Filomat 32:8 (2018), 2995–3030
https://doi.org/10.2298/FIL1808995S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New View on AG-Groupoid Theory via Soft Sets for Uncertainty Modeling

Aslıhan Sezgin[a]

[a] *Department of Elementary Education, Amasya University, 05100 Amasya, Turkey*

Abstract. In this paper, the concepts soft union AG-groupoids, soft union left (right, two-sided) ideals, (generalized) bi-ideals, interior and quasi-ideals in AG-groupoids are introduced with many illustrating examples and their properties and interrelations are given. Moreover, regular, intra-regular, completely regular, weakly regular and quasi-regular AG-groupoids are characterized by the properties of these soft union ideals.

1. Introduction

In 1999, Molodtsov [1] proposed an approach for modeling vagueness and uncertainty, called soft set theory. Since its inception, works on soft set theory has been progressing rapidly with a wide range-applications especially in the mean of algebraic structures and it has provided a naturel framework for generalizing several basic notions of algebra such as groups [2, 3], semirings [4], rings [5], BCK/BCI-algebras [6–8], BL-algebras [9], near-rings [10] and soft substructures and union soft substructures [11, 12].

The structures of soft sets, operations of soft sets and some related concepts have been studied since 1999. Maji et al. [13] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [14] introduced several operations of soft sets and Sezgin and Atagün [15] and Ali et al. [16] studied on soft set operations as well.

Moreover, the theory of soft set continues to experience tremendous growth and diversification such as computer science and soft decision making as in the following studies:[17–19, 21–23] and some other fields as [24–26].

In [27, 28], Sezer studied soft LA-semigroups with the concept of soft intersection ideals. However in this paper, a new approach to the AG-groupoid theory via soft set theory with the concept of soft union AG-groupoids and soft union ideals of AG-groupoids are made. First, some basic definitions about soft sets, AG-groupoids, soft union product and soft characteristic function are reminded. Then, soft union AG-groupoids, soft union left (right, two-sided) ideals, (generalized) bi-ideals, interior ideals, quasi-ideals in AG-groupoid are introduced and studied with respect to soft set operations and soft union product. In the following sections, regular, intra-regular, completely regular, weakly regular and quasi-regular AG-groupoids are characterized by the properties of these soft union ideals.

2010 *Mathematics Subject Classification*. Primary 03EE99; Secondary 03E75

Keywords. Soft set, Soft union left (right, two-sided, interior, quasi) ideal, Soft union (generalized)-bi ideal, Regular AG-gropoids.
Received: 30 June 2014; Accepted: 09 September 2018
Communicated by Dragan S. Djordjević

Email address: aslihan.sezgin@amasya.edu.tr (Aslıhan Sezgin)

2. Preliminaries

The idea of generalization of a commutative semigroup (which we call the left almost semigroup) was introduced by M.A. Kazim and M. Naseeruddin in 1972 [29]. They introduced braces on the left of the tenrary commutative law $abc = cba$ to get a new pseudo associative law, that is

$$(ab)c = (cb)a \text{ for all } a, b, c.$$

It is since then called the left invertive law. A groupoid satisfying the left invertive law is called a left almost semigroup and is abbreviated by LA-semigroup. P. Holgate call it simple invertive groupoid [30]. It is also known as Abel-Grassmann's groupoid [31]. It is a mid structure between a groupoid and a commutative semigroup, having many applications in the theory of flocks [32]. From now on, S denotes an AG-groupoid. There can be a unique left identity in an AG-groupoid [33]. For further information about AG-groupoids, fuzzy AG-groupoids, we invite the reader to [34–39]. From now on, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

Definition 2.1. *([1, 18]) A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

Note that throughout this paper, the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2. *[18] Let $f_A, f_B \in S(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Definition 2.3. *[18] Let f_A, $f_B \in S(U)$. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A\tilde{\cup}B}$, where $f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$. Intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A\tilde{\cap}B}$, where $f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.*

Definition 2.4. *[18] Let f_A, $f_B \in S(U)$. Then, $\wedge$-product of f_A and f_B, denoted by $f_A \wedge f_B$, is defined as $f_A \wedge f_B = f_{A\wedge B}$, where $f_{A\wedge B}(x, y) = f_A(x) \cap f_B(y)$ for all $(x, y) \in E \times E$.*

Definition 2.5. *[40] Let f_A and f_B be soft sets over the common universe U and Ψ be a function from A to B. Then, soft anti image of f_A under Ψ, denoted by $\Psi^\star(f_A)$, is a soft set over U by*

$$(\Psi^\star(f_A))(b) = \begin{cases} \bigcap\{f_A(a) \mid a \in A \;\; and \;\; \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & otherwise \end{cases}$$

for all $b \in B$. And soft pre-image (or soft inverse image) of f_B under Ψ, denoted by $\Psi^{-1}(f_B)$, is a soft set over U by $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$ for all $a \in A$.

Definition 2.6. *[41] Let f_A be a soft set over U and $\alpha \subseteq U$. Then, lower α-inclusion of f_A, denoted by $\mathcal{L}(f_A; \alpha)$, is defined as*

$$\mathcal{L}(f_A : \alpha) = \{x \in A \mid f_A(x) \subseteq \alpha\}.$$

3. Soft union product and soft characteristic function

In this section, soft union product and soft characteristic function is defined and their properties are studied. From now on, the soft sets the parameter set of which are restricted to S will be denoted by $S(S)$. In [42], Sezgin defined soft union product for soft sets the parameter set of which is a semigroup. Now, soft union product is defined for soft sets the parameter set of which is an AG-groupoid.

Definition 3.1. *Let f_S and g_S be soft sets over the common universe U. Then, soft union product $f_S * g_S$ is defined by*

$$(f_S * g_S)(x) = \begin{cases} \bigcap_{x=yz}\{f_S(y) \cup g_S(z)\}, & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ U, & \text{otherwise} \end{cases}$$

for all $x \in S$.

Note that soft union product is abbreviated by soft uni-product in what follows.

Example 3.2. *Consider the AG-groupoid $S = \{a, b, c, d\}$ defined by the following table:*

.	a	b	c	d
a	d	a	b	c
b	c	d	a	b
c	b	c	d	a
d	a	b	c	d

Let $U = D_3 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ be the universal set. Let f_S and g_S be soft sets over U such that $f_S(a) = \{e, x, yx\}$, $f_S(b) = \{e, x, y\}$, $f_S(c) = \{e, y, yx\}$, $f_S(d) = \{e, y\}$ and $g_S(a) = \{e, x, y, yx\}$, $g_S(b) = \{x, y, yx\}$, $g_S(c) = \{e, x\}$, $g_S(d) = \{x, y\}$. Since $d = aa = bb = cc = dd$, then

$$(f_S * g_S)(d) = \{f_S(a) \cup g_S(a)\} \cap \{f_S(b) \cup g_S(b)\} \cap \{f_S(c) \cup g_S(c)\} \cap \{f_S(d) \cup g_S(d)\} = \{e, x, y\}.$$

Similarly, $(f_S * g_S)(a) = \{e, x, y\}$, $(f_S * g_S)(b) = \{e, x\}$, $(f_S * g_S)(c) = \{e, x, y\}$.

The proof of the following theorem is similar to those in [42].

Theorem 3.3. *Let $f_S, g_S, h_S \in S(U)$. Then,*

i) $(f_S * g_S) * h_S = f_S * (g_S * h_S)$.
ii) $f_S * g_S \neq g_S * f_S$, *generally.*
iii) $f_S * (g_S \widetilde{\cup} h_S) = (f_S * g_S) \widetilde{\cup} (f_S * h_S)$ *and* $(f_S \widetilde{\cup} g_S) * h_S = (f_S * h_S) \widetilde{\cup} (g_S * h_S)$.
iv) $f_S * (g_S \widetilde{\cup} h_S) = (f_S * g_S) \widetilde{\cup} (f_S * h_S)$ *and* $(f_S \widetilde{\cup} g_S) * h_S = (f_S * h_S) \widetilde{\cup} (g_S * h_S)$.
v) *If* $f_S \tilde{\subseteq} g_S$, *then* $f_S * h_S \tilde{\subseteq} g_S * h_S$ *and* $h_S * f_S \tilde{\subseteq} h_S * g_S$.
vi) *If* $t_S, l_S \in S(U)$ *such that* $t_S \tilde{\subseteq} f_S$ *and* $l_S \tilde{\subseteq} g_S$, *then* $t_S * l_S \tilde{\subseteq} f_S * g_S$.

Proposition 3.4. *Let S be an AG-groupoid. Then, the set $((S(S), *)$ is an AG-groupoid.*

Proof. Obviously, $S(S)$ is closed. Let $f_S, g_S, h_S \in S(S)$. Let x be any element of S such that it is not expressible as the product of two elements in S. Then, we have

$$((f_S * g_S) * h_S)(x) = ((h_S * g_S) * f_S)(x) = U$$

Let s be the element that can be written as $s = yz$. Then, we have

$$\begin{aligned} ((f_S * g_S) * h_S)(s) &= \bigcap_{s=yz}\{(f_S * g_S)(y) \cup h_S(z)\} \\ &= \bigcap_{s=yz}\{\bigcap_{y=pq}\{f_S(p) \cup g_S(q)\} \cup h_S(z)\} \\ &= \bigcap_{s=(pq)z}\{f_S(p) \cup g_S(q) \cup h_S(z)\} \\ &= \bigcap_{s=(zq)p}\{h_S(z) \cup g_S(q) \cup f_S(p)\} \\ &= \bigcap_{s=wp}\{\bigcap_{w=zq}\{h_S(z) \cup g_S(q)\} \cup f_S(p)\} \\ &= \bigcap_{s=wp}\{(h_S * g_S)(w) \cup f_S(p)\} \\ &= ((h_S * g_S) * f_S)(s) \end{aligned}$$

Hence, $(S(S), *)$ is an AG-groupoid. $\square$

Proposition 3.5. *Let S be an AG-groupoid. Then, the medial law holds in $S(S)$.*

Proof. Let f_S, g_S, h_S, k_S be any elements of $S(S)$. Then, applying the left invertive law,

$$(f_S * g_S) * (h_S * k_S) = ((h_S * k_S) * g_S) * f_S = ((g_S * k_S) * h_S) * f_S = (f_S * h_S) * (g_S * k_S)$$

$\square$

Theorem 3.6. *Let S be an AG-groupoid with left identity and f_S, g_S, h_S be any elements of $S(S)$. Then, the following properties hold in $S(S)$:*

i) $f_S * (g_S * h_S) = g_S * (f_S * h_S)$.
ii) $(f_S * g_S) * (h_S * k_S) = (k_S * h_S) * (g_S * f_S)$.

Proof. *i*) Let $s \in S$. If s is not expressible as a product of two elements in S, then

$$(f_S * (g_S * h_S))(s) = (g_S * (f_S * h_S))(s) = U.$$

Otherwise, let there exist $y, z \in S$ such that $s = yz$. Then;

$$\begin{aligned}
(f_S * (g_S * h_S))(s) &= \bigcap_{s=yz}\{f_S(y) \cup (g_S * h_S)(z)\} \\
&= \bigcap_{s=yz}\{f_S(y) \cup \bigcap_{z=pq}\{g_S(p) \cup h_S(q)\}\} \\
&= \bigcap_{s=y(pq)}\{f_S(y) \cup g_S(p) \cup h_S(q)\} \\
&= \bigcap_{s=p(yq)}\{g_S(p) \cup f_S(y) \cup h_S(q)\} \\
&= \bigcap_{s=pw}\{g_S(p) \cup \bigcap_{w=yq}\{f_S(y) \cup h_S(q)\}\} \\
&= \bigcap_{s=pw}\{g_S(p) \cup (f_S * h_S)(w)\} \\
&= (g_S * (f_S * h_S))(s)
\end{aligned}$$

Thus, $(f_S * (g_S * h_S)) = (g_S * (f_S * h_S))$. If s is not expressible as product of two elements in S, then $(f_S * (g_S * h_S))(s) = (g_S * (f_S * h_S))(s) = U$. Hence, $(f_S * (g_S * h_S))(s) = (g_S * (f_S * h_S))(s)$ for all $s \in S$.

ii) If any element of s of S is not expressible as product of two elements, then $((f_S * g_S)) * (h_S * k_S))(s) =$

$((k_S * h_S)) * (g_S * f_S))(s) = U$. Let there exist elements $y, z \in S$ such that $s = yz$. Then,

$$\begin{aligned}
((f_S * g_S)) * (h_S * k_S))(s) &= \bigcap_{s=yz}\{(f_S * g_S)(y) \cup (h_S * k_S)(z)\} \\
&= \bigcap_{s=yz}\{\bigcap_{y=pq}\{f_S(p) \cup g_S(p)\} \cup \bigcap_{z=uv}\{h_S(u) \cup k_S(v)\}\} \\
&= \bigcap_{s=(pq)(uv)}\{f_S(p) \cup g_S(q) \cup h_S(u) \cup k_S(v)\} \\
&= \bigcap_{s=(vu)(qp)}\{k_S(p) \cup h_S(q) \cup g_S(u) \cup f_S(v)\} \\
&= \bigcap_{s=mn}\{\bigcap_{m=vu}\{k_S(p) \cup h_S(u)\} \cup \bigcap_{n=qp}\{g_S(q) \cup f_S(p)\}\} \\
&= \bigcap_{s=mn}\{(k_S * h_S)(m) \cup (g_S * f_S)(n)\} \\
&= ((k_S * h_S)) * (g_S * f_S))(s)
\end{aligned}$$

□

Proposition 3.7. *An AG-groupoid S with $S(S) = (S(S))^2$ is a commutative semigroup if and only if $(f_S * g_S) * h_S = f_S * (h_S * g_S)$ holds for all soft sets f_S, g_S and h_S.*

Proof. Let an AG-groupoid S be commutative. For any soft sets f_S, g_S and h_S, we have

$$(f_S * g_S) * h_S = (h_S * g_S) * f_S = f_S * (h_S * g_S).$$

Conversely, let $(f_S * g_S) * h_S = f_S * (h_S * g_S)$ for all soft sets f_S, g_S and h_S. We show that an AG-groupoid $S(S)$ is commutative semigroup. Since $S(S) = (S(S))^2$, there exist h_S, k_S such that $f_S = h_S * k_S$. Now

$$f_S * g_S = (h_S * k_S) * g_S = (g_S * k_S) * h_S = g_S * (h_S * k_S) = g_S * h_S.$$

Thus, commutative law holds in $S(S)$. Moreover, $(f_S * g_S) * k_S = (k_S * g_S) * f_S = f_S * (k_S * g_S) = f_S * (g_S * k_S)$. □

Theorem 3.8. *Let S be an AG-groupoid. Then, $\vartheta = \{f_S : f_S \in S(S), f_S * h_S = f_S \text{ where } h_S = h_S * h_S\}$ is a commutative semigroup.*

Proof. Since $h_S * h_S = h_S$, $\emptyset \neq \vartheta \subseteq S(S)$. Let $f_S, g_S \in \vartheta$. Then, $f_S * h_S = f_S$ and $g_S * h_S = g_S$. Thus

$$f_S * g_S = (f_S * h_S) * (g_S * h_S) = (f_S * g_S) * (h_S * h_S) = (f_S * g_S) * h_S$$

implying that ϑ is closed.

Moreover, $f_S * g_S = (f_S * h_S) * g_S = (g_S * h_S) * f_S = g_S * f_S$ implying that commutative law holds in ϑ. Associative law holds from Theorem 3.3. This completes the proof. □

In [42], Sezgin defined soft characteristic function of the complement X which is a subset of a semigroup. Now, soft characteristic function of the complement X which is a subset of an AG-groupoid is given.

