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A New Pointfree Form of Subspaces

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Abstract. This paper concerns sub-topolocales as a generalization of subspaces which are determined by their lattice of open sets. We first study topolocales (dual of topoframes), and sub-topolocales with the connectivity properties of them. Then we show that every sub-topolocale of a regular (resp., completely regular) topolocale is regular (resp., completely regular). Finally, we show that sub-topolocales of a normal topolocale are not necessary normal unless we rewrite this for the special cases.

1. Introduction

Modern topology originates, in principle, from Hausdorff's "Mengenlehre" [16] in 1914. One year earlier there was a paper by Caratheodory [7] containing the idea of a point as an entity localized by a special system of diminishing sets; this is also of relevance for the modern point-free thinking. The category of locales as a substitute for the category of topological spaces was introduced by John Isbell (see [17]). Of course, the idea of regarding frames as generalized topological spaces is a good deal older than this (see the works of Menger [19], and McKinsey and Tarski [21] in the 1940s). Then many authors [1–6, 8–11, 14, 15, 20, 22] have worked on variants of the idea. Frames (pointfree topologies) are complete lattices in which the meet distributes over all joins. In general, there are two different approaches to pointfree spaces and subspaces. In [12] and [26], we developed the theory of frames (pointfree topology), since we need it for introducing an f-ring. For this, by embedding a frame τ in a larger frame L in order that all members of τ are complemented in L, we get a pair (L, τ) called a topoframe. This is the second version of pointfree topology. In a topoframe (L, τ), we have both open and closed elements (the members of τ and their complements, respectively). So the extensive category of topoframes is actually generalized pointfree topology (see also [25]).

When one adopts a localic formulation of topology, sublocales of a locale are of central importance because they correspond to subspaces. In locale theory, the class of all sublocales of a locale as an extension of a subspace is more important; for instance, the closure of a sublocale S is defined as the least closed sublocale containing S (see [23]). In this paper, we define a sub-topolocale whose definition is similar to that of a sublocale (to compare, see 2.2(2) and 3.3(4)). But here, we regard a sub-topolocale as a class of open elements, closed elements and others, and therefore a sub-topolocale is itself a new pointfree form of subspaces. However, one might view the class of all sub-topolocales of a topolocale as an extension of subspaces in future.

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We now briefly summarize the established topolocale theory.

In Section 2, we collect a few facts that will be relevant for our discussion.

In Section 3, we first define a sub-topolocale as the image of a topo-nucleus on a topoframe. Then, to obtain two types of sub-topolocales, we divide the topo-nuclei on a topoframe according to whether they are kernel or closure operators. This section terminates in the fact that the open and closed elements of a sub-topolocale of a topolocale (L, τ) are signified by the open and closed element of (L, τ), respectively.

In Section 4, we define connective topoframes (topolocales) and connected elements of a topoframe. Meanwhile, we find the relation between a dense topo-nucleus and a dense sub-topolocale so that this is in some sense fairly well understood, that is dense sub-topolocales are actually a mirror of dense subspaces.

Finally, in Section 5, preserving some structural properties of a topolocale such as regularity, complete regularity and normality by the respective topo-nuclei will be considered.

2. A Bit of Background

Recall that a *frame* L is a complete lattice in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s)$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of a frame L by 1_L and 0_L respectively. A *frame map* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. An element a of a frame L is said to be *coprime* if $a \neq 0$ and $a \leq x \vee y$ implies $a \leq x$ or $a \leq y$.

Definition 2.1. ([18]) A *nucleus* on a locale (frame) A is a function $j : A \longrightarrow A$ satisfying

(i) $j(a \wedge b) = j(a) \wedge j(b)$

(ii) $a \leq j(a)$

(iii) $j(j(a)) \leq j(a)$

for all $a, b \in A$.

