\eta\Lambda = (\kappa_2' - \kappa_1')T_1 \cdot T_2 + (\kappa_2 - \kappa_1)^2 T_1 \cdot N_2,$$

$$\eta\eta\eta\Lambda = [(\kappa_2'' - \kappa_1'') - (\kappa_2 - \kappa_1)^3]T_1 \cdot T_2 + 3(\kappa_2 - \kappa_1)(\kappa_2' - \kappa_1')T_1 \cdot N_2$$

and

$$Hess\Lambda = \begin{pmatrix} -\kappa_1' N_1 \cdot N_2 + \kappa_1^2 T_1 \cdot N_2 & \kappa_1\kappa_2 N_1 \cdot T_2 \\ \kappa_1\kappa_2 N_1 \cdot T_2 & \kappa_2' T_1 \cdot T_2 + \kappa_2^2 T_1 \cdot N_2 \end{pmatrix}.$$

At $(0, 0)$ we have $\Lambda_s(0, 0) = \kappa_1(0), \Lambda_t(0, 0) = -\kappa_2(0), \eta\Lambda(0, 0) = \kappa_1(0) - \kappa_2(0), \eta\eta\Lambda(0, 0) = \kappa_1'(0) - \kappa_2'(0)$, $\eta\eta\eta\Lambda(0, 0) = \kappa_1''(0) - \kappa_2''(0) + (\kappa_2(0) - \kappa_1(0))^3$ and $det(Hess\Lambda(0, 0)) = -\kappa_1'(0)\kappa_2'(0)$. Thus applying the Criteria 2.2 the results hold. Similarly, we prove the results when $T_1(0) = T_2(0)$, and in this case we choose $\eta = \frac{\partial}{\partial s_1} - \frac{\partial}{\partial s_2}$. □

In [9] the second author and Graham Revee study the λ-equidistant ,associated to a smooth plane curve γ, which is the set of all points of the form $(1 - \lambda)p + \lambda q$ for fixed λ and parallel tangents at p and q.

3. Examples

In this section we give examples of the last three parts of theorem 2.4. To do so it is easier to work locally by considering two segments of curve as the following.

(1) We choose $\gamma_1(t_1) = (t_1, 3t_1^3 + t_1^4)$ and $\gamma_2(t_2) = (t_2, 1 - 2t_2^3 + t_2^4)$. Direct calculations show that $T_1(0) = T_2(0)$, $\kappa_1(0) = \kappa_2(0) = 0$, $\kappa_1'(0) = 18$ and $\kappa_2'(0) = -12$. Therefore, m is $\mathcal{A}$-equivalent to lips (see Figure 1).

(2) For beaks we choose $\gamma_1(t_1) = (t_1, -2t_1^3 + t_1^4)$ and $\gamma_2(t_2) = (t_2, 1 - 3t_2^3 + t_1^4)$. In this case we have $T_1(0) = T_2(0)$, $\kappa_1(0) = \kappa_2(0) = 0$, $\kappa_1'(0) = -12$ and $\kappa_2'(0) = -18$. Therefore, m is $\mathcal{A}$-equivalent to beaks (see Figure 2).

(3) For swallowtail we take $\gamma_1(t_1) = (t_1, 2t_1^2 + 3t_1^3 + 4t_1^4)$ and $\gamma_2(t_2) = (t_2, 1 - 2t_2^2 - 3t_2^3 + t_2^4)$. We have $T_1(0) = T_2(0)$, $\kappa_1(0) = 4$, $\kappa_2(0) = -4$, $\kappa_1'(0) = \kappa_2'(0) = -18$, $\kappa_1''(0) = -168$ and $\kappa_2''(0) = 216$. Therefore, m is $\mathcal{A}$-equivalent to swallowtail.

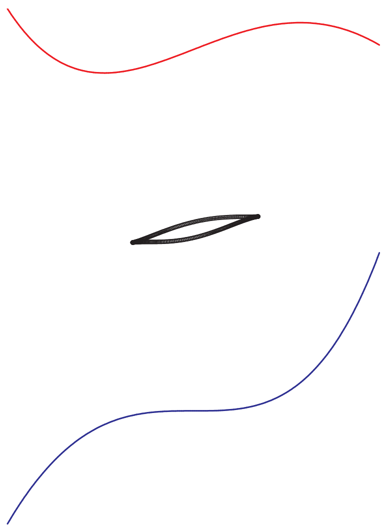

Figure 1: An example of a perturbed midpoint in the case of a lips singularity

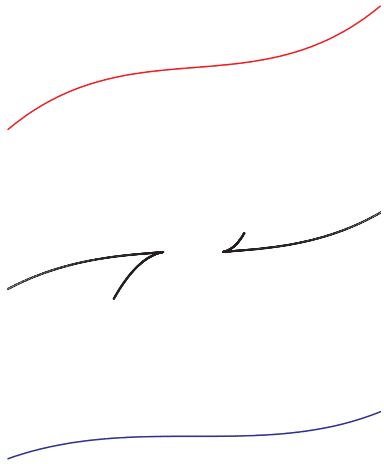

Figure 2: An example of a perturbed midpoint in the case of a beaks singularity

4. Singularity of the midlocus map associated to a space curve as a map from $\mathbb{R}^2$ to $\mathbb{R}^3$

In this section we define the midlocus associated to a smooth space curve γ to be the image of the midpoint map where we use all pairs of point of γ. Also, the geometric conditions for the midlocus of a space curve to have a crosscap and an $S_1^{\pm}$ singularity will be investigated.

Proposition 4.1. *Let $\gamma : I \to \mathbb{R}^3$ be a smooth space curve embedded in $\mathbb{R}^3$ (where I is an open interval or a circle), and let $p_1 = \gamma(t_1)$ and $p_2 = \gamma(t_2)$ be two distinct points of the curve. Then there is a sphere or plane in $\mathbb{R}^3$ tangent to γ at these two points (a bitangent sphere or plane). There are infinitely many such spheres if and only if there is a plane containing both p_1 and p_2 and perpendicular to the tangent lines at those points.*

Proof. The centres of spheres tangent to γ at p_1 all lie on the plane π_1 through p_1 perpendicular to the tangent vector $\gamma'(t_1)$ there; similarly there is a plane π_2 perpendicular to $\gamma'(t_2)$ at p_2. The remaining condition, that one sphere should be tangent at both points requires the centre to lie on the perpendicular bisector plane π_{12} of the chord joining p_1 and p_2. We require the condition that these three planes meet in a single point, which will then be the centre of the unique bitangent sphere. The three normals to the planes are the two tangents to γ at p_1, p_2 and the chord between these two points; the three planes meet in a single point, if and only if the two tangents and the chord are not coplanar.