Definition 3.9. *Let X be a subset of S. We denote by $\mathcal{S}_{X^c}$ the soft characteristic function of the complement X and define as*

$$\mathcal{S}_{X^c}(x) = \begin{cases} \emptyset, & \text{if } x \in X, \\ U, & \text{if } x \in S \setminus X \end{cases}$$

The proof of the following theorem is similar to those in [42].

Theorem 3.10. *Let X and Y be nonempty subsets of an AG-groupoid S. Then, the following properties hold:*

i) *If $Y \subseteq X$, then $\mathcal{S}_{X^c} \tilde{\subseteq} \mathcal{S}_{Y^c}$.*
ii) $\mathcal{S}_{X^c} \tilde{\cap} \mathcal{S}_{Y^c} = \mathcal{S}_{X^c \cap Y^c}$, $\mathcal{S}_{X^c} \tilde{\cup} \mathcal{S}_{Y^c} = \mathcal{S}_{X^c \cup Y^c}$.

4. Soft union AG-groupoid

In this section, soft union AG-groupoids are defined, their basic properties with respect to soft operations and soft uni-product are studied.

Definition 4.1. *Let S be an AG-groupoid and f_S be a soft set over U. Then, f_S is called a soft union AG-groupoid of S, if*

$$f_S(xy) \subseteq f_S(x) \cup f_S(y)$$

for all $x, y \in S$.

For the sake of brevity, soft union AG-groupoid is abbreviated by SU-AG-groupoid in what follows.

Example 4.2. *Let $S = \{a, b, c, d\}$ be the AG-groupoid in Example 3.2 and f_S be a soft set over $U = \mathbb{Z}$. If we construct a soft set such that*

$$f_S(a) = \{0, 2, 4, 6, 8\},\ f_S(b) = \{0, 2, 4, 6\},\ f_S(c) = \{0, 2, 4, 6, 8\}, f_S(d) = \{0, 2, 4\}$$

then, one can easily show that f_S is an SU-AG-groupoid over U.

Now, let $U = \left\{\begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix} \mid x \in \mathbb{Z}_4\right\}$, 2×2 matrices with $\mathbb{Z}_4$ terms, be the universal set. We construct a soft set g_S over U by

$$g_S(a) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\},$$
$$g_S(b) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\},$$
$$g_S(c) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}\right\},$$
$$g_S(d) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}\right\}.$$

Then, since

$$g_S(bb) = g_S(d) \nsubseteq g_S(b) \cup g_S(b),$$

g_S is not an SU-AG-groupoid over U.

note 4.3. *It is easy to see that if $f_S(x) = \emptyset$ for all $x \in S$, then f_S is an SU-AG-groupoid over U. We denote such a kind of SU-AG-groupoid by $\widetilde{\theta}$. Namely, $\widetilde{\theta}(x) = \emptyset$ for all $x \in S$.*

Proposition 4.4. *Let f_S be any SU-AG-groupoid over U. Then, we have the followings:*

i) $\widetilde{\theta} * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$.
ii) $f_S * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$ *and* $\widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta}$.
iii) $f_S \widetilde{\cap} \widetilde{\theta} = \widetilde{\theta}$ *and* $f_S \widetilde{\cup} \widetilde{\theta} = f_S$.

Proposition 4.5. *Let $S(S)$ be an AG-groupoid with left identity. Then,*

$$\widetilde{\theta} * \widetilde{\theta} = \widetilde{\theta}.$$

Proof. Since every element $x \in S$ can be written as $x = ex$, where e is the left identity in S, we have

$$\begin{aligned} (\widetilde{\theta} * \widetilde{\theta})(x) &= \bigcap_{x=yz} (\widetilde{\theta}(y) \cup \widetilde{\theta}(z)) \\ &\subseteq \{\widetilde{\theta}(e) \cup \widetilde{\theta}(x)\} \\ &= \widetilde{\theta}(x) \end{aligned}$$

and since $\widetilde{\theta} * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$ always hold, $\widetilde{\theta} * \widetilde{\theta} = \widetilde{\theta}$. $\square$

Theorem 4.6. *Let f_S be a soft set over U. Then, f_S is an SU-AG-groupoid over U if and only if*

$$f_S * f_S \widetilde{\supseteq} f_S$$

Proof. Assume that f_S is an SU-AG-groupoid over U. Let $a \in S$. If $(f_S * f_S)(a) = U$, then it is obvious that

$$(f_S * f_S)(a) \supseteq f_S(a), \text{ thus } f_S * f_S \widetilde{\supseteq} f_S.$$

Otherwise, there exist elements $x, y \in S$ such that $a = xy$. Then, since f_S is an SU-AG-groupoid over U, we have:

$$\begin{aligned}
(f_S * f_S)(a) &= \bigcap_{a=xy} (f_S(x) \cup f_S(y)) \\
&\supseteq \bigcap_{a=xy} f_S(xy) \\
&= \bigcap_{a=xy} f_S(a) \\
&= f_S(a)
\end{aligned}$$

Thus, $f_S * f_S \widetilde{\supseteq} f_S$.

Conversely, assume that $f_S * f_S \widetilde{\supseteq} f_S$. Let $x, y \in S$ and $a = xy$. Then, we have:

$$\begin{aligned}
f_S(xy) &= f_S(a) \\
&\subseteq (f_S * f_S)(a) \\
&= \bigcap_{a=xy} (f_S(x) \cup f_S(y)) \\
&\subseteq f_S(x) \cup f_S(y)
\end{aligned}$$

Hence, f_S is an SU-AG-groupoid over U. This completes the proof. □

Theorem 4.7. *A non-empty subset A of an AG-groupoid of S is an AG-subgroupoid of S if and only if the soft subset f_S defined by*

$$f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus A, \\ \beta, & \text{if } x \in A \end{cases}$$

is an SU-AG-groupoid, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Suppose A is an AG-subgroupoid of S and $x, y \in S$. If $x, y \in A$, then $xy \in A$. Hence, $f_S(xy) = f_S(x) = f_S(y) = \beta$ and so, $f_S(xy) \subseteq f_S(x) \cup f_S(y)$. If $x, y \notin A$, then $xy \in A$ or $xy \notin A$. In any case, $f_S(xy) \subseteq f_S(x) \cup f_S(y) = \alpha$. Thus, f_S is an SU-semigroup.

Conversely assume that f_S is an SU-AG-groupoid of S. Let $x, y \in A$. Then, $f_S(xy) \subseteq f_S(x) \cup f_S(y) = \beta$. This implies that $f_S(xy) = \beta$. Hence, $xy \in A$ and so A is an AG-subgroupoid of S. □

Theorem 4.8. *Let X be a nonempty subset of an AG-groupoid S. Then, X is an AG-subgroupoid of S if and only if $\mathcal{S}_{X^c}$ is an SU-AG-groupoid of S.*

Proof. Since

$$\mathcal{S}_{X^c}(x) = \begin{cases} U, & \text{if } x \in S \setminus X, \\ \emptyset, & \text{if } x \in X \end{cases}$$

and $U \supseteq \emptyset$, the rest of the proof follows from Theorem 4.7. □

Proposition 4.9. *Let f_S and f_T be SU-AG-groupoid over U. Then, $f_S \vee f_T$ is an SU-AG-groupoid over U.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in S \times T$. Then,

$$\begin{aligned} f_{S\vee T}((x_1, y_1)(x_2, y_2)) &= f_{S\vee T}(x_1x_2, y_1y_2) \\ &= f_S(x_1x_2) \cup f_T(y_1y_2) \\ &\subseteq (f_S(x_1) \cup f_S(x_2)) \cup (f_T(y_1) \cup f_T(y_2)) \\ &= (f_S(x_1) \cup f_T(y_1)) \cup (f_S(x_2) \cup f_T(y_2)) \\ &= f_{S\vee T}(x_1, y_1) \cup f_{S\vee T}(x_2, y_2) \end{aligned}$$

Therefore, $f_S \vee f_T$ is an SU-AG-groupoid over U. □

Proposition 4.10. *If f_S and h_S are SU-AG-groupoids over U, then so is $f_S \widetilde{\cup} h_S$ over U.*

Proof. Let $x, y \in S$, then

$$\begin{aligned} (f_S \widetilde{\cup} h_S)(xy) &= f_S(xy) \cup h_S(xy) \\ &\subseteq (f_S(x) \cup f_S(y)) \cup (h_S(x) \cup h_S(y)) \\ &= (f_S(x) \cup h_S(x)) \cup (f_S(y) \cup h_S(y)) \\ &= (f_S \widetilde{\cup} h_S)(x) \cup (f_S \widetilde{\cup} h_S)(y) \end{aligned}$$

Therefore, $f_S \widetilde{\cup} h_S$ is an SU-AG-groupoid over U. □

Proposition 4.11. *Let f_S be a soft set over U and α be a subset of U such that $\alpha \in Im(f_S)$, where $Im(f_S) = \{\alpha \subseteq U : f_S(x) = \alpha,\ for\ x \in S\}$. If f_S is an SU-AG-groupoid over U, then $\mathcal{L}(f_S; \alpha)$ is an AG-subgroupoid of S.*

Proof. Since $f_S(x) = \alpha$ for some $x \in S$, then $\emptyset \neq \mathcal{L}(f_S; \alpha) \subseteq S$. Let $x, y \in \mathcal{L}(f_S; \alpha)$, then $f_S(x) \subseteq \alpha$ and $f_S(y) \subseteq \alpha$. We need to show that $xy \in \mathcal{L}(f_S; \alpha)$ for all $x, y \in \mathcal{L}(f_S; \alpha)$. Since f_S is an SU-AG-groupoid over U, it follows that

$$f_S(xy) \subseteq f_S(x) \cup f_S(y) \subseteq \alpha \cup \alpha = \alpha$$

implying that $xy \in \mathcal{L}(f_S; \alpha)$. Thus, the proof is completed. □

Definition 4.12. *Let f_S be an SU-AG-groupoid over U. Then, the AG-subgroupoids $\mathcal{L}(f_S; \alpha)$ are called lower α-AG-subgroupoids of f_S.*

Proposition 4.13. *Let f_S be a soft set over U, $\mathcal{L}(f_S; \alpha)$ be lower α-AG-subgroupoids of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an SU-AG-groupoid over U.*

Proof. Let $x, y \in S$ and $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathcal{L}(f_S; \alpha_1)$ and $y \in \mathcal{L}(f_S; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x, y \in \mathcal{L}(f_S; \alpha_1)$ and since $\mathcal{L}(f_S; \alpha)$ is an AG-subgroupoid of S for all $\alpha \subseteq U$, it follows that $xy \in \mathcal{L}(f_S; \alpha_1)$. Hence, $f_S(xy) \subseteq \alpha_1 = \alpha_1 \cup \alpha_2 = f_S(x) \cup f_S(y)$. Thus, f_S is an SU-AG-groupoid over U. □

Proposition 4.14. *Let f_S and f_T be soft sets over U and Ψ be an AG-groupoid isomorphism from S to T. If f_S is an SU-AG-groupoid over U, then so is $\Psi^\star(f_S)$.*

Proof. Let $t_1, t_2 \in T$. Since Ψ is surjective, then there exist $s_1, s_2 \in S$ such that $\Psi(s_1) = t_1$ and $\Psi(s_2) = t_2$. Then,

$$\begin{aligned} &(\Psi^\star(f_S))(t_1t_2) \\ &= \bigcap\{f_S(s) : s \in S, \Psi(s) = t_1t_2\} \\ &= \bigcap\{f_S(s) : s \in S, s = \Psi^{-1}(t_1t_2)\} \\ &= \bigcap\{f_S(s) : s \in S, s = \Psi^{-1}(\Psi(s_1s_2)) = s_1s_2\} \\ &= \bigcap\{f_S(s_1s_2) : s_i \in S, \Psi(s_i) = t_i, i = 1, 2\} \\ &\subseteq \bigcap\{f_S(s_1) \cup f_S(s_2) : s_i \in S, \Psi(s_i) = t_i, i = 1, 2\} \\ &= (\bigcap\{f_S(s_1) : s_1 \in S, \Psi(s_1) = t_1\}) \cup (\bigcap\{f_S(s_2) : s_2 \in S, \Psi(s_2) = t_2\}) \\ &= (\Psi^\star(f_S))(t_1) \cup (\Psi^\star(f_S))(t_2) \end{aligned}$$

Hence, $\Psi^\star(f_S)$ is an SU-AG-groupoid over U. □

Proposition 4.15. *Let f_S and f_T be soft sets over U and Ψ be an AG-groupoid homomorphism from S to T. If f_T is an SU-AG-groupoid over U, then so is $\Psi^{-1}(f_T)$.*

Proof. Let $s_1, s_2 \in S$. Then,

$$\begin{aligned}(\Psi^{-1}(f_T))(s_1 s_2) &= f_T(\Psi(s_1 s_2)) \\ &= f_T(\Psi(s_1)\Psi(s_2)) \\ &\subseteq f_T(\Psi(s_1)) \cup f_T(\Psi(s_2)) \\ &= (\Psi^{-1}(f_T))(s_1) \cup (\Psi^{-1}(f_T))(s_2)\end{aligned}$$

Hence, $\Psi^{-1}(f_T)$ is an SU-AG-groupoid over U. □

5. Soft union left (right, two-sided) ideals of AG-groupoids

In this section, soft union left (right, two-sided) ideals of AG-groupoids are defined and their basic properties related with soft set operations and soft uni-products are obtained.

Definition 5.1. *A soft set over U is called a soft union left (right) ideal of S over U if*

$$f_S(ab) \subseteq f_S(b) \ (f_S(ab) \subseteq f_S(a))$$

for all $a, b \in S$. A soft set over U is called a soft union two-sided ideal (soft union ideal) of S if it is both soft union left and soft union right ideal of S over U.

For the sake of brevity, soft union left (right) ideal is abbreviated by SU-left (right) ideal in what follows.

Example 5.2. *Consider the AG-groupoid $S = \{1, 2, 3\}$ defined by the following table:*

.	1	2	3
1	1	1	1
2	3	3	3
3	1	1	1

Let f_S be a soft set over S such that $f_S(1) = \{1, 2\}$, $f_S(2) = \{1, 2, 3\}$, $f_S(3) = \{1, 2\}$. Then, one can easily show that f_S is an SU-ideal of S over U. However if we define a soft set h_S over S such that $h_S(1) = \{1, 2, 3\}$, $h_S(2) = \{1, 2\}$, $h_S(3) = \{1, 2\}$, then, $h_S(3 \cdot 1) = h_S(1) \nsubseteq h_S(3)$. Thus, h_S is not an SU-right ideal of S over S.

Theorem 5.3. *Let f_S be a soft set over U. Then, f_S is an SU-left ideal of S over U if and only if*

$$\widetilde{\theta} * f_S \widetilde{\supseteq} f_S.$$

Proof. First assume that f_S is an SU-left ideal of S over U. Let $s \in S$. If

$$(\widetilde{\theta} * f_S)(s) = U,$$

then it is clear that $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. Otherwise, there exist elements $x, y \in S$ such that $s = xy$. Then, since f_S is an SU-left ideal of S over U, we have:

$$\begin{aligned}(\widetilde{\theta} * f_S)(s) &= \bigcap_{s=xy} (\widetilde{\theta}(x) \cup f_S(y)) \\ &\supseteq \bigcap_{s=xy} (\emptyset \cup f_S(xy)) \\ &= \bigcap_{s=xy} (\emptyset \cup f_S(s)) \\ &= f_S(s)\end{aligned}$$

Thus, we have $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$.