Proposition 2.2. ([18]). *Let $j : A \longrightarrow A$ be a nucleus on a locale A. Then*

1. *the members of A which are fixed under j are exactly the members of $j(A)$, and denoted by A_j. That is*

$$j(A) = \{a \in A \mid j(a) = a\}.$$

2. *A_j is a frame called a* sublocale *of A and $j : A \longrightarrow A_j$ is a frame epimorphism.*

A *topoframe* is a pair (L, τ), abbreviated L_τ, containing a frame L with a subframe τ all of whose elements are complemented in L (see [12] and [26]). For any topological space X, $(\mathcal{P}(X), \mathcal{O}(X))$ is a topoframe, where $\mathcal{O}(X)$ is the lattice of open sets of X considered as a subframe of the power set $\mathcal{P}(X)$. A Please, prepare the paper in Filomat format STRICTLY following instructions in the attached template and upload (and send to me, too) Please, prepare the paper in Filomat format STRICTLY following instructions in the attached template and upload (and send to me, too) *topoframe map* f from a topoframe (L_1, τ_1) to a topoframe (L_2, τ_2) is a frame map f from L_1 to L_2 with the property $f(\tau_1) \subseteq \tau_2$. Each member of τ is called *open* and each member of $\tau' := \{t' \mid t \in \tau\}$ is called *closed*.

Definition 2.3. ([25]) If (L, τ) is a topoframe and $p \in L$, the *closure* of p in L is defined by

$$Cl_{(L,\tau)}(p) = \overline{p} := \bigwedge \{q \in \tau' \mid p \leq q\},$$

and the *interior* of p is defined by

$$Int_{(L,\tau)}(p) = p^\circ := \bigvee \{t \in \tau \mid t \leq p\}.$$

Proposition 2.4. ([25]) *Let L_τ be a topoframe. Then for every $p, q \in L$, the following properties hold.*

1. $\overline{0} = 0^\circ = 0$ *and* $\overline{1} = 1^\circ = 1$.
2. *If* $p \leq q$, *then* $\overline{p} \leq \overline{q}$ *and* $p^\circ \leq q^\circ$.
3. $p \leq \overline{p}$ *and* $p^\circ \leq p$.
4. $\overline{\overline{p}} = \overline{p}$ *and* $(p^\circ)^\circ = p^\circ$.
5. $p \in \tau'$ *if and only if* $p = \overline{p}$, *and* $p \in \tau$ *if and only if* $p = p^\circ$.
6. $\overline{p \vee q} = \overline{p} \vee \overline{q}$ *and* $(p \wedge q)^\circ = p^\circ \wedge q^\circ$.

Mimicking the contruction of the f-ring operations on $\mathcal{R}(\tau) := Hom_{Frm}(\mathcal{O}(\mathbb{R}); \tau)$ by Ball and others [1, 2]-or those on the systematic version of $\mathcal{R}(\tau)$ by Banaschewski [3], we showed in [12] that the set of "real-continuous functions" $\mathcal{R}(L_\tau)$, consisting of all frame homomorphisms $f : \mathcal{P}(\mathbb{R}) \longrightarrow L$ such that $f(\mathcal{O}(\mathbb{R})) \subseteq \tau$, with the operator $\diamond \in \{+, ., \wedge, \vee\}$ defined by

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) \mid Y \diamond Z \subseteq X\},$$

where

$$Y \diamond Z = \{y \diamond z \mid y \in Y, z \in Z\},$$

or, equivalently,

$$(f \diamond g)(X) = \bigvee \{f(\{x\}) \wedge g(\{y\}) \mid x \diamond y \in X\} \tag{1}$$

is a sub-f-ring of $\mathcal{R}(\tau)$.

Definition 2.5. ([13]) For every $f \in \mathcal{R}(L_\tau)$, $f(\{0\})$ is called the *zero-element* of f and denoted by $z(f)$. Also the *cozero-element* of f is defined by $coz(f) := z(f)'$.

Let L be a lattice. We say that an element $a \in L$ has a *pseudo-complement* if there exists a largest element a^* of L such that $a \wedge a^* = 0$. An element a of L is said to be rather below an element b, written as $a \prec b$, if there is an element s such that $a \wedge s = 0$ and $s \vee b = 1$; in other words, $a^* \vee b = 1$. For any x, t in a frame L, we say that x *interpolates* t and write $x \prec\prec t$, if there exists a trail $\{a_i\}_{i \in [0,1] \cap \mathbb{Q}} \subseteq L$ such that $a_0 = x$, $a_1 = t$ and for every $p, q \in [0, 1] \cap \mathbb{Q}$ with $p < q$, $a_p \prec a_q$.