It remains to examine the case where this fails. Suppose first that the tangent lines at p_1 and p_2 are parallel but distinct, so that π_1 and π_2 are also parallel. If π_1 and π_2 are distinct then the unique plane containing the tangent lines at p_1 and p_2 is a bitangent plane and there are no bitangent spheres. If $\pi_1 = \pi_2$ then there are infinitely many bitangent spheres with centres on the intersection of $\pi_1 = \pi_2$ with π_{12}.

If the tangent lines at p_1 and p_2 coincide then any plane through the common tangent line is a bitangent plane, and there are no bitangent spheres.

Finally if the tangent lines at p_1 and p_2 are coplanar with the chord joining these two points, but the tangent lines are not parallel, then the plane containing them is a bitangent plane and there are no bitangent spheres. $\square$

Proposition 4.1 motivates the following definition of the midlocus associated to a smooth space curve.

Definition 4.2. *When constructing the midlocus of a space curve γ we use* all *the pairs of points p_1, p_2: the midlocus M is the* image of the midpoint map $m : I \times J \to \mathbb{R}^3$, *where I and J are open intervals of real numbers, if we consider two disjoint curves γ_1, γ_2, or $I = J = S^1$ if we consider a single closed curve γ. In this case call M the* midpoint surface.

Note that M is a compact closed surface M with boundary on the generating space curves, and that it will in general have singularities. Note also that the construction of M, unlike that of the midlocus of a plane curve, is *affinely invariant*.

Remark 4.3. *When $p_2 \to p_1$ in the Proposition 4.1 the bitangent sphere, if there is one, will in the limit have (at least) 4-point contact with γ at p_1 and hence will be the unique sphere of curvature with centre*

$$\gamma(t_1) + \frac{1}{\kappa(t_1)} N(t_1) - \frac{\kappa'(t_1)}{\kappa^2(t_1)\tau(t_1)} B(t_1),$$

provided $\kappa(t_1)$ and $\tau(t_1)$ are nonzero. (See[4, §2.34].)

The simple singularities of map germs $(\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ have been classified by Mond [13]. As an application of Mond's classification we give the geometric conditions for the midlocus surface to have a crosscap singularity (resp. $S_1^{\pm}$ singularity) with normal form (x, xy, y^2) (resp. $(x, y^2, y(x^2 \pm y^2))$). We present the criteria for a surface in $\mathbb{R}^3$ to have such singularities and for more details we refer reader to [15]. If a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ has a corank one singularity at 0, then there exist two independent vector fields ξ and η near the origin satisfying $df_0(\eta_0) = 0$ and $\xi_0, \eta_0 \in T_0\mathbb{R}^2$. The function which plays a central role for the criteria is defined by $\varphi : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $\varphi = \det(\xi f, \eta f, \eta\eta f) = (\xi f \wedge \eta f) \cdot \eta\eta f$, where ζf is the directional derivative of f by ζ.

Criteria 4.4. *[15] Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map germ and 0 a corank one singular point. Then*

1. *f at 0 is $\mathcal{A}$-equivalent to the crosscap if and only if $\xi\varphi(0) \neq 0$.*
2. *f at 0 is $\mathcal{A}$-equivalent to S_1^- if and only if φ has a critical point at 0, and $\det(\mathrm{Hess}\varphi(0)) > 0$.*
3. *f at 0 is $\mathcal{A}$ equivalent to S_1^+ if and only if φ has a critical point at 0 and $\det(\mathrm{Hess}\varphi(0)) < 0$ and the vectors $\xi f(0)$ and $\eta\eta f(0)$ are linearly independent.*

Through the rest of this article the curvature and torsion of the curve γ_i are denoted by κ_i and τ_i respectively. Moreover, the Serret- Frenet frame of γ_i is denoted by $\{T_i, N_i, B_i\}$, where T_i, N_i and B_i are the unit tangent, the unit principle normal and the unit binormal respectively.

Lemma 4.5. *Let γ_1 and γ_2 be two regular space curves. If $T_1 = \pm T_2$, then $N_1 \cdot B_2 = \mp N_2 \cdot B_1$ and $N_1 \cdot N_2 = \pm B_1 \cdot B_2$, where $\{T_i, N_i, B_i\}$ is the Serret - Frenet frame of γ_i, $i = 1, 2$.*

Proof. The proof of this Lemma is obvious. □

Before we state and prove the main results of this section, which are related to the singularities of the midlocus of a space curve, we state the essential lemma.

Lemma 4.6. *Let M be the midlocus associated to a smooth space curve γ with non-vanishing curvature.*

1. *The midlocus is smooth at $M(t_1, t_2)$ if and only if the tangents of γ at $\gamma(t_1)$ and $\gamma(t_2)$ are not parallel.*
2. *The midlocus is parametrized by a corank one singularity at $M(t_1, t_2)$ if and only if the tangents of γ at $\gamma(t_1)$ and $\gamma(t_2)$ are parallel.*

Proof. The proof of this lemma is obvious. □

Theorem 4.7. *Let M be the midlocus associated to a smooth space curve γ with non-vanishing curvature. If γ has parallel tangents at t_1 and t_2, then the midlocus has a crosscap singularity at the mid-point of the the chord joining $\gamma(t_1)$ and $\gamma(t_2)$ if and only if*
$N(t_1) \cdot B(t_2) \neq 0$. That means γ does not have parallel Serret - Frenet frames at $\gamma(t_1)$ and $\gamma(t_2)$.