Conversely, assume that $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. Let $x, y \in S$ and $s = xy$. Then, we have:

$$\begin{aligned} f_S(xy) &= f_S(s) \\ &\subseteq (\widetilde{\theta} * f_S)(s) \\ &= \bigcap_{s=mn} (\widetilde{\theta}(m) \cup f_S(n)) \\ &\subseteq \widetilde{\theta}(x) \cup f_S(y) \\ &= \emptyset \cup f_S(y) \\ &= f_S(y) \end{aligned}$$

Hence, f_S is an SU-left ideal over U. This completes the proof. □

Theorem 5.4. *Let f_S be a soft set over U. Then, f_S is an SU-right ideal of S over U if and only if*

$$f_S * \widetilde{\theta} \widetilde{\supseteq} f_S$$

Proof. Similar to the proof of Theorem 5.3. □

Theorem 5.5. *Let f_S be a soft set over U. Then, f_S is an SU-ideal of S over U if and only if*

$$f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \text{ and } \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$$

Corollary 5.6. *$\widetilde{\theta}$ is both SU-right and SU-left ideal of S.*

Proof. Follows from Lemma 4.4-(i). □

Proposition 5.7. *In an AG-groupoid S with left identity, for every SU-left ideal f_S of S, $\widetilde{\theta} * f_S = f_S$.*

Proof. It suffices to show that $\widetilde{\theta} * f_S \widetilde{\subseteq} f_S$. Since every element $x \in S$ can be written as $x = ex$, where e is the left identity in S, we have

$$\begin{aligned} (\widetilde{\theta} * f_S)(x) &= \bigcap_{x=yz} (\widetilde{\theta}(y) \cup f_S(z)) \\ &\subseteq \{\widetilde{\theta}(e) \cup f_S(x)\} \\ &= f_S(x) \end{aligned}$$

Hence, $\widetilde{\theta} * f_S = f_S$. □

Proposition 5.8. *In an AG-groupoid S with left identity, for every SU-right ideal g_S of S, $g_S * \widetilde{\theta} = g_S$.*

Proof. It suffices to show that $g_S * \widetilde{\theta} \widetilde{\subseteq} g_S$. Since every element $a \in S$ can be written as $a = ea = (ee)a = (ae)e$, where e is the left identity in S, we have

$$\begin{aligned} (g_S * \widetilde{\theta})(a) &= \bigcap_{a=(ae)e} (g_S(ae) \cup \widetilde{\theta}(e)) \\ &\subseteq \{g_S(ae) \cup \widetilde{\theta}(e)\} \\ &\subseteq \{g_S(a) \cup \widetilde{\theta}(e)\} \\ &= g_S(a) \end{aligned}$$

Hence, the proof is completed. □

Corollary 5.9. *In an AG-groupoid S with left identity, $\widetilde{\theta} * \widetilde{\theta} = \widetilde{\theta}$.*

Proposition 5.10. *Let S be an AG-groupoid with left identity, f_S be any soft set and k_S be SU-left ideal of S. Then, for any soft set h_S and SU-left ideal g_S of S, $f_S * g_S = h_S * k_S$ implies that $g_S * f_S = k_S * h_S$.*

Proof. Since g_S and k_S are SU-left ideals of S, by Proposition 5.7, $\widetilde{\theta} * g_S = g_S$ and $\widetilde{\theta} * k_S = k_S$. Then,

$$g_S * f_S = (\widetilde{\theta} * g_S) * f_S = (f_S * g_S) * \widetilde{\theta} = (h_S * k_S) * \widetilde{\theta} = (\widetilde{\theta} * k_S) * h_S = k_S * h_S.$$

□

Proposition 5.11. *Every idempotent SU-left ideal of an AG-groupoid S is an SU-ideal of S.*

Proof. Let f_S be an SU-left ideal of S which is idempotent. Then,

$$f_S * \widetilde{\theta} = (f_S * f_S) * \widetilde{\theta} = (\widetilde{\theta} * f_S) * f_S \widetilde{\supseteq} f_S * f_S = f_S.$$

Hence, f_S is a SU-right ideal of S and so SU-ideal of S. □

Proposition 5.12. *Let f_S be an idempotent element in an AG-groupoid S with left identity. Then, $\widetilde{\theta} * f_S$ is an idempotent element.*

Proof. Let f_S be an idempotent element in an AG-groupoid S with left identity. Then, by using medial law,

$$(\widetilde{\theta} * f_S) * (\widetilde{\theta} * f_S) = (\widetilde{\theta} * \widetilde{\theta}) * (f_S * f_S) = \widetilde{\theta} * f_S.$$

□

Proposition 5.13. *Let f_S be an idempotent element in an AG-groupoid S with left identity. Then, every SU-left ideal g_S of S commutes with f_S.*

Proof. Let f_S be an idempotent element in an AG-groupoid S with left identity. Then,

$$f_S * g_S = (f_S * f_S) * g_S = (g_S * f_S) * f_S \widetilde{\supseteq} (g_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} g_S * f_S.$$

Also,

$$g_S * f_S = g_S * (f_S * f_S) = f_S * (g_S * f_S) \widetilde{\supseteq} f_S * (g_S * \widetilde{\theta}) \widetilde{\supseteq} f_S * g_S.$$

□

Theorem 5.14. *Let S be an AG-groupoid with left identity, then the collection of all SU-left ideals of S, which are idempotent forms a commutative monoid.*

Proof. Let $\widetilde{H}$ denote the all SU-left ideals which are idempotent in S. Since $\widetilde{\theta} * \widetilde{\theta} = \widetilde{\theta}$, $\widetilde{H} \neq \emptyset$. Now we show that commutative law holds in $\widetilde{H}$. Let $f_S, g_S \in \widetilde{H}$. Then, by using medial law,

$$(f_S * g_S) * (f_S * g_S) = (f_S * f_S) * (g_S * g_S) = f_S * g_S$$

similarly, $(g_S * f_S) * (g_S * f_S) = g_S * f_S$. Also, by using Theorem 3.6,

$$f_S * g_S = (f_S * g_S) * (f_S * g_S) = (g_S * f_S) * (g_S * g_S) = g_S * f_S.$$

Now, for any $f_S, g_S, h_S \in \widetilde{H}$, we have

$$(f_S * g_S) * h_S = (h_S * g_S) * f_S = f_S * (h_S * g_S) = f_S * (g_S * h_S).$$

Finally, since $\widetilde{\theta} * f_S = f_S$ for every SU-left ideal of S and by commutativity $\widetilde{\theta} * f_S = f_S * \widetilde{\theta} = f_S$, which implies that $\widetilde{\theta}$ is identity in $\widetilde{H}$. This completes the proof. □

Proposition 5.15. *Let S be an AG-groupoid with left identity. Then, every SU-right ideal of S is an SU-ideal of S.*

Proof. Let f_S be an SU-right ideal of S. Then $f_S * \widetilde{\theta}\widetilde{\supseteq} f_S$. Thus,

$$\widetilde{\theta} * f_S = (\widetilde{\theta} * \widetilde{\theta}) * f_S = (f_S * \widetilde{\theta}) * \widetilde{\theta}\widetilde{\supseteq} f_S * \widetilde{\theta}\widetilde{\supseteq} f_S.$$

So, f_S is an SU-left ideal, hence an SU-ideal of S. □

Proposition 5.16. *If f_S is an SU-left ideal of an AG-groupoid S with left identity, then*

$$f_S\widetilde{\cap}(f_S * \widetilde{\theta})$$

is an SU-ideal of S over U.

Proof. Assume that f_S is an SU-left ideal of S. Then,

$$(f_S\widetilde{\cap}(f_S * \widetilde{\theta})) * \widetilde{\theta} = (f_S * \widetilde{\theta})\widetilde{\cap}((f_S * \widetilde{\theta}) * \widetilde{\theta}) = (f_S * \widetilde{\theta})\widetilde{\cap}((\widetilde{\theta} * \widetilde{\theta}) * f_S) = (f_S * \widetilde{\theta})\widetilde{\cap}(\widetilde{\theta} * f_S) = (f_S * \widetilde{\theta})\widetilde{\cap} f_S = f_S\widetilde{\cap}(f_S * \widetilde{\theta}).$$

Hence, $f_S\widetilde{\cap}(f_S * \widetilde{\theta})$ is an SU-right ideal of S and by Proposition 8.5, it is an SU-ideal of S. □

Proposition 5.17. *If f_S is an SU-right ideal of an AG-groupoid S with left identity, then*

$$f_S\widetilde{\cap}(\widetilde{\theta} * f_S)$$

is an SU-ideal of S over U.

Proof. Assume that f_S is an SU-right ideal of S. Then,

$$(f_S\widetilde{\cap}(\widetilde{\theta} * f_S)) * \widetilde{\theta} = (f_S * \widetilde{\theta})\widetilde{\cap}((\widetilde{\theta} * f_S) * \widetilde{\theta})\widetilde{\supseteq} f_S\widetilde{\cap}((\widetilde{\theta} * f_S) * (\widetilde{\theta} * \widetilde{\theta})) = f_S\widetilde{\cap}((\widetilde{\theta} * \widetilde{\theta}) * (f_S * \widetilde{\theta})) = f_S\widetilde{\cap}(\widetilde{\theta} * (f_S * \widetilde{\theta})) = f_S\widetilde{\cap}(f_S * (\widetilde{\theta} * \widetilde{\theta})) = f_S\widetilde{\cap}(f_S * \widetilde{\theta})\widetilde{\supseteq} f_S\widetilde{\supseteq} f_S\widetilde{\cap}(\widetilde{\theta} * f_S).$$

Also,

$$\widetilde{\theta} * (f_S\widetilde{\cap}(\widetilde{\theta} * f_S)) = (\widetilde{\theta} * f_S)\widetilde{\cap}(\widetilde{\theta} * (\widetilde{\theta} * f_S)) = (\widetilde{\theta} * f_S)\widetilde{\cap}((\widetilde{\theta} * \widetilde{\theta}) * (\widetilde{\theta} * f_S)) = (\widetilde{\theta} * f_S)\widetilde{\cap}((\widetilde{\theta} * f_S) * (\widetilde{\theta} * \widetilde{\theta}))\widetilde{\supseteq}(\widetilde{\theta} * f_S)\widetilde{\cap}(f_S * (\widetilde{\theta} * \widetilde{\theta})) = (\widetilde{\theta} * f_S)\widetilde{\cap}(f_S * \widetilde{\theta})\widetilde{\supseteq}(\widetilde{\theta} * f_S)\widetilde{\cap} f_S = f_S\widetilde{\cap}(\widetilde{\theta} * f_S).$$

Hence, $f_S\widetilde{\cap}(\widetilde{\theta} * f_S)$ is an SU-ideal of S. □

Theorem 5.18. *In an AG-groupoid with left identity, the following conditions are equivalent:*

1) *If f_S and g_S are SU-ideals of S, then $(f_S)^2\widetilde{\supseteq} g_S$ implies that $f_S\widetilde{\supseteq} g_S$.*
2) *If f_S is an SU-right ideal of S and g_S is an SU-ideal of S, then $(f_S)^2\widetilde{\supseteq} g_S$ implies that $f_S\widetilde{\supseteq} g_S$.*
3) *If f_S is an SU-left ideal of S and g_S is an SU-ideal of S, then $(f_S)^2\widetilde{\supseteq} g_S$ implies that $f_S\widetilde{\supseteq} g_S$.*

Proof. Let (1) hold and f_S be an SU-left ideal of S and g_S be an SU-ideal of S. Then, as shown above, $f_S\widetilde{\cap}(f_S * \widetilde{\theta})$ is an SU-ideal of S. Thus, by assumption, $(f_S\widetilde{\cap}(f_S * \widetilde{\theta}))^2\widetilde{\supseteq} g_S$ implies that $(f_S\widetilde{\cap}(f_S * \widetilde{\theta}))\widetilde{\supseteq} g_S$, which further implies that $f_S\widetilde{\supseteq} g_S$. (3) implies (2) and (2) implies (1) is obvious. □

Theorem 5.19. *A non-empty subset L of an AG-groupoid of S is a left (right) ideal of S if and only if the soft subset f_S defined by*

$$f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus L, \\ \beta, & \text{if } x \in L \end{cases}$$

is an SU-left (right) ideal of S, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Suppose L is a left ideal of S and $x, y \in S$. If $y \in L$, then $xy \in L$. Hence, $f_S(xy) = f_S(y) = \beta$. If $y \notin L$, then $xy \in L$ or $xy \notin L$. In any case, $f_S(xy) \subseteq f_S(y) = \alpha$. Thus, f_S is an SU-left ideal of S.

Conversely assume that f_S is an SU-left ideal of S. Let $y \in L$ and $x \in S$. Then, $f_S(xy) \subseteq f_S(y) = \beta$. This implies that $f_S(xy) = \beta$. Hence, $xy \in L$ and so L is a left ideal of S. □

Theorem 5.20. *Let X be a nonempty subset of an AG-groupoid S. Then, X is a left (right, two-sided) ideal of S if and only if $\mathcal{S}_{X^c}$ is an SU-left (right, two-sided) ideal of S over U.*

Proof. It follows from Theorem 5.19. □

Proposition 5.21. *Let f_S be a soft set over U. Then, f_S is an SU-ideal of S over U if and only if*

$$f_S(xy) \subseteq f_S(x) \cap f_S(y)$$

for all $x, y \in S$.

Proof. Let f_S be an SU-ideal of S over U. Then,

$$f_S(xy) \subseteq f_S(x) \text{ and } f_S(xy) \subseteq f_S(y)$$

for all $x, y \in S$. Thus, $f_S(xy) \subseteq f_S(x) \cap f_S(y)$. Conversely suppose that $f_S(xy) \subseteq f_S(x) \cap f_S(y)$ for all $x, y \in S$. It follows that

$$f_S(xy) \subseteq f_S(x) \cap f_S(y) \subseteq f_S(x) \text{ and } f_S(xy) \subseteq f_S(x) \cap f_S(y) \subseteq f_S(y)$$

so f_S is an SU-ideal of S over U. □

Theorem 5.22. *Let f_S be a soft set over U. Then, if f_S is an SU-left (right, two-sided) ideal of S over U, f_S is an SU-AG-groupoid over U.*

Proof. We give the proof for SU-left ideals. Let f_S be an SU-left ideal of S over U. Then, $f_S(xy) \subseteq f_S(y)$ for all $x, y \in S$. Thus, $f_S(xy) \subseteq f_S(y) \subseteq f_S(x) \cup f_S(y)$, so f_S is an SU-AG-groupoid over U. □

Theorem 5.23. *Let f_S be an SU-right ideal of S over U and g_S be an SU-left ideal of S over U. Then*

$$f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$$

Proof. Let f_S and g_S be SU-right and SU-left ideal of S over U, respectively. Then, since $f_S, g_S \widetilde{\supseteq} \widetilde{\theta}$ always holds, we have:

$$f_S * g_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \text{ and } f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S.$$

It follows that $f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$. □

Now, we show that if f_S is an SU-right ideal of S over U and g_S is an SU-left ideal of S over U, then

$$f_S * g_S \widetilde{\nsubseteq} f_S \widetilde{\cap} g_S$$

with the following example:

Example 5.24. *Consider the AG-groupoid S and SU-ideal f_S in Example 5.2. Let g_S be a soft set over S such that $g_S(1) = \{2\}$, $g_S(2) = \{1, 2\}$, $g_S(3) = \{2\}$. One can easily show that g_S is an SU-ideal of S over U. However,*

$$(f_S * g_S)(3) = \bigcap_{3=ab} (f_S(a) \cup g_S(b)) = \{1, 2, 3\} \nsubseteq (f_S \widetilde{\cap} g_S)(3) = \{2\}.$$

Proposition 5.25. *Let f_S, g_S be SU-left (right, two-sided) ideals of S, where S is an AG-groupoid with left identity. Then, $f_S * g_S$ is an SU-left (right, two-sided) ideal of S over U.*