Lemma 2.6. ([23]) *For any frame homomorphism $h : L \longrightarrow M$, $a \prec b$ in L implies $h(a) \prec h(b)$ in M, and consequently $a \prec\prec b \Rightarrow h(a) \prec\prec h(b)$.*

Definition 2.7. ([25]) Let τ be a topoframe on L. For any $a \in L$, define

$$a^\perp := \bigvee \{t \in \tau \mid a \wedge t = 0\}.$$

Lemma 2.8. ([25]) *Let L_τ be a topoframe. Then*

1. *for any* $a \in \tau$, $a^\perp = a^* = (\overline{a})' = (a')^\circ$,
2. *if* $a, b \in \tau$, *then* $\overline{a} \leq b$ *if and only if* $a \prec b$ *in* τ,

where the peudo-complement of a is formed in τ.

Proof. 1. Using Proposition 2.4, we have $a \wedge (\overline{a})' \leq \overline{a} \wedge (\overline{a})' = 0$. So $(\overline{a})'$ belongs to the set

$$\{t \in \tau \mid a \wedge t = 0\}.$$

On the other hand, let $t \in \tau$ in order that $a \wedge t = 0$. Then $a \leq t'$ by the distributiviy of L, and $\overline{a} \leq \overline{t'} = t'$ by Proposition 2.4. So $t \leq (\overline{a})'$, and hence $a^{\perp} = a^{*} = (\overline{a})'$. Moreover,

$$(a')^{\circ} = \bigvee\{t \in \tau \mid t \leq a'\} = (\bigwedge\{t' \in \tau' \mid a \leq t'\})' = (\overline{a})'.$$

2. Using part 1, we have $\overline{a} \leq b$ if and only if $(\overline{a})' \vee b = 1$ if and only if $a^{*} \vee b = 1$ if and only if $a \prec b$ in τ. □

The following well-known theorems are stored on the source files in Shahid Beheshti University by M.M. Ebrahimi and M. Mahmoudi. Also, some results in the next section initiates from their counterparts in those manuscripts.

Theorem 2.9. *Let P, Q be posets considered as categories and let $F : P \to Q$, $G : Q \to P$ be functors. Then the following statements are equivalent:*

1. *F is a left adjoint to G.*
2. *For every $(a, b) \in P \times Q$, $F(a) \leq b$ if and only if $a \leq G(b)$.*
3. *For every $(a, b) \in P \times Q$, $a \leq GF(a)$ and $FG(b) \leq b$.*

   Moreover, these conditions imply
4. *$FGF = F$ and $GFG = G$.*
5. *GF and FG are idempotent.*

Theorem 2.10. *Let P, Q be posets and $f : P \to Q$, $g : Q \to P$ be order-preserving maps.*

1. *If P is a complete poset and f preserves all joins then f has a right adjoint (given by $b \mapsto \bigvee\{a \in P : f(a) \leq b\}$).*
2. *If Q is a complete poset and g preserves all meets then g has a left adjoint (given by $a \mapsto \bigwedge\{b \in Q : g(b) \geq a\}$).*

3. Topolocales and Sub-topolocales

Let X be a topological space. It is known that a subspace of X can be written in exactly one way in the form $(E, \{E \cap V \mid V \in \mathcal{O}(X)\})$ for some $E \subseteq X$. The subspace $(E, \{E \cap V \mid V \in \mathcal{O}(X)\})$ is actually a homomorphic image of $(\mathcal{P}(X), \mathcal{O}(X))$ under the idempotent, monotone self-map $j := E \cap (-)$ on $\mathcal{P}(X)$ with $j(A) \subseteq A$ for all $A \subseteq X$. It gives the idea of regarding new topoframes as the homomorphic images of topoframes under special maps. Such a generalization would yield any new topoframes called sub-topolocales which are presented in this section.

Let $(L, \vee, \wedge, 0, 1)$ be a frame, M a set and j a map from L to M. Then evidently, $j(L)$, under the partial ordering inherited from L, is a bounded partially ordered set with the bottom element $j(0)$ and top element $j(1)$. For every subset $\emptyset \neq S$ of L, define a join $\bigvee^{j(L)}\{j(a) \mid a \in S\} := j(\bigvee^{L} S)$ and a meet $\bigwedge^{j(L)}\{j(a) \mid a \in S\} := j(\bigwedge^{L} S)$, so that $(j(L), \vee^{j(L)}, \wedge^{j(L)}, j(0), j(1))$ is a complete lattice. The arguments for any order-preserving map that assigns to each element of a poset (partially ordered set) an element of a topoframe are similar as follows.