Proof. To prove this theorem we use Criteria 4.4. Let $T(t_1) = -T(t_2)$ and consider two pieces γ_1 and γ_2 of γ around t_1 and t_2. We parameterize γ_1 by its arc-length s and γ_2 by its arc-length t such that $t_1 = 0$ and $t_2 = 0$ in the new coordinates. The midlocus associated to γ_1 and γ_2 is defined by $M = \frac{1}{2}(\gamma_1 + \gamma_2)$. By our assumption we have $T_1(0) = -T_2(0)$ and in this case M is singular at $(0, 0)$. Since $dM_0(\eta_0) = 0$ we choose $\eta = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$ and $\xi = \frac{\partial}{\partial s} - \frac{\partial}{\partial t}$. We define the function $\varphi = det(\xi M, \eta M, \eta\eta M)$. Direct calculations show that $\xi M = \frac{1}{2}(T_1 - T_2)$, $\eta M = \frac{1}{2}(T_1 + T_2)$ and $\eta\eta M = \frac{1}{2}(\kappa_1 N_1 + \kappa_2 N_2)$. Thus

$$\begin{aligned}\varphi &= det(\xi M, \eta M, \eta\eta M)\\ &= (\xi M \wedge \eta M)\cdot \eta\eta M\\ &= \frac{1}{4}(\kappa_2 T_1 \cdot B_2 - \kappa_1 T_2 \cdot B_1).\end{aligned}$$

M has a crosscap singularity at $(0, 0)$ if and only if $\xi\varphi \neq 0$. $\xi\varphi = \dfrac{\partial\varphi}{\partial s} - \dfrac{\partial\varphi}{\partial t}$ and direct calculations show that

$$\xi\varphi = \frac{1}{4}(\kappa_1\kappa_2 N_1 \cdot B_2 - \kappa_1' T_2 \cdot B_1 + \kappa_1\tau_1 T_2 \cdot N_1) - \frac{1}{4}(\kappa_2' T_1 \cdot B_2 - \kappa_2\tau_2 T_1 \cdot N_2 - \kappa_1\kappa_2 N_2 \cdot B_1).$$

At $s = 0$ and $t = 0$ we have $T_1 = -T_2$ thus

$$\xi\varphi|_{(0,0)} = \frac{\kappa_1\kappa_2}{4}(N_1 \cdot B_2 + N_2 \cdot B_1)$$

and from Lemma 4.5 we have $N_1 \cdot B_2 = N_2 \cdot B_1$. Therefore, $\xi\varphi|_{(0,0)} \neq 0$ if and only if $N_1 \cdot B_2 \neq 0$. Similarly, we prove the results when $T(t_1) = T(t_2)$, in this case we choose $\eta = \frac{\partial}{\partial s} - \frac{\partial}{\partial t}$, and $\xi = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$. □

Remark 4.8. *From the Theorem 4.7 and its proof it can be easily shown that if the space curve γ has a parallel tangents at $\gamma(t_1)$ and $\gamma(t_2)$ and γ has zero curvature at $\gamma(t_1)$ or at $\gamma(t_2)$, then the midlocus does not have a crosscap singularity.*

Now assume that γ has non-vanishing curvature and the midlocus does not have a crosscap singularity. In this case we have $N(t_1) \cdot B(t_2) = 0$. We will give the geometric conditions for the midlocus to have $S_1^{\pm}$ singularities and to do so we are going to use Criteria 4.4. Before starting our aim in the rest of this section we state the following elementary lemma.

Lemma 4.9. *Let γ_1 and γ_2 be two regular space curves. Suffix 1 or 2 refers to the curve γ_1 or γ_2 respectively.*

1. *If $T_1 = -T_2$ and $N_1 \cdot B_2 = 0$, then one and only one of the following is true*
   (a) $N_1 = -N_2$ and $B_1 = B_2$. (b) $N_1 = N_2$ and $B_1 = -B_2$.
2. *If $T_1 = T_2$ and $N_1 \cdot B_2 = 0$, then one and only one of the following is true*
   (c) $N_1 = N_2$ and $B_1 = B_2$. (d) $N_1 = -N_2$ and $B_1 = -B_2$.

Now we state the main theorem of the rest of this section.

Theorem 4.10. *Let M be the midlocus associated to a smooth space curve γ with curvature κ and torsion τ. Suppose that γ has parallel tangents at t_1 and t_2 and $N(t_1) \cdot B(t_2) = 0$.*

1. *If $T_1 = -T_2$, then M has an S_1^+ singularity if and only if*

$$\tau_1\tau_2(\kappa_1^2 + \kappa_2^2)B_1 \cdot B_2 + \kappa_1\kappa_2(\tau_1^2 + \tau_2^2) > 0.$$

2. *If $T_1 = T_2$, then M has an S_1^+ singularity if and only if*

$$\tau_1\tau_2(\kappa_1^2 + \kappa_2^2)B_1 \cdot B_2 - \kappa_1\kappa_2(\tau_1^2 + \tau_2^2) < 0.$$

3. *If $T_1 = -T_2$, then M has an S_1^- singularity if and only if*

$$\tau_1\tau_2(\kappa_1^2 + \kappa_2^2)B_1 \cdot B_2 + \kappa_1\kappa_2(\tau_1^2 + \tau_2^2) < 0.$$

4. *If $T_1 = T_2$, then M has an S_1^- singularity if and only if*

$$\tau_1\tau_2(\kappa_1^2 + \kappa_2^2)B_1 \cdot B_2 - \kappa_1\kappa_2(\tau_1^2 + \tau_2^2) > 0.$$

Proof. We will follow the same procedure of the proof of Theorem 4.7. Let $T_1 = -T_2$, then we have $\varphi = \frac{1}{4}(\kappa_2 T_1 \cdot B_2 - \kappa_1 T_2 \cdot B_1)$. Direct calculations show that

$$\varphi_s = \frac{1}{4}(\kappa_1\kappa_2 N_1 \cdot B_2 - \kappa_1' T_2 \cdot B_1 + \kappa_1\tau_1 T_2 \cdot N_1),$$

and

$$\varphi_t = \frac{1}{4}(\kappa_2' T_1 \cdot B_2 - \kappa_2\tau_2 T_1 \cdot N_2 - \kappa_1\kappa_2 N_2 \cdot B_1).$$