Proof. Let f_S, g_S be SU-left ideals of a AG-groupoid S with identity. Then, $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$ and $\widetilde{\theta} * g_S \widetilde{\supseteq} g_S$. Thus,

$$\widetilde{\theta} * (f_S * g_S) = f_S * (\widetilde{\theta} * g_S) \widetilde{\supseteq} f_S * g_S.$$

Also,

$$(f_S * g_S) * \widetilde{\theta} = (f_S * g_S) * (\widetilde{\theta} * \widetilde{\theta}) = (f_S * \widetilde{\theta}) * (g_S * \widetilde{\theta}) \widetilde{\supseteq} f_S * g_S.$$

This completes the proof. □

Proposition 5.26. *Let f_S be an SU-left (right, two-sided) ideals of S, where S is an AG-groupoid with left identity. Then, $f_S * f_S$ is an SU-ideal of S over U.*

Proof. Let f_S be an SU-left ideal of a AG-groupoid S with identity. Then,

$$\widetilde{\theta} * (f_S * f_S) = (\widetilde{\theta} * \widetilde{\theta}) * (f_S * f_S) = (\widetilde{\theta} * f_S) * (\widetilde{\theta} * f_S) \widetilde{\supseteq} f_S * f_S.$$

Also,

$$(f_S * f_S) * \widetilde{\theta} = (\widetilde{\theta} * f_S) * f_S \widetilde{\supseteq} f_S * f_S.$$

This completes the proof. □

Definition 5.27. *An SU-ideal f_S of an AG-groupoid S is said to be strongly irreducible if and only if for every SU-ideals g_S and h_S of S, $g_S \widetilde{\cup} h_S \widetilde{\supseteq} f_S$ implies that $g_S \widetilde{\supseteq} f_S$ or $h_S \widetilde{\supseteq} f_S$.*

Definition 5.28. *An SU-ideal h_S of an AG-groupoid S is said to be soft prime ideal if for any SU-ideals f_S and g_S of S, $f_S * g_S \widetilde{\supseteq} h_S$ implies that $f_S \widetilde{\supseteq} h_S$ or $g_S \widetilde{\supseteq} h_S$. An AG-groupoid S is called fully soft prime if every SU-ideal is soft prime in S.*

Definition 5.29. *An SU-left ideal is called SU-quasi-prime ideal if for any two SU-left ideals f_S and g_S of S, $f_S * g_S \widetilde{\supseteq} h_S$ implies that $f_S \widetilde{\supseteq} h_S$ or $g_S \widetilde{\supseteq} h_S$. An AG-groupoid S is called fully soft quasi-prime if every SU-left ideal is quasi-prime in S.*

Definition 5.30. *An SU-left ideal f_S of an AG-groupoid is called SU-semiprime left ideal of S if for any SU-left ideal g_S of S, $(g_S)^2 \widetilde{\supseteq} f_S$ implies that $g_S \widetilde{\supseteq} f_S$.*

Definition 5.31. *The set of SU-ideals of an AG-groupoid is called totally ordered under inclusion if for any SU-ideals f_S and g_S of S, either $f_S \widetilde{\supseteq} g_S$ or $g_S \widetilde{\supseteq} f_S$.*

Theorem 5.32. *An AG-groupoid S with left identity is fully soft prime if and only if every SU-ideal is idempotent and SU-ideals are totally ordered by inclusion.*

Proof. Let S be fully soft prime and f_S be an SU-ideal of S. Then, it is obvious that $f_S * f_S \widetilde{\supseteq} f_S$, since every SU-ideal is SU-AG-subgroupoid. Now, we show that $f_S * f_S \widetilde{\supseteq} f_S$. Since $f_S * f_S$ is an SU-ideal of S by Proposition 8.5 and so by hypothesis $f_S * f_S$ is a soft prime ideal, $f_S * f_S \widetilde{\supseteq} f_S * f_S$ implies that $f_S \widetilde{\supseteq} f_S * f_S$ or $f_S \widetilde{\supseteq} f_S * f_S$. Thus, $f_S \widetilde{\supseteq} f_S * f_S$. Hence $f_S * f_S = f_S$. Now, we show that S is totally ordered by inclusion. We have

$$f_S * g_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \text{ and } f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S, \text{ and so, } f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S,$$

where $f_S \widetilde{\cup} g_S$ is an SU-ideal. By hypothesis, $f_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$ or $g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$, which implies that $f_S \widetilde{\supseteq} g_S$ or $g_S \widetilde{\supseteq} f_S$.

Conversely, let every SU-ideal is idempotent and SU-ideals are totally ordered by inclusion. Let f_S be an SU-ideal of S such that $g_S * h_S \widetilde{\supseteq} f_S$ where g_S are h_S are SU-ideals of S. Since the SU-ideals are totally ordered by inclusion, so for g_S and h_S, either $g_S \widetilde{\supseteq} h_S$ or $h_S \widetilde{\supseteq} g_S$. Let $g_S \widetilde{\supseteq} h_S$. Since g_S is idempotent, $g_S = g_S * g_S \widetilde{\supseteq} g_S * h_S \widetilde{\supseteq} f_S$. Similarly, if $h_S \widetilde{\supseteq} g_S$, then we have $h_S \widetilde{\supseteq} f_S$. Thus, S is fully soft prime. □

Proposition 5.33. *Let S be an AG-groupoid with left identity. If S is fully soft quasi-prime, then every SU-left ideal is idempotent.*

Proof. Let f_S be an SU-left ideal of an AG-groupoid S with left identity. Then, f_S is an SU-AG-subgroupoid and so, $f_S * f_S \widetilde{\supseteq} f_S$. Now, since $f_S * f_S$ is an SU-left ideal of S by Proposition 8.5 and so by hypothesis $f_S * f_S$ is a soft quasi-prime ideal, $f_S * f_S \widetilde{\supseteq} f_S * f_S$ implies that $f_S \widetilde{\supseteq} f_S * f_S$ or $f_S \widetilde{\supseteq} f_S * f_S$. Thus, $f_S \widetilde{\supseteq} f_S * f_S$. Hence $f_S * f_S = f_S$. □

Proposition 5.34. *Let S be an AG-groupoid with left identity. If S is fully soft quasi-prime, then for every SU-left ideals f_S and g_S of S, $f_S * g_S = f_S \widetilde{\cup} g_S$.*

Proof. Let f_S and g_S be SU-left ideals of an AG-groupoid S with left identity, where S is fully soft quasi-prime. Then,

$$g_S * f_S \widetilde{\supseteq} \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$$

and

$$f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S,$$

thus, $g_S * f_S \widetilde{\supseteq} f_S$ and $f_S * g_S \widetilde{\supseteq} g_S$. Moreover, by Proposition 5.33,

$$g_S * f_S = (g_S * g_S) * (f_S * f_S) = (f_S * f_S) * (g_S * g_S) = f_S * g_S,$$

which implies that $f_S * g_S \widetilde{\supseteq} f_S$ and $f_S * g_S \widetilde{\supseteq} g_S$, and so

$$f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S.$$

Since $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S$ and $f_S \widetilde{\cup} g_S \widetilde{\supseteq} g_S$,

$$(f_S \widetilde{\cup} g_S) * (f_S \widetilde{\cup} g_S) \widetilde{\supseteq} f_S * g_S.$$

Since $f_S \widetilde{\cup} g_S$ is an SU-left ideal of S, and so idempotent by Proposition 5.33,

$$f_S \widetilde{\cup} g_S \widetilde{\subseteq} f_S * g_S.$$

Thus, $f_S \widetilde{\cup} g_S = f_S * g_S$. □

Corollary 5.35. *The set of all soft quasi-prime ideals of an AG-groupoid with identity forms a semilattice structure.*

Theorem 5.36. *Let S be an AG-groupoid with left identity, then the followings are equivalent:*

i) *Each SU-left ideal of S is idempotent.*
ii) *For each SU-left ideals f_S and g_S of S, $f_S * g_S = f_S \widetilde{\cup} g_S$.*
iii) *Each SU-left ideal of S is soft union semiprime SU-left ideal.*

Proof. (i) implies (ii) follows from Proposition 5.33 and Proposition 5.34. Let (ii) hold and f_S be an SU-left ideal of S. Then,

$$f_S * f_S = f_S \widetilde{\cup} f_S = f_S,$$

hence $(f_S)^2 = f_S \widetilde{\supseteq} f_S$ implies that $f_S \widetilde{\supseteq} f_S$, and so f_S is a soft union semiprime SU-left ideal. Now, let (iii) hold and f_S be an SU-left ideal of S. Then,

$$\widetilde{\theta} * (f_S * f_S) = f_S * (\widetilde{\theta} * f_S) = f_S * f_S.$$

Hence, $f_S * f_S$ is an SU-left ideal of S, and so soft union semiprime by hypothesis. Thus, $f_S * f_S \widetilde{\supseteq} f_S * f_S$ implies that $f_S \widetilde{\supseteq} f_S * f_S$. Moreover, since f_S is an SU-left ideal of S, $f_S * f_S \widetilde{\supseteq} f_S$. Thus, $f_S * f_S = f_S$. This completes the proof. □

6. Soft union bi-ideals of AG-groupoids

In this section, soft union bi-ideals are defined and their properties as regards soft set operations and soft uni-product are studied.

Definition 6.1. *An SU-AG-groupoid f_S over U is called a soft union bi-ideal of S over U if*

$$f_S((xy)z) \subseteq f_S(x) \cup f_S(z)$$

for all $x, y, z \in S$.

For the sake of brevity, soft union bi-ideal is abbreviated by SU-bi-ideal in what follows.

Example 6.2. *Let $S = \{a, b, c\}$ be the AG-groupoid with the operation table given below.*

.	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Define the soft set f_S over $U = S_3$, symmetric group such that $f_S(a) = \{(1), (12)\}$, $f_S(b) = \{(1), (12), (13)\}$, $f_S(c) = \{(1), (12), (23)\}$. Then, one can easily show that f_S is an SU bi-ideal of S over U. However if we define a soft set h_S over S_3 such that $h_S(a) = \{(1), (12), (13), (123)\}$, $h_S(b) = \{(1), (123)\}$, $h_S(c) = \{(1), (12), (123)\}$,

then since $f_S((cb)c) = f_S(a) \nsubseteq f_S(c) \cup f_S(c)$, f_S is not an SU-bi-ideal of S over U.

Theorem 6.3. *Let f_S be a soft set over U. Then, f_S is an SU-bi-ideal of S over U if and only if*

$$f_S * f_S \widetilde{\supseteq} f_S \text{ and } (f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S$$

Proof. First assume that f_S is an SU-bi-ideal of S over U. Since f_S is an SU-AG-groupoid over U, we have

$$f_S * f_S \widetilde{\supseteq} f_S.$$

Let $s \in S$. In the case, when $((f_S * \widetilde{\theta}) * f_S)(s) = U$, then it is clear that $(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S$, Otherwise, let a be any element of S. If there exist elements $x, y \in S$ such that

$$a = xy$$

then, we have

$$\begin{aligned}
((f_S * \widetilde{\theta}) * f_S)(a) &= \bigcap_{a=xy} [(f_S * \widetilde{\theta})(x) \cup f_S(y)] \\
&= \bigcap_{a=xy} [(\bigcap_{x=pq} (f_S(p) \cup \widetilde{\theta}(q)) \cup f_S(y)] \\
&= \bigcap_{s=(pq)y} (f_S(p) \cup f_S(y)) \\
&\supseteq \bigcap_{s=(pq)y} f_S((pq)y) \\
&= f_S(a)
\end{aligned}$$

Hence, $(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S$.

Conversely, assume that $f_S * f_S \widetilde{\supseteq} f_S$. By Theorem 4.6, f_S is an SU-AG-groupoid of S. Let $x, y, z \in S$. Then, since $(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S$, we have

$$\begin{aligned}
f_S((xy)z) &\subseteq ((f_S * \widetilde{\theta}) * f_S)((xy)z) \\
&= \bigcap_{((xy)z)=ab} [(f_S * \widetilde{\theta})(a) \cup f_S(b)] \\
&= [\bigcap_{((xy)z)=ab} [\bigcap_{a=pq} \{(f_S(p) \cup \widetilde{\theta}(q)] \cup f_S(b)\} \\
&= \bigcap_{((xy)z)=(pq)b} \{(f_S(p) \cup f_S(b)\} \\
&\subseteq f_S(x) \cup f_S(z)
\end{aligned}$$

Thus, for all $x, y, z \in S$, $f_S((xy)z) \subseteq f_S(x) \cup f_S(z)$. Hence, f_S is an SU-bi-ideal of S over U. This completes the proof. □

Theorem 6.4. *A non-empty subset B of an AG-groupoid of S is a bi-ideal of S if and only if the soft subset f_S defined by*

$$f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus B, \\ \beta, & \text{if } x \in B \end{cases}$$

is an SU-bi-ideal of S, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Similar to Theorem 4.7. □

Theorem 6.5. *Let X be a nonempty subset of an AG-groupoid S. Then, X is a bi-ideal of S if and only if $\mathcal{S}_{X^c}$ is an SU-bi-ideal of S over U.*

Proof. It follows from Theorem 6.4. □

Theorem 6.6. *Every SU-left (two sided) ideal of an AG-groupoid S over U is an SU-bi-ideal of S over U.*

Proof. Let f_S be an SU-left (two sided) ideal of S over U and $x, y, z \in S$. Then, f_S is as SU-AG-groupoid by Theorem 5.22. Moreover,

$$f_S((xy)z) \subseteq f_S(z) \subseteq f_S(x) \cup f_S(z)$$

Thus, f_S is an SU-bi-ideal of S. □

Theorem 6.7. *Let f_S and g_S be any SU-right ideal of an AG-groupoid S with identity. Then, the soft uni-products $f_S * g_S$ and $g_S * f_S$ are SU-bi-ideals of S over U.*

Proof. We show the proof for $f_S * g_S$. To see that $f_S * g_S$ is an SU-bi-ideal of S over U, first we need to show that $f_S * g_S$ is an SU-AG-groupoid over U. Thus,

$$(f_S * g_S) * (f_S * g_S) = (f_S * f_S) * (g_S * g_S) \widetilde{\supseteq} f_S * g_S$$

Hence, by Theorem 4.6, $f_S * g_S$ is an SU-AG-groupoid over U. Moreover we have:

$$\begin{aligned} ((f_S * g_S) * \widetilde{\theta}) * (f_S * g_S) &= ((f_S * g_S) * (\widetilde{\theta} * \widetilde{\theta})) * (f_S * g_S) \\ &= ((f_S * \widetilde{\theta}) * (g_S * \widetilde{\theta})) * (f_S * g_S) \\ &\widetilde{\supseteq} (f_S * g_S) * (f_S * g_S) \\ &\widetilde{\supseteq} f_S * g_S \end{aligned}$$

Thus, it follows that $f_S * g_S$ is an SU-bi-ideal of S over U. It can be seen in a similar way that $g_S * f_S$ is an SU-bi-ideal of S over U. This completes the proof. □

Theorem 6.8. *Let f_S be any SU-left ideal of an AG-groupoid S with identity. Then, the soft uni-product $f_S * f_S$ is an SU-bi-ideals of S over U.*

Proof. Since every SU-ideal of S is an SU-bi-ideal of S, the rest of the proof follows from Proposition 8.5. □

7. Soft union interior ideals of AG-groupoids

In this section, soft union interior ideals of AG-groupoids are defined and their basic properties with respect to soft operations and soft uni-product are studied.