Lemma 3.1. *Let (L, τ) be a topoframe. Let M be a poset and j be an order-preserving map from L to M. Then*

1. *$j(L)$ with the definitions $\bigvee^{j(L)}$, $\wedge^{j(L)}$, $j(0)$ and $j(1)$ mentioned above is a frame;*
2. *$j(\tau)$ is a subframe of $j(L)$;*
3. *$(j(L), j(\tau))$ is a topoframe, and hence we can say j is a topoframe map.*

Proof. By the preceding discussion, it suffices to prove part (3): every subset $j(S)$ of $j(\tau)$, where $S \subseteq \tau$, has the join $j(\bigvee^L S)$ in $j(\tau)$, since $\bigvee^L S \in \tau$, and every finite subset $j(F)$ of $j(\tau)$, where $F \subseteq \tau$, has a meet $j(\bigwedge^L F)$ in $j(\tau)$, since $\bigwedge^L F \in \tau$. Also, every element $j(t) \in j(\tau)$ has the complement $j(t')$ in $j(L)$, since

$$j(t) \wedge^{j(L)} j(t') = j(t \wedge^L t') = j(0) \quad \text{and} \quad j(t) \vee^{j(L)} j(t') = j(t \vee^L t') = j(1)$$

for every $t \in \tau$. Finally, on $j(L)$ and consequently on $j(\tau)$ the binary meet distributes over arbitrary joins, since for every $a \in L$ and $S \subseteq L$, we have

$$\begin{aligned} j(a) \wedge^{j(L)} \bigvee_{s\in S}^{j(L)} j(s) &= j(a) \wedge^{j(L)} j(\bigvee S) \\ &= j(a \wedge^L \bigvee^L S) \\ &= j(\bigvee^L \{a \wedge^L s \mid s \in S\}) \qquad \text{since } \tau \text{ is a frame} \\ &= \bigvee^{j(L)} \{j(a \wedge^L s) \mid s \in S\} \\ &= \bigvee^{j(L)} \{j(a) \wedge^{j(L)} j(s) \mid s \in S\}. \end{aligned}$$

□

Generalizing the algebraic properties of the pair $(\mathcal{P}(X), \mathcal{O}(X))$ of a topological space X, we get the abstract notion of a topoframe. But what we have overlooked is the fact that for any continuous map $f : X \longrightarrow Y$ between topological spaces X and Y we have the map $f^{-1} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$, where $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$ for all $A \in \mathcal{P}(Y)$, of topoframes in the reverse direction. Because of this reversal direction, and to get a more direct generalization of the notion of a topology, we consider the opposite of the category **TFrm** of topoframes which is denoted by **TLoc** and its objects are called *topolocales*.

Since **TLoc** is the dual of the category **TFrm**, a topolocale is just a topoframe, but a map between topolocales (also called a continuous map) is a topoframe map in the opposite direction. In other words, so long as we are working only with objects, the terms “topoframe” and “topolocale” are synonymous.

Definition 3.2. A *topo-nucleus* on a topoframe (L, τ) is a function $j : L \longrightarrow L$ satisfying
(i) $j(a \wedge b) = j(a) \wedge j(b)$
(ii) $j(j(a)) = j(a)$
for all $a, b \in L$. A topo-nucleus on a topoframe (L, τ) is called a $\nearrow$ *topo-nucleus* on (L, τ) (or a nucleus on L), if $a \leq j(a)$ for all $a \in L$ and a $\searrow$ *topo-nucleus* on (L, τ), if $j(a) \leq a$ for all $a \in L$.

Proposition 3.3. *Let (L, τ) be a topoframe. Suppose that j is a topo-nucleus on L. Then*

1. *j is order preserving;*
2. *the members of L fixed under j, denoted by L_j, are exactly the members of $j(L)$;*
3. *the members of $j(\tau)$ (resp. $j(\tau')$) are fixed under j;*
4. *$(L_j, j(\tau))$, with $\wedge^L$ and the join induced by j, is a topoframe which in **TLoc** we call a* sub-topolocale *of (L, τ);*
5. *$j : (L, \tau) \longrightarrow (L_j, j(\tau))$ is an epimorphism in **TFrm** (or a monomorphism in **TLoc**).*

Proof. 1. Since j preserves finite meets.