Now at $(0,0)$ we have $T_1 = -T_2$ and $N_1 \cdot B_2 = N_2 \cdot B_1 = 0$. Thus φ has a critical point at $(0,0)$. Also, we have

$$\begin{aligned}\varphi_{ss} = \frac{1}{4}\{&\kappa_2\kappa_1' N_1 \cdot B_2 - \kappa_2\kappa_1^2 T_1 \cdot B_2 + \kappa_2\kappa_1\tau_1 B_1 \cdot B_2\\ &- \kappa_1'' T_2 \cdot B_1 + 2\kappa_1'\tau_1 T_2 \cdot N_1 + \kappa_1\tau_1' T_2 \cdot N_1\\ &- \kappa_1^2\tau_1 T_1 \cdot T_2 + \kappa_1\tau_1^2 T_2 \cdot B_1\},\end{aligned}$$

$$\varphi_{ts} = \frac{1}{4}\{\kappa_1\kappa_2' N_1 \cdot B_2 - \kappa_1\kappa_2\tau_2 N_1 \cdot N_2 - \kappa_2\kappa_1' N_2 \cdot B_1 + \kappa_1\kappa_2\tau_1 N_1 \cdot N_2\},$$

and

$$\begin{aligned}\varphi_{tt} = \frac{1}{4}\{&\kappa_2'' T_1 \cdot B_2 - 2\kappa_2'\tau_2 T_1 \cdot N_2 - \kappa_2\tau_2' T_1 \cdot N_2\\ &+ \kappa_2^2\tau_2 T_1 \cdot T_2 - \kappa_2\tau_2^2 T_1 \cdot B_2 - \kappa_1\kappa_2' N_2 \cdot B_1\\ &+ \kappa_1\kappa_2^2 T_2 \cdot B_1 - \kappa_1\kappa_2\tau_2 B_1 \cdot B_2\},\end{aligned}$$

where $Z_1' = \dfrac{dZ_1}{ds}$ and $Z_2' = \dfrac{dZ_2}{dt}$. Now at $s = 0, t = 0$ we have $T_1 = -T_2$ and $B_1 \cdot N_2 = B_2 \cdot N_1 = 0$, thus we have $\varphi_{ss} = \dfrac{\kappa_1\tau_1}{4}(\kappa_2 B_1 \cdot B_2 - \kappa_1 T_1 \cdot T_2)$, $\varphi_{ts} = \dfrac{\kappa_1\kappa_2}{4}(\tau_1 - \tau_2)N_1 \cdot N_2$ and $\varphi_{tt} = \dfrac{\kappa_2\tau_2}{4}(\kappa_2 T_1 \cdot T_2 - \kappa_1 B_1 \cdot B_2)$. Therefore, $\varphi_{ss} = \dfrac{\kappa_1\tau_1}{4}(\kappa_2 B_1 \cdot B_2 + \kappa_1)$, and $\varphi_{tt} = \dfrac{-\kappa_2\tau_2}{4}(\kappa_2 + \kappa_1 B_1 \cdot B_2)$. The necessary and sufficient condition for the midlocus to have an S_1^+ singularity is $\varphi_{ss}\varphi_{tt} - \varphi_{ts}^2 < 0$ if and only if

$$-\kappa_1\kappa_2\{\tau_1\tau_2(\kappa_2 B_1 \cdot B_2 + \kappa_1)(\kappa_2 + \kappa_1 B_1 \cdot B_2) + \kappa_1\kappa_2(\tau_1 - \tau_2)^2\} < 0$$

if and only if

$$\kappa_1\kappa_2\{\tau_1\tau_2(\kappa_2 B_1 \cdot B_2 + \kappa_1)(\kappa_2 + \kappa_1 B_1 \cdot B_2) + \kappa_1\kappa_2(\tau_1 - \tau_2)^2\} > 0.$$

Also, the condition for the midlocus to have an S_1^- singularity is $\varphi_{ss}\varphi_{tt} - \varphi_{ts}^2 > 0$ if and only if

$$\kappa_1\kappa_2\{\tau_1\tau_2(\kappa_2 B_1 \cdot B_2 + \kappa_1)(\kappa_2 + \kappa_1 B_1 \cdot B_2) + \kappa_1\kappa_2(\tau_1 - \tau_2)^2\} < 0.$$

Similarly we prove the results when $T_1 = T_2$ and in this case $\eta = \frac{\partial}{\partial s} - \frac{\partial}{\partial t}$ and $\xi = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$. Thus $\varphi = \frac{1}{4}(\kappa_1 T_2 \cdot B_1 - \kappa_2 T_1 \cdot B_2)$. Therefore, by the same procedure of the first case we prove the results. □

Now we present examples to illustrate our results in section 4.

5. Examples

(1) Let $\gamma(t) = (\cos t, \sin t, \sin 2t)$. (We can for example change the third coordinate to $\sin 2t + a\cos t + b\sin t$, $a, b \in \mathbb{R}$ (an affine transformation of $\mathbb{R}^3$) without affecting the results.) Then it is easy to show that parallel tangents occur exactly for $(t_1, t_2) = (\pm\frac{1}{4}\pi, \mp\frac{3}{4}\pi)$, and that the binormals at these four points are parallel to:

$$t = \pm\tfrac{1}{4}\pi : (\mp 2\sqrt{2}, -2\sqrt{2}, 1); \quad t = \pm\tfrac{3}{4}\pi : (\pm 2\sqrt{2}, 2\sqrt{2}, 1).$$

Hence the binormals at the parallel tangent pairs are not parallel and using Theorem 4.7 M will have a crosscap singularity at each point. The midpoint surface M is shown in Figure 3.

(2) In order to give examples of the non-crosscap cases it is easier to work locally, that is consider two segments of curve, say

$$\gamma_1(t) = (x, y, z) = (t, t^2, t^3); \quad \gamma_2(u) = (x, y, z) = (au, bu^2, 1 + cu^3),$$

for t, u close to 0. These curves have parallel tangent lines $y = z = 0$ and parallel osculating planes $z = 0$. The binormals, curvature and torsion at the basepoints $t = 0, u = 0$ are:

$$B_1 = (0, 0, 1),\ \kappa_1 = 2, \tau_1 = 3; B_2 = (0, 0, \mathrm{sign}(ab)), \kappa_2 = \frac{2|b|}{a^2}, \tau_2 = \frac{3c}{ab}.$$

Therefore, If we take $\gamma_1(t) = (t, t^2, t^3)$ and $\gamma_2(u) = (2u, -u^2, \frac{-1}{9}u^3 + 1)$, then the associated midpoint of γ_1 and γ_2 has an S_1^+ singularity at $(0, 0)$. If we take $\gamma_1(t) = (t, t^2, t^3)$ and $\gamma_2(u) = (\frac{1}{2}u, u^2, \frac{1}{2}u^3 + 1)$, then the associated midpoint of γ_1 and γ_2 has an S_1^- singularity at $(0, 0)$ see Figure 4.