Definition 7.1. *Let f_S be an SU-AG-groupoid over U. Then, f_S is called a soft union interior ideal of S, if*

$$f_S((xa)y) \subseteq f_S(a)$$

for all $x, y, a \in S$.

For the sake of brevity, soft union interior ideal is abbreviated by SU-interior ideal in what follows.

Example 7.2. *Consider the AG-groupoid $S = \{a, b, c\}$ with the following operation table:*

.	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

Let $U = D_3 = \{< x, y >: x^3 = y^2 = e, xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ be the universal set and f_S be soft set over U such that

$$f_S(a) = \{e, x, y, yx^2\},\ f_S(b) = \{e, yx^2\},\ f_S(c) = \{e, x, yx^2\}.$$

Then, one can easily show that f_S is an SU-interior ideal over U.

Now, let $U = S_2$ be the symmetric group. If we construct a soft set g_S over U such that

$$g_S(a) = \{(1), (12)\},\ g_S(b) = \{(1), (12)\},\ g_S(c) = \{(1)\},$$

then, since

$$g_S((bc)c) = g_S(b) \nsubseteq g_S(c),$$

g_S is not an SU-interior ideal over U.

note 7.3. *It is easy to see that if $f_S(x) = \emptyset$ for all $x \in S$, then f_S is an SU-interior ideal over U. We denote such a kind of SU-interior ideal by $\widetilde{\theta}$, that is, $\widetilde{\theta}(x) = \emptyset$ for all $x \in S$.*

Theorem 7.4. *Let f_S be a soft set over U. Then, f_S is an SU-interior ideal over U if and only if*

$$(\widetilde{\theta} * f_S) * \widetilde{\theta} \widetilde{\supseteq} f_S$$

Proof. Assume that f_S is an SU-interior ideal over U. Let $a \in S$. If $((\widetilde{\theta} * f_S) * \widetilde{\theta})(a) = U$, then it is obvious that

$$((\widetilde{\theta} * f_S) * \widetilde{\theta})(a) \supseteq f_S(a),\ \text{thus } (\widetilde{\theta} * f_S) * \widetilde{\theta} \widetilde{\supseteq} f_S.$$

Otherwise, if there exist elements y, z, u and v of S such that $x = yz$ and $y = uv$, then, since f_S is an SU-interior ideal of S, we have

$$f_S(x) = f_S(yz) = f_S((uv)z) \subseteq f_S(v).$$

Thus,

$$\begin{aligned}
((\widetilde{\theta} * f_S) * \widetilde{\theta})(x) &= \{\bigcap_{x=yz} (\widetilde{\theta} * f_S)(y) \cup \widetilde{\theta}(z)\} \\
&= \bigcap_{x=yz} \{(\bigcap_{y=uv} (\widetilde{\theta}(u) \cup f_S(v))) \cup \widetilde{\theta}(z)\} \\
&= \bigcap_{x=yz} \{(\bigcap_{y=uv} (\emptyset \cup f_S(v))) \cup \emptyset\} \\
&\supseteq \bigcap_{x=yz} \{(\bigcap_{y=uv} (\emptyset \cup f_S((uv)z))) \cup \emptyset\} \\
&= f_S(x)
\end{aligned}$$

Thus, $(\widetilde{\theta} * f_S) * \widetilde{\theta}\widetilde{\supseteq} f_S$. Note that if $y \neq uv$, then $(\widetilde{\theta} * f_S)(y) = U$, and so $((\widetilde{\theta} * f_S) * \widetilde{\theta})(x) = U \supseteq f_S(x)$.

Conversely, assume that $(\widetilde{\theta} * f_S) * \widetilde{\theta}\widetilde{\supseteq} f_S$. Let x, a, y be any element of S. Then, we have:

$$\begin{aligned} f_S((xa)y) &\subseteq ((\widetilde{\theta} * f_S) * \widetilde{\theta})((xa)y) \\ &= \bigcap_{(xa)y=pq} \{(\widetilde{\theta} * f_S)(p) \cup \widetilde{\theta}(q)\} \\ &\subseteq (\widetilde{\theta} * f_S)(xa) \cup \widetilde{\theta}(y) \\ &= (\widetilde{\theta} * f_S)(xa) \cup \emptyset \\ &= \bigcap_{xa=mn} \{\widetilde{\theta}(m) \cup f_S(n)\} \\ &\subseteq \widetilde{\theta}(x) \cup f_S(a) \\ &= f_S(a) \end{aligned}$$

Hence, f_S is an SU-interior ideal over U. This completes the proof. □

Theorem 7.5. *A non-empty subset I of an AG-groupoid of S is an interior ideal of S if and only if the soft subset f_S defined by*

$$f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus I, \\ \beta, & \text{if } x \in I \end{cases}$$

is an SU-interior ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Suppose I is an interior ideal of S and $x, a, y \in S$. If $a \in I$, then $xay \in I$. Hence, $f_S(xay) = f_S(a) = \beta$. If $a \notin I$, then $xay \in I$ or $xay \notin I$. In any case, $f_S(xay) \subseteq f_S(a) = \alpha$. Thus, f_S is an SU-interior ideal of S.

Conversely assume that f_S is an SU-interior ideal of S. Let $a \in I$ and $x, y \in S$. Then, $f_S(xay) \subseteq f_S(a) = \beta$. This implies that $f_S(xay) = \beta$. Hence, $xay \in I$ and so I is an interior ideal of S. □

Theorem 7.6. *Let X be a nonempty subset of an AG-groupoid S. Then, X is an interior ideal of S if and only if $\mathcal{S}_{X^c}$ is an SU-interior ideal of S.*

It is obvious that every two-sided ideal of S is an interior ideal of S. Moreover, we have the following:

Proposition 7.7. *Let f_S be a soft set over U. Then, if f_S is an SU-ideal of S over U, f_S is an SU-interior ideal of S over U.*

Proof. Let f_S be an SU-ideal of S over U and $x, y \in S$. Then,

$$f_S((xy)z) \subseteq f_S(xy) \subseteq f_S(y).$$

Hence, f_S is an SU-interior ideal of S over U. □

The following example shows that the converse of this property does not hold in general:

Example 7.8. *Consider the SU-interior ideal f_S in Example 7.2. Since*

$$f_S(ab) = f_S(c) \not\subseteq f_S(b)$$

f_S is not an SU-left ideal of S, that is, it is not an SU-ideal of S.

Proposition 7.9. *Every soft set f_S of an AG-groupoid S with left identity is an SU-right ideal if and only if it is an SU-interior ideal.*

Proof. Let every soft set f_S of S be an SU-right ideal of S. For $x, a, y \in S$, consider

$$f_S((xa)y) \subseteq f_S(xa) \subseteq f_S((ex)a) = f_S((ax)e) \subseteq f_S(ax) \subseteq f_S(a)$$

which implies that f_S is an SU-interior ideal of S.

Conversely, for any $x, y \in S$, we have

$$f_S(xy) = f_S((ex)y) = f_S(x).$$

□

It is known that an AG-groupoid S is called *left (right) simple* if it contains no proper left (right) ideal of S and is called *simple* if it contains no proper ideal.

Definition 7.10. *An AG-groupoid S is called soft union left (right) simple if every SU-left (right) ideal of S is a constant function and is called soft union simple if every SU-ideal of S is a constant function.*

Theorem 7.11. *For a semigroup S, the following conditions are equivalent:*

1) *S is left (right) simple.*
2) *S is soft union left (right) simple.*

Proof. First assume that S is left simple. Let f_S be any SU-left ideal of S and a and b be any element of S. Then, it follows that there exist elements $x, y \in S$ such that $b = xa$ and $a = yb$. Hence, since S is an SU-left ideal of S,

$$f_S(a) = f_S(yb) \subseteq f_S(b) = f_S(xa) \subseteq f_S(a)$$

and so $f_S(a) = f_S(b)$. Since a and b be any elements of S, this means that f_S is a constant function. Thus, we obtain that S is soft union left simple and (1) implies (2).

Conversely, assume that (2) holds. Let A be any left ideal of S. Then, $\mathcal{S}_{A^c}$ is an SU-left ideal of S. By assumption, $\mathcal{S}_{A^c}$ is a constant function. Let x be any element of S. Then, since $A \neq \emptyset$,

$$\mathcal{S}_{A^c}(x) = \emptyset$$

and so $x \in A$. This implies that $S \subseteq A$, and so $S = A$. Hence, S is left simple and (2) implies (1). In the case, when S is soft union right simple, the proof follows similarly. □

Theorem 7.12. *For an AG-groupoid with left identity S, the following conditions are equivalent:*

1) *S is simple.*
2) *S is soft union right simple.*
3) *Every SU-interior ideal of S is constant function.*

Proof. The equivalence of (1) and (2) follows from Theorem 7.11. Assume that (2) holds. Let f_S be any SU-interior ideal of S and a and b be any element of S. Then, since S is simple, that there exist elements x and y in S such that

$$a = (xb)y.$$

Then, since f_S is an SU-interior ideal of S, we have

$$f_S(a) = f_S((xb)y) \subseteq f_S(b).$$

One can similarly show that $f_S(b) \subseteq f_S(a)$. Thus, $f_S(a) = f_S(b)$. Since a and b be any elements of S, f_S is a constant function and so (2) implies (3). Since every SU-interior ideal of S is an SU-ideal of S since S is an AG-groupoid with left identity, (3) implies (2). □

Definition 7.13. *A soft set f_S over U is called soft semiprime* if for all $a \in S$,*

$$f_S(a) \subseteq f_S(a^2).$$

Proposition 7.14. *Let f_S be a soft semiprime* SU-interior ideal of an AG-groupoid S. Then, $f_S(a^n) \subseteq f_S(a^{n+1})$ for all positive integers n.*

Proof. Let n be any positive integer. Then,

$$f_S(a^n) \subseteq f_S(a^{2n}) \subseteq f_S(a^{4n}) = f_S((a^{3n-2}a^{n+1})a) \subseteq f_S(a^{n+1}).$$

□

Definition 7.15. *An AG-groupoid S is called archimedean if for all $a, b \in S$, there exists a positive integer n such that $a^n \in (Sb)S$.*

Proposition 7.16. *Let S be an archimedean AG-groupoid. Then, every soft semiprime* SU-interior ideal of S is a constant function.*

Proof. Let f_S be any soft semiprime* SU-interior ideal of S and a, b any element of S. Since S is archimedean, there exist elements $x, y \in S$ such that

$$a^n = (xb)y.$$

Thus, we have $f_S(a) \subseteq f_S(a^n) = f_S((xb)y) \subseteq f_S(b)$. Similarly, we have $f_S(b) \subseteq f_S(a)$ and so $f_S(a) = f_S(b)$. Since a and b be any elements of S, f_S is a constant function. □

8. Soft union quasi-ideals of AG-groupoids

In this section, soft union quasi-ideals are studied and their properties as regards soft set operations, soft union product and certain kinds of soft union ideals are studied.

Definition 8.1. *A soft set over U is called a soft union quasi-ideal of S over U if*

$$(f_S * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * f_S)\widetilde{\supseteq} f_S.$$

For the sake of brevity, soft union quasi-ideal is abbreviated by SU-quasi-ideal in what follows.

Proposition 8.2. *Every SU-quasi ideal of S is an SU-AG-groupoid of S.*

Proof. Let f_S be any SU-quasi-ideal of S. Then, since $f_S \widetilde{\supseteq} \widetilde{\theta}$,

$$f_S * f_S \widetilde{\supseteq} \widetilde{\theta} * f_S \text{ and } f_S * f_S \widetilde{\supseteq} f_S * \widetilde{\theta}.$$

Hence,

$$f_S * f_S \widetilde{\supseteq} (\widetilde{\theta} * f_S)\widetilde{\cup}(f_S * \widetilde{\theta})\widetilde{\supseteq} f_S$$

as f_S is an SU-quasi-ideal of S. That is, f_S is an SU-AG-groupoid over U. □

Proposition 8.3. *Each one-sided SU-ideal of S is an SU-quasi-ideal of S.*

Proof. Let f_S be an SU-left ideal of S. Then, since $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$, we have

$$(\widetilde{\theta} * f_S)\widetilde{\cup}(f_S * \widetilde{\theta})\widetilde{\supseteq} \widetilde{\theta} * f_S \widetilde{\supseteq} f_S.$$

Thus, f_S is an SU-quasi-ideal of S. □

The converse of Proposition 8.3 does not hold in general as shown in the following example:

Example 8.4. *Consider the AG-groupoid $S = \{1, 2, 3, 4\}$ with the following operation table:*

.	1	2	3	4
1	1	2	3	4
2	4	3	3	3
3	3	3	3	3
4	2	3	3	3

Let $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ be the universal set and f_S be the soft set over U such that

$$f_S(1) = \{e, x, yx, yx^2\},\ f_S(2) = \{e, x, yx^2\},\ f_S(3) = \{x\},\ f_S(4) = \{e, x\}.$$

Then, one can easily show that f_S is an SU-quasi-ideal of S, but since

$$f_S(4 \cdot 1) = f_S(2) \nsubseteq f_S(4)$$

f_S is not an SU-right ideal of S and so it is not an SU-ideal of S.

Proposition 8.5. *In an AG-groupoid S, every idempotent SU-quasi-ideal of S is an SU-bi-ideal of S.*

Proof. Let f_S be an SU-quasi-ideal of S. Then, f_S is an SU-AG-groupoid by Proposition 8.2. Moreover,

$$(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} (\widetilde{\theta} * \widetilde{\theta}) * f_S \widetilde{\supseteq} \widetilde{\theta} * f_S \text{ and } (f_S * \widetilde{\theta}) * f_S = (f_S * \widetilde{\theta}) * (f_S * f_S) = (f_S * f_S) * (\widetilde{\theta} * f_S) \widetilde{\supseteq} f_S * \widetilde{\theta}$$

and so $(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} (\widetilde{\theta} * f_S) \widetilde{\cup} (f_S * \widetilde{\theta}) \widetilde{\supseteq} f_S$, as f_S is an SU-quasi-ideal of S. Hence,

$$(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S.$$

Thus, f_S is an SU-bi-ideal of S. □

Theorem 8.6. *Let X be a nonempty subset of an AG-groupoid S. Then, X is a quasi-ideal of S if and only if $\mathcal{S}_{X^c}$ is an SU-quasi-ideal of S over U.*

Theorem 8.7. *Let f_S and g_S be any idempotent SU-quasi-ideals of S over U, where S is an AG-groupoid with left identity. Then, the soft union product $f_S * g_S$ is an SU-bi-ideal of S over U.*

Proof. Let f_S and g_S be any idempotent SU-quasi-ideals of S. Then,

$$(f_S * g_S) * (f_S * g_S) = (f_S * f_S) * (g_S * g_S) = f_S * g_S$$

and

$$((f_S * g_S) * \widetilde{\theta}) * (f_S * g_S) = ((\widetilde{\theta} * g_S) * f_S) * (f_S * g_S) \widetilde{\supseteq} ((\widetilde{\theta} * \widetilde{\theta}) * f_S) * (f_S * g_S) \widetilde{\supseteq} (\widetilde{\theta} * f_S) * (f_S * g_S) \widetilde{\supseteq} (g_S * f_S) * (f_S * \widetilde{\theta}) = ((f_S * \widetilde{\theta}) * f_S) * g_S \widetilde{\supseteq} f_S * g_S.$$

since f_S is an SU-bi-ideal of S by Proposition 8.5. Thus, it follows that $f_S * g_S$ is an SU-bi-ideal of S over U. □

Proposition 8.8. *Let f_S be any SU-right ideal of S and g_S be any SU-left ideal of S. Then, $f_S \widetilde{\cup} g_S$ is an SU-quasi-ideal of S.*