2. Clearly $\{a \in L \mid j(a) = a\} \subseteq j(L)$. Also, $j(L) \subseteq \{a \in L \mid j(a) = a\}$, since $j(j(a)) = j(a)$ for every $j(a) \in j(L)$. So $j(L) = \{a \in L \mid j(a) = a\}$.

3. Since the members of $j(L)$ are fixed under j and $j(\tau) \subseteq j(L)$, the members of $j(\tau)$ are also fixed under j and similarly for $j(\tau')$.

4. It follows from Lemma 3.1 and that $j(a \wedge b) = j(a) \wedge j(b)$, for all $a, b \in L$.

5. By Lemma 3.1, $j : (L, \tau) \longrightarrow (L_j, j(\tau))$ is a topoframe map. It is clear that $j : (L, \tau) \longrightarrow (L_j, j(\tau))$ is onto, so that j is an epimorphism of topoframes. □

Let P be a poset. It is known that an idempotent, monotone self-map $\pi : P \to P$ is called a *kernel operator* if and only if $\pi(p) \leq p$, and a *closure operator* if and only if $p \leq \pi(p)$, where p stands for arbitrary element in P.

Proposition 3.4. *Let (L, τ) be a topoframe and j a $\nearrow$ topo-nucleus on (L, τ). Then*

1. *j is a closure operator;*
2. *$j(1) = 1$.*

Proof. 1. Since for every $a, b \in L$, $j(a \wedge b) = j(a) \wedge j(b)$, j is order-preserving and since $j(j(a)) = j(a)$, for all $a \in L$, j is idempotent. Finally, and most importantly, $a \leq j(a)$ for all $a \in L$. Hence j is a closure operator.
2. $j(1) = 1$, since, by definition, $j(1) \geq 1$. □

Proposition 3.5. *Let (L, τ) be a topoframe and j a $\searrow$ topo-nucleus on (L, τ). Then*

1. *j is a kernel operator;*
2. *$j(0) = 0$.*

Proof. 1. Trivial.
2. $j(0) = 0$, since, by definition, $j(0) \leq 0$. □

Example 3.6. Let $f : M \longrightarrow L$ be a monomorphism between topolocales (L, A) and (M, B). Let the corresponding frame map be $f^* : L \longrightarrow M$ and let f_* be the right adjoint of f. Then $f^* f_* : M \longrightarrow M$ is a $\searrow$ topo-nucleus on M. The first two properties of a $\searrow$ topo-nucleus for $f^* f_*$ follows from the fact that f^* is a left adjoint to f_* (see Theorem 2.9). Further $f^* f_*$ preserves binary meets since both f_* and f^* do (this is because f^* is a frame map and f_* is a right adjoint).

By the following example, we call a sub-topolocale $(L_j, j(\tau))$ induced by a $\nearrow$topo-nucleus j on (L, τ) a $\nearrow$*sub-topolocale*.

Example 3.7. For any $e \in L$, a sub-topolocale of a topolocale (L, τ), induced by the self-map $e \vee (-) : L \longrightarrow \uparrow e$ that takes any x to $e \vee x$ is a $\nearrow$ sub-topolocale denoted by $(\uparrow e, \tau^e)$, where $\uparrow e := \{x \vee e \mid x \in L\}$ and $\tau^e = \{t \vee e \mid t \in \tau\}$.

Similarly, by the following example, we call a sub-topolocale $(L_j, j(\tau))$ induced by a $\searrow$topo-nucleus j on (L, τ) a $\searrow$*sub-topolocale*.

Example 3.8. For any $e \in L$, a sub-topolocale of a topolocale (L, τ), induced by the self-map $e \wedge (-) : L \longrightarrow \downarrow e$ sending any x to $e \wedge x$ is a $\searrow$ sub-topolocale of (L, τ) denoted by $(\downarrow e, \tau_e)$, where $\downarrow e := \{x \wedge e \mid x \in L\}$ and $\tau_e = \{t \wedge e \mid t \in \tau\}$. In this context, we call $(\downarrow e, \tau_e)$, a *pointfree subspace* of (L, τ).