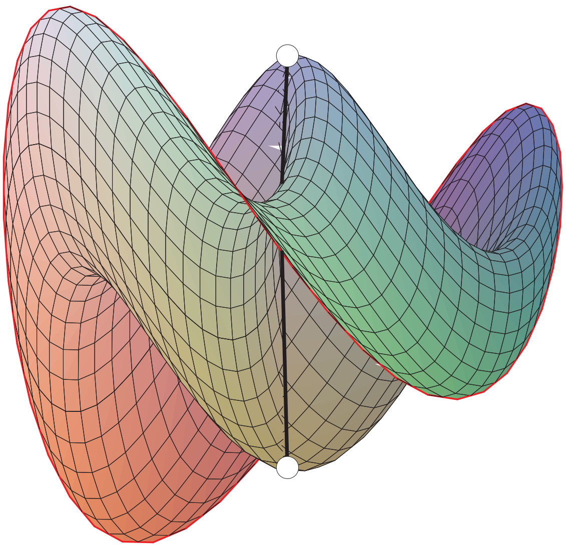

Figure 3: The midpoint surface for the curve in Example (1) which has two crosscaps, when $t \in (0, 2\pi)$. Two crosscaps marked by a white circle.

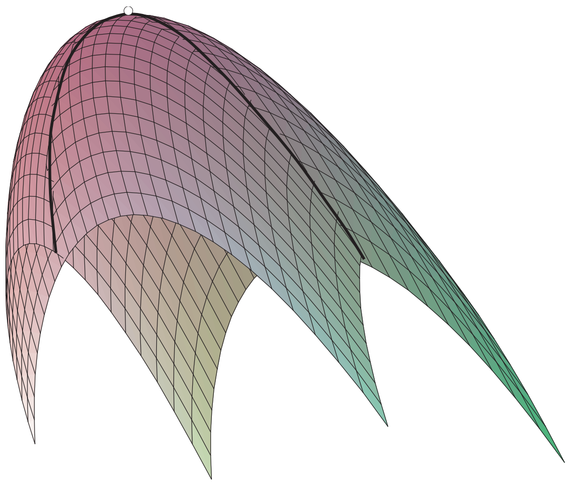

Figure 4: The midpoint surface for the curves in Example (2) which has an S_1^- singularity. The self-intersection curve is emphasized by a dark line.

6. λ-point map

In this section we study the λ-point map associated to space curves which is more general than the midpoint map. Our main task in this section is to study the singularity of this map and to recognize the special values of λ. The λ-point map associated to two regular space curves γ_1 and γ_2 (or one curve) is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$M(t_1, t_2) = (1-\lambda)\gamma_1(t_1) + \lambda\gamma_2(t_2). \tag{1}$$

In [17] the author classifies the local singularities of the envelope of this 2-parameter family of chords, calling it the chord set. Away from γ_1 and γ_2 themselves this is the ruled surface consisting of lines joining points p_1 and p_2 of γ_1 and γ_2 for which the tangents at p_1 and p_2 and the chord are coplanar. This contrasts with our investigation which studies the locus of points at a fixed ratio along the chords. It is obvious for the reader to recognize that the image of the λ-point map is γ_1 when $\lambda = 0$ and γ_2 when $\lambda = 1$. In our case we assume that $\lambda \neq 0, 1$ and this will be taken in the rest of this section. Without loss of generality we may assume that γ_1 and γ_2 are parametrized by their arc-lengths s and t respectively. It is clear that M is singular at $M(s_0, t_0)$ if and only $T_1(s_0)$ and $T_2(t_0)$ are parallel. By similar calculations to those in section 4 we have the following result.

Theorem 6.1. *Let γ_1 and γ_2 be two regular space curves with non-vanishing curvatures such that $T_1(s_0) = \pm T_2(t_0)$. The λ- point map given by equation (1) is $\mathcal{A}$-equivalent to crosscap if and only if the osculating planes of γ_1 and γ_2 at $\gamma_1(s_0)$ and $\gamma_2(t_0)$ are not parallel.*

This theorem tells us that when the osculating planes are not parallel then all values of λ ($\neq 0, 1$) give the same map up to $\mathcal{A}$-equivalence.
In the following we study the case when $T_1(0) = -T_2(0)$ and the osculating planes are parallel; the case $T_1(0) = T_2(0)$ is similar. If γ_1 and γ_2 have non-vanishing curvatures and torsion, then by a similar method used in Theorem 4.10, the determinant of the Hessian of the function φ at $(0, 0)$ is given by

$$\rho = -\Big(\frac{1-\lambda}{\lambda}\Big)^2 \kappa_1\kappa_2 \left\{\tau_1\tau_2\left(\kappa_1^2 + \kappa_2^2\Big(\frac{1-\lambda}{\lambda}\Big)^2\right) B_1 \cdot B_2 + \kappa_1\kappa_2\left(\tau_1^2 + \tau_2^2\Big(\frac{1-\lambda}{\lambda}\Big)^2\right)\right\}. \tag{2}$$

Using criteria 4.4, the λ-point map is $\mathcal{A}$-equivalent to $S_1^{\pm}$ if and only if $\rho \neq 0$. The interesting question rises now when $\rho = 0$ is, which type of singularity can occur? It is obvious from equation (2) that $\rho = 0$ if and only if

$$\Big(\frac{1-\lambda}{\lambda}\Big)^2 = -\delta\frac{\kappa_1\tau_1}{\kappa_2\tau_2}, \tag{3}$$

where δ is the sign of $(B_1 \cdot B_2)$.

Definition 6.2. *The values of λ given by equation (3) will be called* special values *of λ when the osculating planes are parallel.*

From Lemma (4.9), when the osculating planes are parallel, $B_1 = \pm B_2$. Therefore, the existence of the special values of λ depends on the signs of τ_1 and τ_2. The following remark gives the situation when the special values of λ exist.

Remark 6.3. *Let γ_1 and γ_2 be two regular space curves with non-vanishing curvatures and torsions. Let $T_1(0) = -T_2(0)$ and γ_1 and γ_2 have parallel osculating planes at $\gamma_1(0)$ and $\gamma_2(0)$.*

1. *If $B_1(0) = B_2(0)$, then the special values of λ exist if and only if $\tau_1(0)$ and $\tau_2(0)$ have opposite signs.*
2. *If $B_1(0) = -B_2(0)$, then the special values of λ exist if and only if $\tau_1(0)$ and $\tau_2(0)$ have the same sign.*

Now we have the following theorem.