Proof. Let f_S be any SU-right ideal of S and g_S be any SU-left ideal of S. Then,

$$((f_S \widetilde{\cup} g_S) * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * (f_S \widetilde{\cup} g_S)) \widetilde{\supseteq} (f_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * g_S) \widetilde{\supseteq} f_S \widetilde{\cup} g_S.$$

□

Proposition 8.9. *Let f_S and g_S be any SU-quasi-ideals of S. Then, $f_S\widetilde{\cup}g_S$ is an SU-quasi-ideal of S.*

Proof. Let f_S and g_S be any SU-quasi-ideals of S. Then,

$$((f_S\widetilde{\cup}g_S) * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * (f_S\widetilde{\cup}g_S))\widetilde{\supseteq}(f_S * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * f_S)\widetilde{\supseteq}f_S$$

and

$$((f_S\widetilde{\cup}g_S) * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * (f_S\widetilde{\cup}g_S))\widetilde{\supseteq}(g_S * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * g_S)\widetilde{\supseteq}g_S.$$

Thus,

$$((f_S\widetilde{\cup}g_S) * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * (f_S\widetilde{\cup}g_S))\widetilde{\supseteq}f_S\widetilde{\cup}g_S.$$

□

Proposition 8.10. *Let f_S be a soft set over U and α be a subset of U such that $\alpha \in Im(f_S)$. If f_S is an SU-quasi-ideal of S over U, then $\mathcal{L}(f_S;\alpha)$ is a quasi-ideal of S.*

Proof. Since $f_S(x) = \alpha$ for some $x \in S$, then $\emptyset \neq \mathcal{L}(f_S;\alpha) \subseteq S$. Let $a \in (S \cdot \mathcal{L}(f_S;\alpha) \cup \mathcal{L}(f_S;\alpha) \cdot S)$. Then, there exist $x, y \in \mathcal{L}(f_S;\alpha)$ and $s, r \in S$ such that

$$a = sx = yr.$$

Thus, $f_S(x) \subseteq \alpha$ and $f_S(y) \subseteq \alpha$. Since

$$\begin{aligned}
(\widetilde{\theta} * f_S)(a) &= \{\bigcap_{a=cd}\{\widetilde{\theta}(c) \cup f_S(d)\} \\
&\subseteq \widetilde{\theta}(s) \cup f_S(x) \\
&= f_S(x) \\
&\subseteq \alpha
\end{aligned}$$

and

$$\begin{aligned}
(f_S * \widetilde{\theta})(a) &= \{\bigcap_{a=nm}\{f_S(n) * \widetilde{\theta}(m)\} \\
&\subseteq f_S(y) \cup \widetilde{\theta}(r) \\
&= f_S(y) \\
&\subseteq \alpha
\end{aligned}$$

Since f_S is an SU-quasi-ideal of S, we have

$$f_S(a) \subseteq (\widetilde{\theta} * f_S)(a) \cup (f_S * \widetilde{\theta})(a) \subseteq \alpha,$$

thus $a \in \mathcal{L}(f_S;\alpha)$. This shows that $\mathcal{L}(f_S;\alpha)$ is a quasi-ideal of S. □

Definition 8.11. *Let f_S be an SU-quasi-ideal of S over U. Then, the quasi-ideals $\mathcal{L}(f_S;\alpha)$ are called lower α-quasi-ideals of f_S.*

Proposition 8.12. *Let f_S be any SU-quasi-ideal of a commutative AG-groupoid S and a be any element of A. Then,*

$$f_S(a^n) \subseteq f_S(a^{n+1})$$

for every positive integer n.

Proof. For any positive integer n, we have

$$\begin{aligned}(f_S * \widetilde{\theta})(a^{n+1}) &= \bigcap_{a^{n+1}=xy} (f_S(x) \cup \widetilde{\theta}(y)) \\ &\subseteq f_S(a^n) \cup \widetilde{\theta}(a) \\ &= f_S(a^n).\end{aligned}$$

Similarly,

$$(\widetilde{\theta} * f_S)(a^{n+1}) \subseteq f_S(a^n).$$

Thus, since f_S is an SU-quasi-ideal of S

$$\begin{aligned}f_S(a^{n+1}) &\subseteq ((f_S * \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} * f_S))(a^{n+1}) \\ &= (f_S * \widetilde{\theta})(a^{n+1}) \cup (\widetilde{\theta} * f_S))(a^{n+1}) \\ &\subseteq f_S(a^n) \cup f_S(a^n) \\ &= f_S(a^n)\end{aligned}$$

This completes the proof. □

9. Soft union generalized bi-ideals of AG-groupoids

In this section, soft union generalized bi-ideals are defined and their properties with regard to soft set operations and soft union product are studied.

Definition 9.1. *A soft set over U is called a soft union generalized bi-ideal of S over U if*

$$f_S((xy)z) \subseteq f_S(x) \cup f_S(z)$$

for all $x, y, z \in S$.

For the sake of brevity, soft union generalized bi-ideal is abbreviated by SU-generalized bi-ideal in what follows.

It is clear that every SU-bi-ideal of S is an SU-generalized bi-ideal of S, but the converse of this statement does not hold in general. This is shown by the following example:

Example 9.2. *Consider the AG-groupoid $S = \{a, b, c\}$ in Example 7.2. Let f_S be a soft set over S such that $f_S(a) = \{a, c\}$, $f_S(b) = \{c\}$, $f_S(c) = \{a, b, c\}$. Then, one can easily show that f_S is an SU-generalized bi-ideal of S over U. However, since $h_S(a \cdot a) = h_S(c) \nsubseteq h_S(a)$. Thus, h_S is not an SU-bi-ideal of S.*

Theorem 9.3. *Let f_S be a soft set over U. Then, f_S is an SU-generalized bi-ideal of S over U if and only if*

$$(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S$$

Theorem 9.4. *Let X be a nonempty subset of an AG-groupoid S. Then, X is a generalized bi-ideal of S if and only if $\mathcal{S}_X$ is an SU-generalized bi-ideal of S over U.*

Theorem 9.5. *Every SU-left (two-sided) ideal of an AG-groupoid S over U is an SU-generalized bi-ideal of S over U.*

Theorem 9.6. *Let f_S and g_S be any SU-right ideal of an AG-groupoid S with identity. Then, the soft uni-products $f_S * g_S$ and $g_S * f_S$ are SU-generalized bi-ideals of S over U.*

Theorem 9.7. *Let f_S be any SU-left ideal of an AG-groupoid S with identity. Then, the soft uni-product $f_S * f_S$ is an SU-generalized bi-ideals of S over U.*

Proposition 9.8. *If S is AG-groupoid S, then $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) * f_S$ for every SU-generalized bi-ideal f_S of S and SU-interior ideal g_S of S over U.*

The following propositions are similar to those in Section 4.

Proposition 9.9. *Let f_S and f_T be SU-left (right, two-sided, bi, generalized bi-ideal, interior) ideals over U. Then, $f_S \vee f_T$ is an SU-left (right, two-sided, bi, generalized bi-ideal, interior) of $S \times T$ over U.*

Proposition 9.10. *If f_S and h_S are two SU-left (right, two-sided, bi, generalized bi-ideal, interior) of S over U, then so is $f_S \widetilde{\cup} h_S$ of S over U.*

Proposition 9.11. *Let f_S be a soft set over U and α be a subset of U such that $\alpha \in Im(f_S)$. If f_S is an SU-left (right, two-sided, bi, generalized bi-ideal, interior) of S over U, then $\mathcal{L}(f_S;\alpha)$ is a left (right, two-sided, bi, generalized bi-ideal, interior) ideal of S.*

Definition 9.12. *If f_S is an SU-left (right, two-sided, bi, generalized bi-ideal, interior) of S over U, then left (right, two-sided, bi, generalized bi-ideal, interior) ideals $\mathcal{L}(f_S;\alpha)$ are called lower α-left (right, two-sided, bi, generalized bi-ideal, interior) ideals of f_S.*

Proposition 9.13. *Let f_S be a soft set over U, $\mathcal{L}(f_S;\alpha)$ be upper α-left (right, two-sided, bi, generalized bi-ideal, interior) of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an SU-left (right, two-sided, bi, generalized bi-ideal, interior) ideal of S over U.*

Proposition 9.14. *Let f_S and f_T be soft sets over U and Ψ be an AG-groupoid isomorphism from S to T. If f_S is an SU-left (right, two-sided, bi, generalized bi-ideal, interior) ideal of S over U, then so is $\Psi^\star(f_S)$ of T over U.*

Proposition 9.15. *Let f_S and f_T be soft sets over U and Ψ be an AG-groupoid homomorphism from S to T. If f_T is an SU-left (right, two-sided, bi, generalized bi-ideal, interior) ideal of T over U, then so is $\Psi^{-1}(f_T)$ of S over U.*

10. Regular AG-groupoids

In this section, we characterize a regular AG-groupoid in terms of SU-ideals.

An AG-groupoid S is called *regular* if for every element a of S there exists an element x in S such that

$$a = (ax)a.$$

Example 10.1. *[37] Consider the AG-groupoid $S = \{1,2,3,4,5,6,7\}$ defined by the following table:*

.	1	2	3	4	5	6	7
1	7	2	4	6	1	3	5
2	3	5	7	2	4	6	1
3	6	1	3	5	7	2	4
4	2	4	6	1	3	5	7
5	5	7	2	4	6	1	3
6	1	3	5	7	2	4	6
7	4	6	1	3	5	7	2

Clearly, S is an AG-groupoid also $(1 \cdot 4 \cdot 6) \neq (1 \cdot (4 \cdot 6)$, so S is non-associative and S is regular, since $1 = (1 \cdot 4) \cdot 1, 2 = (2 \cdot 7) \cdot 2, 3 = (3 \cdot 3) \cdot 3, 4 = (4 \cdot 6) \cdot 4, 5 = (5 \cdot 2) \cdot 5, 6 = (6 \cdot 2) \cdot 6$ and $7 = (7 \cdot 1) \cdot 1$. Note that in a regular AG-groupoid, $S^2 = S$.

Theorem 10.2. *Let S be an AG-groupoid. If S is regular, then $f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-right ideal f_S of S over U and SU-left ideal g_S of S over U.*

Proof. Let S be a regular AG-groupoid and f_S be an SU-right ideal of S and g_S be an SU-left ideal of S over U. Then,

$$f_S * g_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \text{ and } f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S,$$

thus,

$$f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$$

for every SU-right ideal f_S of S and SU-left ideal g_S of S over U. Therefore, it suffices to show that $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$. Let s be any element of S. Then, since S is regular, there exists an element x in S such that $s = (sx)s$. Thus, we have

$$\begin{aligned}
(f_S * g_S)(s) &= \bigcap_{s=ab} (f_S(a) \cup g_S(b)) \\
&\subseteq f_S(sx) \cup g_S(s) \\
&\subseteq f_S(s) \cup g_S(s) \\
&= (f_S \widetilde{\cup} g_S)(s)
\end{aligned}$$

Thus, $f_S * g_S = f_S \widetilde{\cup} g_S$. □

Corollary 10.3. *Let S be an AG-groupoid. If S is regular, then $f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-ideals f_S and g_S of S over U.*

Corollary 10.4. *Let S be an AG-groupoid. If S is regular, then $f_S * g_S = g_S * f_S$ for every SU-ideals f_S and g_S of S over U.*

Proposition 10.5. *Every SU-right ideal of a regular AG-groupoid is idempotent.*

Proof. Let h_S be an SU-right ideal of S. Then,

$$h_S * h_S \widetilde{\supseteq} h_S * \widetilde{\theta} \widetilde{\supseteq} h_S.$$

Now, we show that $h_S \widetilde{\supseteq} h_S * h_S$. Since S is regular, there exists an element $x \in S$ such that $a = (ax)a$ for all $a \in S$. So, we have;

$$\begin{aligned}
(h_S * h_S)(a) &= \bigcap_{a=(ax)a} (h_S(ax) \cup h_S(a)) \\
&\subseteq h_S(a) \cup h_S(a) \\
&= h_S(a)
\end{aligned}$$

Hence, $h_S \widetilde{\supseteq} h_S * h_S$ and so $(h_S)^2 = h_S * h_S = h_S$. □

Corollary 10.6. *Every SU-ideal of a regular AG-groupoid is idempotent.*

Corollary 10.7. *Every SU-ideal of a regular AG-groupoid is soft union semiprime.*

Corollary 10.8. *The set of all SU-ideals of a regular AG-groupoid S forms a semilattice under the soft uni-product.*

Proposition 10.9. *Let S be a regular AG-groupoid. Then every SU-right ideal of S is SU-left ideal of S.*

Proof. Let f_S be an SU-right ideal of S. Since S is regular, for any $x \in S$, there exist $n \in S$ such that $x = (xn)x$. Thus,

$$f_S(xy) = f_S((xn)x)y) = f_S((yx)(xn)) \subseteq f_S(yx) \subseteq f_S(y)$$

Thus, f_S is an SU-left ideal of S. □

Proposition 10.10. *Let the set of all SU-ideals of S be a regular AG-groupoid of S under the soft uni-product. Then, every SU-ideal of S has the form* $f_S = (f_S * \widetilde{\theta}) * f_S$.

Proof. Let f_S be an SU-ideal of S. Then, by assumption, there exists an SU-ideal g_S of S such that

$$f_S = (f_S * g_S) * f_S.$$

Thus, we have

$$f_S = (f_S * g_S) * f_S \widetilde{\supseteq} (f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} (f_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * f_S) \widetilde{\supseteq} f_S \widetilde{\cup} f_S = f_S,$$

since

$$(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} (f_S * \widetilde{\theta}) * \widetilde{\theta} \widetilde{\supseteq} f_S * \widetilde{\theta}$$

and

$$(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} (\widetilde{\theta} * \widetilde{\theta}) * f_S \widetilde{\supseteq} \widetilde{\theta} * f_S.$$

Hence, $f_S = (f_S * \widetilde{\theta}) * f_S$. □

Proposition 10.11. *In a regular AG-groupoid S, an SU-right ideal is soft strongly irreducible if and only if it is soft prime.*

Proof. It follows from Corollary 10.3, Definition 5.27 and Definition 5.28. □

Proposition 10.12. *Every SU-ideal of a regular AG-groupoid S is soft prime if and only if the set of SU-ideals of S is totally ordered under inclusion.*

Proof. It follows from Corollary 10.3, Definition 5.28 and Definition 5.31. □

Proposition 10.13. *If f_S is an SU-interior ideal of S, where S is a regular AG-goupoid, then f_S is an SU-right ideal of S over U.*

Proof. Let a be any elements of S. Then, since S is regular, there exist elements x in S such that

$$a = (ax)a.$$

Then, since f_S is an SU-interior ideal of S, we have

$$f_S(ab) = f_S((ax)a)b) \subseteq f_S(a),$$

This means that f_S is an Su-right ideal of S. □

Proposition 10.14. *Let S be a regular AG-groupoid, f_S be any SU-right ideal of S and g_S be any SU-left ideal of S. Then, $f_S * g_S$ is an SU-quasi-ideal of S.*

Proof. Let S be a regular AG-groupoid and f_S be an SU-right ideal of S and g_S be an SU-left ideal of S. It follows by Proposition 8.8 that $f_S \widetilde{\cup} g_S$ is an SU-quasi-ideal of S. Since S is regular, $f_S * g_S = f_S \widetilde{\cup} g_S$ by Theorem 10.2. Thus, $f_S * g_S$ is an SU-quasi-ideal of S. □

Theorem 10.15. *Let S be an AG-groupoid. If S is regular, then $f_S = (f_S * \widetilde{\theta}) * f_S$ for every SU-bi-ideal (generalized bi-ideal) f_S of S over U.*

Proof. Let f_S be any SU-bi-ideal f_S of S over U and s be any element of S. Then, since S is regular, there exists an element $x \in S$ such that $s = (sx)s$. Thus, we have;

$$\begin{aligned}((f_S * \widetilde{\theta}) * f_S)(s) &= \bigcap_{s=ab}[(f_S * \widetilde{\theta})(a) \cup f_S(b)] \\ &\subseteq (f_S * \widetilde{\theta})(sx) \cup f_S(s) \\ &= \bigcap_{sx=mn}\{(f_S(m) \cup \widetilde{\theta}(n)\} \cup f_S(s) \\ &\subseteq (f_S(s) \cup \widetilde{\theta}(x)) \cup f_S(s) \\ &= (f_S(s) \cup \emptyset) \cup f_S(s) \\ &= f_S(s)\end{aligned}$$

and so, we have $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. Since f_S is an SU-bi-ideal of S, $(f_S * \widetilde{\theta}) * f_S \widetilde{\supseteq} f_S$. Thus, $(f_S * \widetilde{\theta}) * f_S = f_S$. □

11. Intra-regular AG-groupoids

In this section, we characterize an intra-regular AG-groupoid in terms of SU-ideals. An AG-groupoid S is called *intra-regular* if for every element a of S there exist elements x and y in S such that

$$a = (xa^2)y$$

Example 11.1. *[35] Consider the AG-groupoid $S = \{1, 2, 3, 4, 5, 6, 7\}$ defined by the following table:*

.	1	2	3	4	5
1	5	1	2	3	4
2	4	5	1	2	3
3	3	4	5	1	2
4	2	3	4	5	1
5	1	2	3	4	5

Clearly, S is an intra-regular AG-groupoid, since $1 = (2\cdot1^2)\cdot4, 2 = (3\cdot2^2)\cdot4, 3 = (5\cdot3^2)\cdot3, 4 = (5\cdot4^2)\cdot4, 5 = (2\cdot5^2)\cdot3$.