Lemma 3.9. *Let $(\tau; \wedge, \bigvee, 1, 0)$ be a topoframe on $(L; \wedge, \bigvee, 1, 0)$. Then*

$$(\bigvee_{i \in I} a_i)' = \bigwedge_{i \in I} a_i' ,$$

where $\{a_i\}_{i \in I} \subseteq \tau$.

Proof. Straightforward. □

Proposition 3.10. *In a sub-topolocale $(L_j, j(\tau))$ of a topolocale (L, τ),*

1. *$b \in L_j$ is open in L_j if and only if $b = j(t)$, for some open element t in L;*
2. *$b \in L_j$ is closed in L_j if and only if $b = j(t')$, for some closed element t' in L;*
3. *if $(L_j, j(\tau))$ is a $\nearrow$ sub-topolocale, then for every $b \in L_j$,*

$$Int_{(L_j, j(\tau))}(b) = j(Int_{(L,\tau)}(b)) .$$

4. *if* $(L_j, j(\tau))$ *is a* $\searrow$ *sub-topolocale, then for every* $b \in L_j$,

$$Cl_{(L_j, j(\tau))}(b) = j(Cl_{(L,\tau)}(b)).$$

Proof. 1. It is just the definition of a sub-topolocale on L_j.

2. Since L (and consequently L_j) is distributive, it follows directly from the uniqueness of complements in L_j.

3. For every $b \in L_j$,

$$\begin{aligned} Int_{(L_j, j(\tau))}(b) &= \bigvee^{j(\tau)}\{q \in j(\tau) \mid q \leq b\} \\ &= \bigvee^{j(\tau)}\{j(t) \mid t \in \tau, j(t) \leq b\} && \text{by part (1)} \\ &= j(\bigvee\{t \mid t \in \tau, j(t) \leq b\}) \\ &= j(\bigvee\{t \mid t \in \tau, t \leq b\}) && \text{since } t \leq j(t) \text{ and } j(b) = b \\ &= j(Int_{(L,\tau)}(b)). \end{aligned}$$

4. For every $b \in L_j$, we have

$$\begin{aligned} Cl_{(L_j, j(\tau))}(b) &= \bigwedge^{j(\tau)'}\{q \in j(\tau)' \mid b \leq q\} \\ &= \bigwedge^{j(\tau)'}\{j(t') \mid t \in \tau, b \leq j(t')\} && \text{by part (2)} \\ &= \bigwedge^{j(\tau)'}\{j(t)' \mid t \in \tau, b \leq j(t')\} \\ &= (\bigvee^{j(\tau)}\{j(t) \mid t \in \tau, b \leq j(t')\})' && \text{by Lemma 3.9} \\ &= (j(\bigvee\{t \in \tau \mid b \leq j(t')\}))' \\ &= j((\bigvee\{t \in \tau \mid b \leq j(t')\})') \\ &= j(\bigwedge\{t' \mid t \in \tau, b \leq j(t')\}) && \text{by Lemma 3.9} \\ &= j(\bigwedge\{t' \mid t \in \tau, b \leq t'\}) && \text{since } j(t') \leq t' \text{ and } j(b) = b \\ &= j(Cl_{(L,\tau)}(b)). \end{aligned}$$

□

The open elements in a $\searrow$ sub-topolocale $(\downarrow e, \tau_e)$ of (L, τ) are the meets with e of the open elements in L. Most, but not all, of the related topoframe notions are introduced into e in the same way, by meet, as follows.

Corollary 3.11. *Let* (L, τ) *be a topolocale and* $e \in L$. *Then for the* $\searrow$ *sub-topolocale* $(\downarrow e, \tau_e)$ *and the* $\nearrow$ *sub-topolocale* $(\uparrow e, \tau^e)$, *the following statements hold.*

1. $b \in \downarrow e$ *is open in* $(\downarrow e, \tau_e)$ *if and only if* $b = t \wedge e$, *where* t *is open in* L;
2. $c \in \downarrow e$ *is closed in* $(\downarrow e, \tau_e)$ *if and only if* $c = t' \wedge e$ *where* t' *is closed in* L;
3. $b \in \uparrow e$ *is open in* $(\uparrow e, \tau^e)$ *if and only if* $b = t \vee e$, *where* t *is open in* L;
4. $c \in \uparrow e$ *is closed in* $(\uparrow e, \tau^e)$ *if and only if* $c = t' \vee e$ *where* t' *is closed in* L;
5. *For every* $b \geq e$, $Int_{(\uparrow e, \tau^e)}(b) = e \vee Int_{(L,\tau)}(b)$.
6. *For every* $b \leq e$, $Cl_{(\downarrow e, \tau_e)}(b) = e \wedge Cl_{(L,\tau)}(b)$.