Theorem 6.4. *Let γ_1 and γ_2 be two regular space curves with non-vanishing curvatures and torsions (at $s_0 = 0$, and $t_0 = 0$). If $T_1(0) = -T_2(0)$ and the two curves have parallel osculating planes at $\gamma_1(0)$ and $\gamma_2(0)$, then away from the special values of λ the λ-point map at $M(0, 0)$ is $\mathcal{A}$-equivalent to $S_1^{\pm}$.*

This theorem tells us that the type of singularity of the λ-point map, when $T_1(0) = -T_2(0$ and the two curves have parallel osculating planes at $\gamma_1(0)$ and $\gamma_2(0)$, is always $S_1^{\pm}$ at all values of λ except at values of λ satisfying equation (3). For this reason we call the values of λ satisfy equation (3), the special values of λ. In the rest of this section our task is to classify the type of singularity of the λ-point map when λ reaches its special values. Now we use the results of Mond ([13]) to classify the type of singularity of the λ-point map at the special values of λ. Consider two curves γ_1 and γ_2. By an affine transformation we may assume that γ_1 and γ_2 have the form

$$\gamma_1(t) = (t, a_2^2t^2 + a_3t^3 + a_4t^4 + \ldots, b_3^2t^3 + b_4t^4 + b_5t^5 + \ldots) \tag{4}$$

$$\gamma_2(u) = (p - u, q + c_2^2u^2 + c_3u^3 + v_4u^4 + \ldots, r - d_3^2u^3 + d_4u^4 + d_5u^5 + \ldots). \tag{5}$$

Direct calculations show that $T_1(0) = -T_2(0)$, $B_1(0) = -B_2(0)$. For the purpose of calculation we may assume that $b_3 > 0$, $d_3 > 0$, and $d_3 \neq b_3$. In this case the special values of λ are given by $\lambda = \dfrac{d_3}{d_3 \pm b_3}$. In the following we study the case when $\lambda = \dfrac{d_3}{d_3 + b_3}$. By appropriate variable changes in the source and suitable coordinates changes in the target, we find the following proposition.

Proposition 6.5. *Assume that γ_1 and γ_1 are as in equations (4) and (5). If $\lambda = \dfrac{d_3}{d_3 + b_3}$, then the 5-jet of the λ-point map is $\mathcal{A}$-equivalent to*

$$j^5M = (x, y^2, a_{21}x^2y + a_{13}xy^3 + a_{31}x^3y + a_{41}x^4y + a_{23}x^2y^3 + a_{05}y^5). \tag{6}$$

In the appendix we will give a geometric interpretation of the coefficients of the third component of j^5M in terms of curvatures and torsions of γ_1 and γ_2. Now we state the following theorem which was proved by Mond ([13]).

Theorem 6.6. *[13] A map germ $\Omega : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ with $j^2\Omega = (x, y^2, 0)$ is $\mathcal{A}$-equivalent to a germ of the form $(x, y^2, yF(x, y^2))$, for smooth $F(x, y^2)$.*

The following corollary gives the normal form of the λ-point map at the special values of λ.

Corollary 6.7. *Let γ_1 and γ_2 be two regular space curves with non-vanishing curvatures and torsions. Let $T_1(0) = -T_2(0)$ and γ_1 and γ_2 have parallel osculating planes at $\gamma_1(0)$ and $\gamma_2(0)$. The λ-point map at the special values of λ is $\mathcal{A}$-equivalent to a germ of the form $(x, y^2, yF(x, y^2))$, for smooth $F(x, y^2)$.*

Proof. From equation (6), the second jet of the λ-point map is given by $j^2M = (x, y^2, 0)$. Therefore, using Theorem (6.6) the result holds. □

The coefficient a_{21} plays a central role in the type of classification of the λ-point map. We use equation (6) to give the normal form of the λ-point map. Precisely, we give the condition for this map to be $\mathcal{A}$-equivalent to $B_2^{\pm}$, $C_3^{\pm}$, F_4, and $C_4^{\pm}$ with normal forms $(x, y^2, x^2y \pm y^5)$, $(x, y^2, xy^3 \pm x^3y)$, $(x, y^2, x^3y + y^5)$, and $(x, y^2, xy^3 \pm x^4y)$ respectively. Recall that $C_3^{\pm}$ is 4-determined, and the others are 5-determined and for more details in this subject we refer the reader to ([11, 13]).

Case 1 $a_{21} \neq 0$

If $a_{21} \neq 0$, then after suitable coordinates change in the target j^5M can be transformed to $j^5M = (x, y^2, a_{21}x^2y + a_{13}xy^3 + a_{05}y^5)$. Therefore, j^5M is $\mathcal{A}$-equivalent to $B_2^{\pm}$ if and only if $4a_{05}a_{21} - a_{13}^2 \neq 0$.

Case 2 $a_{21} = 0$

In this case the fourth jet of the λ-point map is given by $j^4M = (x, y^2, a_{13}xy^3 + a_{31}x^3y)$. Therefore, j^4M is $\mathcal{A}$-equivalent to $C_3^{\pm}$ if and only if $a_{13} \neq 0$ and $a_{31} \neq 0$. Thus M is $\mathcal{A}$-equivalent to $C_3^{\pm}$ if and only if $a_{13} \neq 0$ and $a_{31} \neq 0$. Now assume that $a_{13} = 0$, then the fifth jet of M is given by $j^5M = (x, y^2, a_{31}x^3y + a_{41}x^4y + a_{23}x^2y^3 + a_{05}y^5)$.

If $a_{31} \neq 0$, then j^5M can be transformed to $j^5M = (x, y^2, a_{31}x^3y + a_{23}x^2y^3 + a_{05}y^5)$. Therefore, j^5M is $\mathcal{A}$-equivalent to F_4 if and only if $a_{05} \neq 0$. Now assume that $a_{31} = 0$. If $a_{13} \neq 0$, then j^5M can be transformed to $j^5M = (x, y^2, a_{13}xy^3 + a_{41}x^4y + a_{05}y^5)$. Therefore, j^5M is $\mathcal{A}$-equivalent to $C_4^{\pm}$ if and only if $a_{41} \neq 0$. We summarize this discussion in the following theorem.