Proposition 11.2. *A soft set f_S of an intra-regular AG-groupoid S is an SU-right ideal if and only if it is an SU-left ideal.*

Proof. Assume that f_S is an SU-right ideal of S. Since S is intra-regular, so for each $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y$. So, by using left invertive law,

$$f_S(ab) = f_S((xa^2)y)b) = f_S((by)(xa^2)) \subseteq f_S(by) \subseteq f_S(b).$$

Thus, f_S is an SU-left ideal of S. Conversely, assume that f_S is an SU-left ideal of S. Then, by using left invertive law,

$$f_S(ab) = f_S(((xa^2)y)b) = f_S((by)(xa^2)) \subseteq f_S(xa^2) \subseteq f_S(a^2) \subseteq f_S(a).$$

Thus, f_S is an SU-right ideal of S. □

Proposition 11.3. *Every SU-two-sided ideal of an intra-regular AG-groupoid S with left identity is idempotent.*

Proof. Assume that f_S is an SU-two-sided ideal of S, then

$$f_S * f_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f.$$

Since S is intra-regular, so for each $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y$. So,

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

Thus, we have

$$\begin{aligned}(f_S * f_S)(a) &= \bigcap_{a=(y(xa))a} \{f_S(y(xa)) \cup f_S(a)\}\\ &\subseteq f_S(y(xa)) \cup f_S(a)\\ &\subseteq f_S(a) \cup f_S(a)\\ &= f_S(a)\end{aligned}$$

Hence, $f_S * f_S = f_S$. □

Corollary 11.4. *Every SU-left ideal of an intra-regular AG-groupoid S with left identity is idempotent.*

Proposition 11.5. *Let S be an intra-regular AG-groupoid with left identity. If S is intra-regular, then $f_S = (\widetilde{\theta} * f_S)^2$ for any SU-left ideal f_S of S.*

Proof. Let f_S be any SU-left ideal of an intra-regular AG-groupoid S with left identity S. Then, $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$ and since $\widetilde{\theta} * f_S$ is an SU-left ideal of S, it is idempotent. Thus, $(\widetilde{\theta} * f_S)^2 = \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. Moreover, $f_S = f_S * f_S \widetilde{\supseteq} \widetilde{\theta} * f_S = (\widetilde{\theta} * f_S)^2$, which implies that $f_S = (\widetilde{\theta} * f_S)^2$. □

Theorem 11.6. *For an AG-groupoid S with left identity, the following conditions are equivalent:*

1) *f_S is an SU-ideal of S.*
2) *f_S is an SU-bi-ideal of S.*

Proof. (1) implies (2) follows from Theorem 9.5. Let f_S be an SU-bi-ideal of S. Since S is intra-regular, so for each $a, b \in S$, there exist x, y and $u, v \in S$ such that $a = (xa^2)y$ and $b = (ub^2)v$. So,

$$\begin{aligned}f_S(ab) &= f_S(((xa^2)y)b) = f_S((by)(xa^2)) = f_S((a^2x)(yb)) = f_S(((yb)x)a^2)\\ &= f_S(((yb)x)(aa)) = f_S((aa)(x(yb))) = f_S(((x(yb))a)a)\\ &= f_S(((x(yb))((xa^2)y))a) = f_S(((xa^2)((x(yb))y))a)\\ &= f_S(((y(x(yb)))(a^2x))a) = f_S((a^2((y(x(yb)))x))a)\\ &= f_S(((aa)((y(x(yb)))x))a) = f_S(((x(y(x(yb))))(aa))a)\\ &= f_S((a((x(y(x(yb))))a))a)\\ &\subseteq f_S(a) \cup f_S(a)\\ &= f_S(a)\end{aligned}$$

and

$$\begin{aligned}f_S(ab) &= f_S(a((ub^2)v)) = f_S((ub^2)(av)) = f_S((va)(b^2u))\\ &= f_S(b^2((va)u)) = f_S((bb)((va)u)) = f_S((((va)u)b)b)\\ &= f_S([((va)u)((ub^2)v)]b) = f_S([(ub^2)(((va)u)v)]b)\\ &= f_S([(v((va)u))(b^2u)]b) = f_S([b^2((v((va)u))u)]b)\\ &= f_S(((bb)((v((va)u))u))b) = f_S(((u(v((va)u)))(bb))b)\\ &= f_S((b((u(v((va)u)))b))b)\\ &\subseteq f_S(b) \cup f_S(b)\\ &= f_S(b)\end{aligned}$$

□

Corollary 11.7. *An SU-right ideal of an AG-groupoid S with left identity is an SU-bi-ideal of S.*

Proposition 11.8. *For a soft set f_S of an intra-regular LA-semigroup S with left identity, the following conditions are equivalent:*

1) *f_S is an SU-ideal of S.*
2) *f_S is an SU-interior ideal of S.*

Proof. (1) implies (2) is clear. Assume that (2) holds. Let a and b be any elements of S. Then, since S is intra-regular, there exist elements x, y, u and v in S such that $a = (xa^2)y$ and $b = (ub^2)v$. Since f_S is an SU-interior ideal of S, we have

$$f_S(ab) = f_S(((xa^2)y)b) = f_S(((by)(xa^2)) = f_S((by)(x(aa))) = f_S((by)(a(xa))) = f_S((ba)(y(xa))) \subseteq f_S(a)$$

and

$$f_S(ab) = f_S(a((ub^2)v)) = f_S((ub^2)(av)) = f_S((b(ub))(av)) = f_S((va)((ub)b)) = f_S((ub)((va)b)) \subseteq f_S(b)$$

Hence, f_S is an SU-ideal of S. □

Proposition 11.9. *For a soft set f_S of an intra-regular AG-groupoid S with left identity, the following conditions are equivalent:*

1) *f_S is an SU-bi-ideal of S.*
2) *f_S is an SU-generalized bi-ideal of S.*

Proof. (1) implies (2) is clear. Assume that (2) holds. Let a be any element of S. Then, since S is intra-regular, there exist elements x, y in S such that $a = (xa^2)y$. Thus, we have

$$f_S(ab) = f_S(((xa^2)y)b = f_S(((xa^2)(ey)b) = f_S(((ye)(a^2x))b) = f_S((a^2((ye)x))b) = f_S(((aa)((ye)x))b) = f_S(((x(ye))(aa))b) = f_S((a((ye))a))b) \subseteq f_S(a) \cup f_S(b).$$

Hence, f_S is an SU-bi-ideal of S. □

Proposition 11.10. *For a soft set f_S of an intra-regular AG-groupoid S with left identity, the following conditions are equivalent:*

1) *f_S is an SU-ideal of S.*
2) *f_S is an SU-quasi-ideal of S.*

Proof. (1) implies (2) is clear. Assume that (2) holds. Let a be any element of S. Then, since S is intra-regular, there exist elements x, y in S such that $a = (xa^2)y$. Thus, we have

$$a = (xa^2)y = (xa^2)(ey) = (xe)(a^2y) = a^2((xe)y) = ((aa)((xe)y) = (ya)((xe)a) = (y(xe))(aa) = a((y((xe))a).$$

Also,

$$\widetilde{\theta} * f_S = (\widetilde{\theta} * \widetilde{\theta}) * f_S = (f_S * \widetilde{\theta}) * \widetilde{\theta}.$$

Therefore,

$$\begin{aligned}(\widetilde{\theta} * f_S)(a) &= ((f_S * \widetilde{\theta}) * \widetilde{\theta})(a) \\ &= \bigcap_{a=a((y(xe))a)} \{(f_S * \widetilde{\theta})(a) \cup \widetilde{\theta}((y(xe))a)\} \\ &\subseteq (f_S * \widetilde{\theta})(a)\end{aligned}$$

Therefore,

$$f_S * \widetilde{\theta} \widetilde{\subseteq} (f_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * f_S) \widetilde{\subseteq} f_S.$$

Hence, f_S is an SU-right ideal of S. And by Proposition 11.2, f_S is an SU-left ideal of S. Hence, f_S is an SU-ideal of S. □

Theorem 11.11. *For a soft set f_S of an intra-regular AG-groupoid S with left identity, the following conditions are equivalent:*

1) *f_S is an SU-right ideal of S.*
2) *f_S is an SU-left ideal of S.*
3) *f_S is an SU-ideal of S.*
4) *f_S is an SU-bi-ideal of S.*
5) *f_S is an SU-generalized bi-ideal of S.*
6) *f_S is an SU-interior ideal of S.*
7) *f_S is an SU-quasi-ideal of S.*

Definition 11.12. *A soft set f_S over U is called soft union semiprime* if for all $a \in S$,*

$$f_S(a) \subseteq f_S(a^2).$$

Theorem 11.13. *For a nonempty A of S, the following conditions are equivalent:*

1) *A is semiprime.*
2) *The soft characteristic function $\mathcal{S}_{A^c}$ is soft union semiprime*.*

Proof. First assume that (1) holds. Let a be any element of S. We need to show that $\mathcal{S}_{A^c}(a) \subseteq \mathcal{S}_{A^c}(a^2)$ for all $a \in S$. If $a^2 \in A$, then since A is semiprime, $a \in A$. Thus,

$$\mathcal{S}_{A^c}(a) = \emptyset = \mathcal{S}_{A^c}(a^2)$$

If $a^2 \notin A$, then

$$\mathcal{S}_{A^c}(a) \subseteq U = \mathcal{S}_{A^c}(a^2)$$

In any case, $\mathcal{S}_{A^c}(a) \subseteq \mathcal{S}_{A^c}(a^2)$ for all $a \in S$. Thus, $\mathcal{S}_{A^c}$ is soft union semiprime*. Hence (1) implies (2).

Conversely assume that (2) holds. Let $a^2 \in A$ and $a \notin A$. Since $\mathcal{S}_{A^c}$ is soft union semiprime*, we have

$$\mathcal{S}_{A^c}(a) = U \subseteq \mathcal{S}_{A^c}(a^2) = \emptyset$$

But, this is a contradiction. Hence, $a \in A$ and so A is semiprime. Thus, (2) implies (1). □

Theorem 11.14. *For any SU-AG-groupoid f_S, the following conditions are equivalent:*

1) *f_S is soft union semiprime*.*
2) *$f_S(a) = f_S(a^2)$ for all $a \in S$.*

Proof. (2) implies (1) is clear. Assume that (1) holds. Let a be any element of S. Since f_S is an SU-AG-groupoid, we have;

$$f_S(a) \subseteq f_S(a^2) = f_S(aa) \subseteq f_S(a) \cup f_S(a) = f_S(a)$$

So, $f_S(a^2) = f_S(a)$ and (1) implies (2). This completes the proof. □

Proposition 11.15. *In an intra-regular LA-semigroup, every SI-interior ideal is soft semiprime*.*

Proof. Let f_S be any SI-interior ideal of S and a be any element of S. Since S is intra-regular, there exist elements x, y in S such that $a = (xa^2)y$. Thus, we have

$$f_S(ab) = f_S((xa^2)y) \subseteq f_S(a^2).$$

Hence, f_S is soft semiprime*. □

Theorem 11.16. *For an AG-groupoid S, the following conditions are equivalent:*

1) *S is intra-regular.*

2) *Every SU-ideal of S is soft union semiprime**.
3) $f_S(a) = f_S(a^2)$ *for all SU-ideal of S and for all* $a \in S$.

Proof. First assume that (1) holds. Let f_S be any SU-ideal of S and a any element of S. Since S is intra-regular, there exist elements x and y in S such that $a = (xa^2)y$. Thus,

$$f_S(a) = f_S((xa^2)y) \subseteq f_S(xa^2) = f_S(x(aa)) \subseteq f_S(aa) \subseteq f_S(a)$$

so, we have $f_S(a) = f_S(a^2)$. Hence, (1) implies (3).

Conversely, assume that (3) holds. It is known that $J[a^2]$ is an ideal of S. Thus, the soft characteristic function $\mathcal{S}_{(J[a^2)^c]}$ is an SU-ideal of S. Since $a^2 \in J[a^2]$, we have;

$$\mathcal{S}_{(J[a^2])^c}(a) = \mathcal{S}_{(J[a^2)^c]}(a^2) = \emptyset$$

Thus, $a \in J[a^2] = (S(aa))S$. Here, one can easily show that S is intra-regular. Hence (3) implies (1).

It is obvious that (3) implies (2). Now, assume that (2) holds. Let f_S be an SU-ideal of S. Since f_S is a soft union semiprime* ideal of S,

$$f_S(a) \subseteq f_S(a^2) = f_S(aa) \subseteq f_S(a)$$

Thus, $f_S(a) = f_S(a^2)$. Hence (2) implies (3). This completes the proof. □

Theorem 11.17. *If S is an intra-regular AG-groupoid with left identity, then* $f_S \widetilde{\cup} g_S = f_S * g_S$ *for every SU-left ideal* f_S *and every SU-right ideal* g_S *of S and the SU-right ideal* g_S *is soft union semiprime**.

Proof. Let f_S and g_S be any SU-left ideal and SU-right ideal of S, respectively. Then, it is obvious that $f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$. Let a any element of S. Since S is intra-regular, there exist elements x and y in S such that $a = (xa^2)y$. Thus,

$$a = (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))(ea) = (ye)((xa)a) = (xa)((ye)a) = (xa)((ae)y).$$

Thus, we have

$$\begin{aligned} (f_S * g_S)(a) &= \bigcap_{a=(xa)((ae)y)} \{f_S(xa) \cup f_S((ae)y)\} \\ &\subseteq f_S(a) \cup g_S(a) \\ &= (f_S \widetilde{\cup} g_S)(a) \end{aligned}$$

Thus, $f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$ and so $f_S * g_S = f_S \widetilde{\cup} g_S$. Moreover,

$$g_S(a) = g_S((xa^2)y) = g_S((xa^2)(ey)) = g_S((ye)(a^2x)) = g_S(a^2((ye)x))) \subseteq g_S(a^2).$$

Hence, g_S is soft union semiprime*. □

Corollary 11.18. *Let S be an intra-regular AG-groupoid S with left identity. Then,* $f_S \widetilde{\cup} g_S = f_S * g_S$ *for every SU-ideals* f_S *and* g_S *of S.*

Theorem 11.19. *The set of all SU-ideals of an intra-regular AG-groupoid S with left identity forms a semilattice structure with identity* $\widetilde{\theta}$.