Theorem 6.8. *Let γ_1 and γ_2 be two regular space curves with non-vanishing curvatures and torsions (necessarily at $t_0 = 0$, and $u_0 = 0$). If $T_1(0) = -T_2(0)$ and the two curves have parallel osculating planes at $\gamma_1(0)$ and $\gamma_2(0)$. At the special values of λ, we have the following.*

1. *If $a_{21} \neq 0$, then M is $\mathcal{A}$-equivalent to $B_2^{\pm}$ if and only if $4a_{05}a_{21} - a_{13}^2 \neq 0$.*
2. *If $a_{21} = 0$, then M is $\mathcal{A}$-equivalent to $C_3^{\pm}$ if and only if $a_{13} \neq 0$ and $a_{31} \neq 0$.*
3. *If $a_{21} = a_{13} = 0$, then M is $\mathcal{A}$-equivalent to F_4 if and only if $a_{31} \neq 0$ and $a_{05} \neq 0$.*
4. *If $a_{21} = a_{31} = 0$, then M is $\mathcal{A}$-equivalent to $C_4^{\pm}$ if and only if $a_{13} \neq 0$ and $a_{41} \neq 0$.*

In the appendix we give the geometric interpretations of the coefficients a_{ij} in terms of the curvatures and torsions of γ_1 and γ_2. In the previous we discuss the possible singularities of the λ-point map when $\tau_1(0) \neq 0$ and $\tau_2(0) \neq 0$. The interesting question now is that what is the type of singularity does the λ-point map may have when $\tau_1(0) = 0$ or $\tau_2(0) = 0$ or $\tau_1(0) = \tau_2(0) = 0$?

Proposition 6.9. *Let γ_1 and γ_2 be two regular space curves with non-vanishing curvatures such that $T_1(s_0) = \pm T_2(t_0)$ and γ_1 and γ_2 have parallel osculating planes at $\gamma_1(0)$ and $\gamma_2(0)$. If $\tau_1(0) = 0$ or $\tau_2(0) = 0$, but not both zero, then the λ-point map is $\mathcal{A}$-equivalent to S_1^+.*

Proof. The proof of this proposition comes directly from equation (2) and Criteria (4.4). □

The following table is a summary of our results when the torsions are not both zero.

Type of singularity	Osculating planes are parallel	Special values	a_{21}	a_{13}	a_{31}	τ_1,τ_2	a_{41}	a_{05}	$4a_{21}a_{05} - a_{13}^2$
Crosscap	No	—	—	—	—	—	—	—	—
$S_1^{\pm}$	Yes	No	—	—	—	Not both zero	—	—	—
$B_2^{\pm}$	Yes	Yes	$\neq 0$	—	—	—	—	—	$\neq 0$
$C_3^{\pm}$	Yes	Yes	$= 0$	$\neq 0$	$\neq 0$	—	—	—	—
$C_4^{\pm}$	Yes	Yes	$= 0$	$\neq 0$	$= 0$	—	$\neq 0$	—	—
F_4	Yes	Yes	$= 0$	$= 0$	$\neq 0$	—	—	$\neq 0$	—

Table 1: This table is the summary of the classifications of λ-point map. The dash — means this term is not involved.

Remark 6.10. *If $\tau_1(0) = \tau_2(0) = 0$ that means $b_3 = d_3 = 0$ in equations (4) and (5). In this case there is another special values of λ. If $\dfrac{1-\lambda}{\lambda} \neq \dfrac{\kappa_1}{\kappa_2}$, then by appropriate variable changes in the source and suitable coordinates changes in the target, it can be shown that the fifth jet of the λ-point map is given by*

$$j^5M = (x, y^2, A_{13}xy^3 + A_{31}x^3y + A_{41}x^4y + A_{23}x^2y^3 + A_{05}y^5). \tag{7}$$

From this equation it is clear that the $B_2^{\pm}$ singularity is not possible for λ-point map when both torsions are zero, whereas the $C_3^{\pm}$, $C_4^{\pm}$ and F_4 singularities are possible.

Example 6.11. *Consider the two curves* $\gamma_1(t) = (t, 4t^2 + 3t^3 - 2t^4 - 5t^5 + 2t^6, 4t^4 - 8t^5 - 2t^6 + 6t^7)$ *and* $\gamma_2(u) = (3 - u, 2 + 9u^2 - 6u^3 - 7u^4 + 3u^5 + 12u^6 + 4u^7, 1 + 6u^4 + u^5 - u^6 + 5u^7)$. *The associated* λ*-point map to these curves* $M(t, u) = (1 - \lambda)\gamma_1(t) + \lambda\gamma_2(u)$ *at* $M(0, 0)$ *is* $\mathcal{A}$*-equivalent to* C_3^+ *when* $\lambda = \frac{1}{2}$ *and to* C_3^- *when* $\lambda = \frac{1}{3}$.

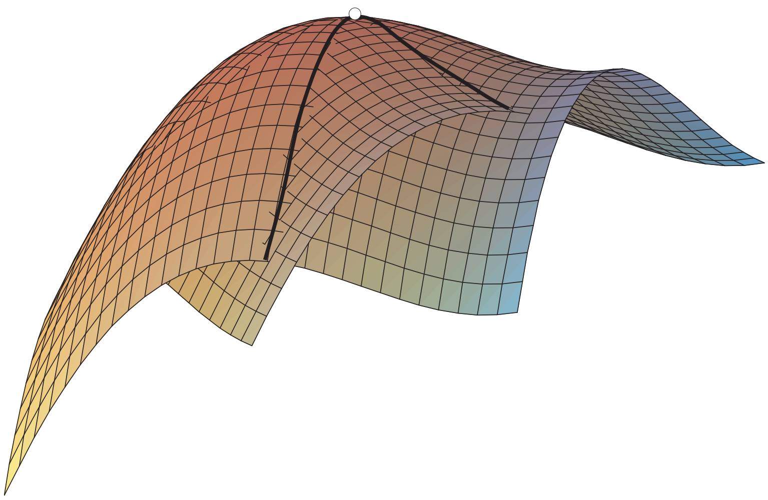

Figure 5: The λ-point map in example 6.11 when $\lambda = \frac{1}{3}$. The self-intersection curve is emphasized by a dark line.

7. Appendix

In this appendix we express the coefficients of the 5-jet of the λ-point map appear in Theorem (6.8) in terms of the curvatures, torsions and their derivatives. Calculations show that the Taylor expansion of the curvature and torsion of γ_1 in terms of the arc-length are given by.

$$\begin{cases} \kappa_1(s) = 2a_2^2 + 6a_3 s_1 - \frac{3(4a_2^8 - 3b_3^4 - 2a_2^2 a_4)}{a_2^2} s_1^2 - \frac{27a_3 b_3^4 - 36a_2^2 b_3^2 b_4 - 20a_2^4 a_5 - 76a_2^8 a_3}{a_2^4} s_1^3 + \dots \\ \tau_1(s) = \frac{3b_3^2}{a_2^2} + \frac{6(2a_2^2 b_4 - 3a_3 b_3^2)}{a_2^4} s + \frac{3(10a_2^4 b_5 - 18b_4 a_3 a_2^2 - 18a_4 b_3^2 a_2^2 + 27a_3^2 b_3^2 - 9b_3^6)}{a_2^6} s^2 + \dots \end{cases} \tag{8}$$

Also, Taylor expansion of the curvature and torsion of γ_2 in terms of the arc-length are given by.

$$\begin{cases} \kappa_2(s_2) = 2c_2^2 + 6c_3 s_2 + \frac{3(4c_2^2 c_4 + 3d_3^4 - 4c_2^8)}{c_2^2} s_2^2 - \frac{(76c_2^8 c_3 - 20c_2^4 c_5 + 36c_2^2 d_3^2 d_4 + 27c_3 d_3^4)}{c_2^4} s_2^3 + \dots \\ \tau_2(s_2) = \frac{3d_3^2}{c_2^2} - \frac{6(2c_2^2 d_4 + 3c_3 d_3^2)}{c_2^4} s_2 - \frac{3(18c_2^2 c_4 d_3^2 - 18c_2^2 c_3 d_4 + 10c_2^4 d_5 + 9d_3^6 - 27c_3^2 d_3^2)}{c_2^6} s_2^2 + \dots \end{cases} \tag{9}$$

Using equation (8) we have the following expressions for the coefficients $a_2, a_3, a_4, a_5, b_3, b_4$ and b_5. All values are calculated at $s_1 = 0$

$$\begin{cases} a_2^2 = \frac{\kappa_1}{2}, \quad a_3 = \frac{\kappa_1'}{6}, \quad a_4 = \frac{\kappa_1'' - \kappa_1\tau_1^2 + 3\kappa_1^3}{24}, \quad a_5 = \frac{\kappa_1''' + 19\kappa_1^2\kappa_1' - 3\kappa_1\tau_1\tau_1' - 3\kappa_1'\tau_1^2}{120}. \\ b_3^2 = \frac{\kappa_1\tau_1}{6}, \quad b_4 = \frac{\kappa_1\tau_1' + 2\kappa_1'\tau_1}{24}, \quad b_5 = \frac{\kappa_1\tau_1'' + 3\kappa_1'\tau_1' + 3\kappa_1''\tau_1 + 9\kappa_1^3\tau_1 - \kappa_1\tau_1^3}{120}. \end{cases} \tag{10}$$

Also, from equation (9) at $s_2 = 0$, we have

$$\begin{cases} c_2^2 = \dfrac{\kappa_2}{2}, \quad c_3 = \dfrac{\kappa_2'}{6}, \quad c_4 = \dfrac{\kappa_2'' - \kappa_2\tau_2^2 + 3\kappa_2^3}{24}, \quad c_5 = \dfrac{\kappa_2''' + 19\kappa_2^2\kappa_2'}{120} - \dfrac{\kappa_2^2\tau_2\tau_2' + \kappa_2'\tau_2^2}{80}. \\ d_3^2 = \dfrac{\kappa_2\tau_2}{6}, \quad d_4 = \dfrac{-\kappa_2^2\tau_2' - 4\kappa_2'\tau_2}{48}, \quad d_5 = \dfrac{\kappa_2\tau_2^3 - \kappa_2\tau_2'' - 3\kappa_2''\tau_2 - 9\kappa_2^3\tau_2}{120} - \dfrac{\kappa_2\kappa_2'\tau_2'}{80}. \end{cases} \tag{11}$$

In calculating j^5M we use the Maple, and the coefficients of j^5M are given by

$$\begin{cases} a_{21} = -3\,\dfrac{(d_3+b_3)d_3\left({a_2}^2d_3-{c_2}^2b_3\right)}{{a_2}^2d_3+{c_2}^2b_3}. \\ a_{13} = -\dfrac{d_3\left(-4\,b_4{d_3}^2{c_2}^2+4\,{a_2}^2d_4{b_3}^2+3\,d_3{b_3}^3c_3+3\,{d_3}^3b_3a_3\right)}{{b_3}^2({a_2}^2d_3+{c_2}^2b_3)}. \\ a_{31} = -\dfrac{1}{2}\,\dfrac{(d_3+b_3)^2d_3\left(-9\,{a_2}^2c_3{c_2}^2{b_3}^2+27\,b_3{a_2}^4c_3d_3+27\,b_3a_3{c_2}^4d_3+8\,{a_2}^6d_4-9\,a_3{d_3}^2{c_2}^2{a_2}^2-8\,b_4{c_2}^6\right)}{({a_2}^2d_3+{c_2}^2b_3)^3}. \\ a_{05} = \dfrac{d_3\left(d_5{b_3}^4{a_2}^2d_3+d_5{b_3}^5{c_2}^2+b_5{d_3}^5{a_2}^2+b_5{d_3}^4{c_2}^2b_3-2\,d_4{b_3}^5c_3-2\,d_4{b_3}^3a_3{d_3}^2-2\,b_4{d_3}^3c_3{b_3}^2-2\,b_4{d_3}^5a_3\right)}{{b_3}^4(d_3+b_3)({a_2}^2d_3+{c_2}^2b_3)}. \end{cases} \tag{12}$$

Calculations show that the coefficient a_{41} is a long equation, but when $a_{21} = 0$, then a_{41} can be simplified to

$$\begin{cases} a_{41} = \dfrac{1}{64}\,\dfrac{\left({c_2}^2+{a_2}^2\right)^3}{{a_2}^{10}{c_2}^{10}}[27\,{a_2}^{10}{c_3}^2{d_3}^2 + 48\,{a_2}^{10}{d_3}^2c_4{c_2}^2 \\ \qquad + 20\,{a_2}^{10}d_5{c_2}^4 + 162\,c_3a_3{d_3}^2{c_2}^4{a_2}^6 \\ \qquad - 72\,{a_2}^4b_4c_3{c_2}^8 - 48\,{a_2}^4a_4{d_3}^2{c_2}^8 \\ \qquad + 135\,{a_2}^2{a_3}^2{d_3}^2{c_2}^8 + 20\,{a_2}^2b_5{c_2}^{12} \\ \qquad - 72\,b_4{c_2}^{12}a_3]. \end{cases} \tag{13}$$

Using equations (10) and (11), $a_{21}, a_{13}, a_{31}, a_{41}$ and a_{05} can be expressed in terms of κ_1, κ_2 τ_1, τ_2 and their derivatives.

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