Proof. Let $\mathcal{I}_S$ be the set of all SU-ideals of an AG-groupoid S and $f_S g_S, h_S \in \mathcal{I}_S$. It is obvious that $\mathcal{I}_S$ is closed by Proposition 5.25. Moreover, we have $f_S = (f_S)^2$ by Proposition 11.3 and by Corollary 11.18, $f_S * g_S = f_S \widetilde{\cup} g_S$, where f_S and g_S are SU-ideals. Obviously, $f_S * g_S = g_S * f_S$. Moreover, by using left invertive law,

$$(f_S * g_S) * h_S = (h_S * g_S) * h_S = f_S * (g_S * h_S).$$

Also, by using left invertive law and Proposition 5.25,

$$f_S * \widetilde{\theta} = (f_S * f_S) * \widetilde{\theta} = (\widetilde{\theta} * f_S) * f_S = f_S * f_S = f_S.$$

□

Theorem 11.20. *If S is an intra-regular AG-groupoid with left identity, then $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) * f_S$ for every SU-right ideal f_S and every SU-left (bi-) ideal g_S of S and SU-right ideal f_S is soft union semiprime**.

Proof. Assume that S is intra-regular. Let f_S and g_S be any SU-right and SU-bi-ideal of S, respectively. Then, since S is intra-regular, for each $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y$. Thus,

$$a = (xa^2)y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a = (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a = ((a(xa))(y(xy)))a = (((y(xy))(xa))a)a = (((ax)((xy)y))a)a.$$

Thus, we have

$$\begin{aligned} ((f_S * g_S) * f_S)(a) &= \bigcap_{a=(((ax)((xy)y))a)a} \{f_S((ax)((xy)y)) \cup g_S(a)) \cup f_S(a)\} \\ &\subseteq f_S((ax)((xy)y)) \cup g_S(a) \cup f_S(a) \\ &\subseteq (f_S(a) \cup g_S(a)) \cup f_S(a) \\ &= (f_S \widetilde{\cup} g_S)(a) \end{aligned}$$

Thus, $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) * f_S$. Moreover,

$$f_S(a) = f_S((xa^2)y) = f_S((xa^2)(ey)) = f_S((ye)(a^2x)) = f_S(a^2((ye)x)) \subseteq f_S(a^2).$$

Hence, f_S is soft union semiprime*. □

Proposition 11.21. *In an intra-regular AG-groupoid S with left identity, an SU-ideal is soft strongly irreducible if and only if it is soft prime.*

Proof. It follows from Corollary 11.18, Definition 5.27 and Definition 5.28. □

Proposition 11.22. *Every SU-ideal of an intra-regular AG-groupoid S is soft prime if and only if the set of SU-ideals of S is totally ordered under inclusion.*

Proof. It follows from Corollary 11.18, Definition 5.28 and Definition 5.31. □

12. Completely regular AG-groupoids

In this section, we characterize a completely regular AG-groupoids in terms of SU-ideals. An element a of an AG-groupoid S is called *left regular* if there exists $x \in S$ such that

$$a = xa^2 = x(aa).$$

and S is called *left regular* if all elements of S are left regular. An element a of an AG-groupoid S is called *right regular* if there exists $x \in S$ such that

$$a = a^2x = (aa)x.$$

and S is called *right regular* if all elements of S are right regular. An element a of an AG-groupoid S is called *completely regular* if a is regular and left and right regular and S is called *completely regular* if all elements of S are completely regular.

Theorem 12.1. *For an AG-groupoid S, the following conditions are equivalent:*

1) *S is left regular.*
2) *For every SU-left ideal f_S of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.*

Proof. First assume that (1) holds. Let f_S be any SU-left ideal of S and a be any element of S. Since S is left regular, there exists an element x in S such that $a = x(aa)$. Thus, we have

$$f_S(a) = f_S(x(aa)) \subseteq f_S(aa) \subseteq f_S(a)$$

implying that $f_S(a) = f_S(a^2)$. Hence (1) implies (2).

Conversely, assume that (2) holds. Let a be any element of S. Since $L[a^2]$ is a left ideal of S, the soft characteristic function $\mathcal{S}_{(L[a^2])^c}$ is an SU-left ideal of S. Since $a^2 \in L[a^2]$, we have

$$\mathcal{S}_{(L[a^2])^c}(a) = \mathcal{S}_{(L[a^2)]^c}(a^2) = \emptyset$$

implying that $a \in L[a^2] = S(aa)$. This obviously means that S is left regular. So (2) implies (1). This completes the proof. □

Theorem 12.2. *For an AG-groupoid S, the following conditions are equivalent:*

1) *S is right regular.*
2) *For every SU-right ideal f_S of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.*

13. Weakly Regular AG-groupoids

In this section, we characterize a weakly regular AG-groupoid in terms of SU-ideals. An element a of an AG-groupoid S is called weakly-regular if there exist $x, y \in S$ such that $a = (ax)(ay)$ and S is called weakly regular if all elements of S are weakly regular.

Theorem 13.1. *Let S be an AG-groupoid. If S is weakly regular, then $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$ for every SU-right ideal f_S of S and for every SU-ideal g_S of S.*

Proof. Let f_S be an SU-right ideal of S, g_S be an SU-left ideal of S and $x \in S$. Then, since S is weakly regular, $x = (xs)(xt)$ for some $s, t \in S$. Hence,

$$\begin{aligned}(f_S * g_S)(x) &= \bigcap_{x=(xs)(xt)} (f_S(xs) \cup g_S(xt)) \\ &\subseteq f_S(x) \cup g_S(x) \\ &= (f_S \widetilde{\cup} g_S)(x)\end{aligned}$$

Since $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$ always holds for every SU-right ideal f_S and SU-left ideal g_S of S, $f_S \widetilde{\cup} g_S = f_S * g_S$. □

14. Quasi-regular AG-groupoids

An element a of an AG-groupoid S is called left (right) quasi-regular if there exist $x, y \in S$ such that $a = (xa)(ya)(a = (ax)(ay))$ and S is called left (right) quasi-regular if all elements of S are left (right) quasi-regular.

Theorem 14.1. *An AG-groupoid S is left (right) quasi-regular if and only if every SU-left (right) ideal is idempotent.*

Proof. Assume that f_S is an SU-left ideal. Then, there exist $x, y \in S$ such that $a = (xa)(ya)$. So, we have;

$$\begin{aligned}(f_S * f_S)(a) &= \bigcap_{a=(xa)(ya)} (f_S(xa) \cup f_S(ya)) \\ &\subseteq f_S(xa) \cup f_S(ya) \\ &\subseteq f_S(a) \cup f_S(a) \\ &= f_S(a)\end{aligned}$$

and so, $f_S * f_S \widetilde{\supseteq} f_S$. Thus, $f_S * f_S = f_S$ and f_S is idempotent.

Conversely, assume that every SU-left ideal of S is idempotent. Let $a \in S$. Then, since $L[a]$ is a principal left ideal of S, the soft characteristic function $\mathcal{S}_{(L[a])^c}$ is an SU-left ideal of S. It is known that $a \in L[a]$ and so

$$\mathcal{S}_{(L[a])^c}(a) = \emptyset$$

and let $a \notin L[a]L[a]$. Thus, there do not exist $y, z \in L[a]$ such that $a = yz$. Then,

$$(\mathcal{S}_{(L[a])^c} * \mathcal{S}_{(L[a])^c})(a) = \bigcap_{a=yz} (\mathcal{S}_{L^c}(y) \cup \mathcal{S}_{L^c}(z)) = \bigcap_{a=yz} (U \cup U) = U,$$

but this is a contradiction. So $a \in L[a]L[a] = (Sa)(Sa)$. Hence, S is left quasi-regular. The case when S is right quasi-regular can be similarly proved. $\square$

Proposition 14.2. *If f_S is an SU-right ideal of an left (right) quasi-regular AG-groupoid S, then f_S is an SU-ideal of S.*

Proof. Let f_S be an SU-right ideal of an left (right) quasi-regular AG-groupoid S. Then, since $\widetilde{\theta}$ itself is an SU-right ideal of S, and by assumption $\widetilde{\theta}$ is idempotent, we have

$$\widetilde{\theta} * f_S = (\widetilde{\theta} * \widetilde{\theta}) * f_S = (f_S * \widetilde{\theta}) * \widetilde{\theta} \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S.$$

$\square$

Theorem 14.3. *Let f_S be an AG-groupoid S. If $f_S = (f_S * \widetilde{\theta})^2 \widetilde{\cup} (\widetilde{\theta} * f_S)^2$ for every SU-ideal f_S of S, then S is quasi-regular.*

Proof. Let f_S be any SU-right ideal of S. Thus, we have

$$f_S = (f_S * \widetilde{\theta})^2 \widetilde{\cup} (\widetilde{\theta} * f_S)^2 \widetilde{\supseteq} (f_S * \widetilde{\theta})^2 \widetilde{\supseteq} f_S * f_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S$$

and so $f_S = (f_S)^2$. It follows that S is right quasi-regular by Theorem 14.1. One can similarly show that S is left quasi-regular. This completes proof. $\square$

15. Conclusion

In this paper, the concepts of soft union AG-groupoids and certain soft ideals of AG-groupoids are introduced and studied. Furthermore important characterizations of regular, intra-regular, completely regular, weakly regular and quasi-regular AG-groupoids are obtained by using the properties of these soft union ideals.

References

[1] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19-31.
[2] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726-2735.
[3] A. Sezgin and A. O. Atagün, Soft groups and normalistic soft groups, Comput. Math. Appl. 62 (2) (2011) 685-698.
[4] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621-2628.
[5] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Comput. Math. Appl., 59 (2010) 3458-3463.
[6] Y.B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408-1413.
[7] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466-2475.
[8] Y. B. Jun, K.J. Lee and J. Zhan, Soft p-ideals of soft BCI-algebras. Comput. Math. Appl. 58 (2009) 2060-2068.
[9] J. Zhan, Y. B. Jun, Soft BL-algebras based on fuzzy sets, Comput. Math. Appl. 59 (6) (2010) 2037-2046.
[10] A. Sezgin, A. O. Atagün and E. Aygün, A note on soft near-rings and idealistic soft near-rings, Filomat 25 (1) (2011) 53-68.
[11] A.O. Atagün, A. Sezgin, Soft substructures of rings, fields and modules, Comput. Math. Appl. 61 (3) (2011) 592-601.
[12] A. Sezgin, A. O. Atagün, N. Çağman, Union soft substructures of near-rings and N-groups, Neural Comput. Appl., DOI: 10.1007/s00521-011-0732-1.

[13] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555-562.
[14] M. I. Ali, F. Feng, X. Liu, W.K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547-1553.
[15] A. Sezgin and A. O. Atagün, On operations of soft sets, Comput. Math. Appl. 61 (5) (2011) 1457-1467.
[16] M.I. Ali, M. Shabir, M. Naz, Algebraic structures of soft sets associated with new operations, Comput. Math. Appl., 61 (9) (2011) 2647-2654.
[17] N. Çağman and S. Enginoğlu, Soft matrix theory and its decision making, Comput. Math. Appl. 59 (2010) 3308-3314.
[18] N. Çağman and S. Enginoğlu, Soft set theory and uni-int decision making, Eur. J. Oper. Res. 207 (2010) 848-855.
[19] P.K. Maji, A.R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077-1083.
[20] M.I. Ali, A note on soft sets, Rough soft sets and fuzzy soft sets, Appl. Soft Comput. 11 (4) (2011) 3329-3332.
[21] F. Feng, Y.M. Li, V. Leoreanu-Fotea, Application of level soft sets in decision making based on interval-valued fuzzy soft sets, Comput. Math. Appl., 2010 60 (6) 1756-1767.
[22] F. Feng, Y.M. Li, N. Çağman, Generalized uni-int decision making schemes based on choice value soft sets, Eur. J. Oper. Res. 220 (2012) 162-170.
[23] T.M. Basu, N.K. Mahapatra, S.K. Mondal, A balanced solution of a fuzzy soft set based decision making problem in medical science, Appl. Soft Comput. 12 (2012) 3260-3275.
[24] F. Feng, Y.B. Jun, X.Y. Liu, L.F. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (2010) 10-20.
[25] F. Feng, C. Li, B. Davvas, M.I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Comput. 14 (6) (2010) 899-911.
[26] F. Feng, X.Y. Liu, V. Leoreanu-Fotea, Y.B. Jun, Soft sets and soft rough sets, Inform. Sci. 181 (2011) 1125-1137.
[27] A. Sezgin Sezer, A new approach to LA-semigroup theory via the soft sets, Journal of Intelligent and Fuzzy Systmes, 26, (5), 2483-2495, 2014.
[28] A. Sezgin Sezer, Certain characterizations of LA-semigroups by soft sets, Journal of Intelligent and Fuzzy Systmes, 27 (2), 1035-1046, 2014.
[29] M. A. Kazim, M. Naseeruddin, On almost semigroups, The Alig. Bull. Math. 2 (1972) 1-7.
[30] P. Holgate, Groupoids satisfying a simple invertive law, The Math. Stud. 1-4 (61) (1992) 101-106.
[31] P. V. Protic, M. Bozinovic, AG-test and some genaral properties of Abel-Grassmann's Discrete Mathematics, Nis, 3 (1995) 879-886
[32] N. Naseeruddin, Some studies in almost semigroups and flocks. Ph. D. Thesis, Aligarh Muslim University, Aligarh, India (1970).
[33] Q. Mushtaq, S. M. Yousuf, On LA-semigroups, The Alig. Bull. Math. 8 (1978) 65-70.
[34] M. Aslam, M. Shabir, A. Mehmood, Some studies in soft LA-semigroups, Journal of Advanced Research in Pure Mathematics, 3 (4) (2011) 128
[35] M. Khan, N. Ahmad, Characterization of left almost semigroups by their ideals, Journal of Advanced Research in Pure Mathematics, 2 (3) (2010) 61-73.
[36] M. Khan, M. N. A. Khan, On Fuzzy Abel-Grassmann's Groupoids, Advances in Fuzzy Mathematics, 5 (3) (2010) 349-360.
[37] M. Khan, T. Asif, Characterization of intra-regular left almost semigroups by their fuzzy ideals, Journal of Mathematics Research, 2 (3) (2010) 87-96.
[38] S. Abdullah, M. Aslam, M. Imran, M Ibrar, Direct product of intuitionistic fuzzy sets in LA-semigroups-II, Annals of Fuzzy Mathematics and Informatics, 2(2) (2011), 151-160.
[39] S. Abdullah, M. Aslam, N. Amin and M. Taimur, Direct product of finite fuzzy subsets in LA-semigroups, Annals of Fuzzy Math. Info., 3 (2) (2012) 281-292.
[40] N. Çağman, F. Çıtak and H. Aktaş, Soft int-groups and its applications to group theory, Neural Computing and Applications, 21 (Issue 1-Supplement) (2012) 151-158.
[41] A. Sezgin, N. Çağman, F. Çıtak, α-inclusions applied to group theory via soft set and logic, Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics, 68 (1) (2018) 334-352.
[42] A. Sezgin, A new approach to semigoup theory I; Soft union semigroup, ideals and bi-ideals , Algebra Letters, Vol 2016, Article ID 3, 1-46, 2016.