4. Connectedness in Topoframes (Topolocales)

The topological study of connectedness is heavily geometric (or visual). But the viewpoint of topoframes of connectedness in lattice theory makes it algebraic. All known examples and counterexamples given in topological spaces are true for the analogous statement in topoframes. Henceforth we avoid presenting them, as much as possible. By the way, almost all of the material developed in this section are analogous to those of general topology stated in [24].

The elements a and b in a bounded lattice L are called *disjoint* if and only if $a \wedge b = 0$. We say that a is *non-zero* if and only f $a \neq 0$. A convenient way of expressing the relation $a \leq b$ in words is to say that a is *below* b, and that b is *above* a, or that b *dominates* a. An element a in a topoframe is called *clopen* if it is both open and closed.

Definition 4.1. A topoframe (L, τ) is called *disconnected* if and only if there are disjoint non-zero elements $a, b \in \tau$ such that $a \vee b = 1$. Clearly, such a and b are clopen and $a' = b$. We then say that 1 is disconnected by two clopen elements a and a'. When no such disconnection exists, L_τ is called *connected*.

Definition 4.2. A topoframe map $h : (L, A) \longrightarrow (M, B)$ is called *dense* if and only if the frame map $h|_A : A \longrightarrow B$ is *dense*; that is $x = 0$ whenever $h(x) = 0$, for all $x \in A$. The notion of a *codense* topoframe map is defined dually; that is $x = 1$ whenever $h(x) = 1$, for all $x \in A$.

The definition of a dense function in frames and topoframes arise from dense images of continuous functions in topological spaces, by the well-known fact that a continuous map $f : X \longrightarrow Y$ is dense if and only if $f^{-1} : O(Y) \longrightarrow O(X)$ is dense as a frame map.

Proposition 4.3. *The image of a disconnected topoframe under a dense topoframe map is disconnected.*

Proof. Let (L_1, τ_1) and (L_2, τ_2) be topoframes. Suppose (L_1, τ_1) is disconnected and f is a dense topoframe map from (L_1, τ_1) onto (L_2, τ_2). If 1_{L_1} were disconnected by a and b, then 1_{L_2} would be disconnected by $f(a)$ and $f(b)$, because $f(a)$ and $f(b)$ are clopen by the continuity of f, and non-zero by the density of f. So (L_2, τ_2) must be disconnected. □

Definition 4.4. Let (L, τ) be a topoframe. Non-zero elements a and b in L are *mutually separated* in (L, τ) if and only if

$$\overline{a} \wedge b = a \wedge \overline{b} = 0 \ .$$

Lemma 4.5. *A topoframe (L, τ) is connected if and only if there are no mutually separated elements a and b in L with $1 = a \vee b$.*

Proof. If 1 is disconnected by non-zero clopen elements a and b in (L, τ), then a and b are mutually separated in (L, τ), since a and b are closed in (L, τ) and hence $\overline{a} \wedge b = a \wedge \overline{b} = a \wedge b = 0$.

Conversely, if a and b are mutually separated in (L, τ) with $1 = a \vee b$, then $1 = a \vee b \leq \overline{a} \vee b$ and $0 = \overline{a} \wedge b$. So $b = \overline{a}' \in \tau$ and hence b is open in L. Similarly we can show that a is open in L, so that a and b are non-zero disjoint open elements in L with $1 = a \vee b$. □

Remark 4.6. It is an immediate consequence of Lemma 4.5 that if (L, τ) is a topoframe (topolocale) and j is a topo-nucleus on L, then the following statements are equivalent.

1. $(L_j, j(\tau))$ is disconnected.
2. There are mutually separated elements a and b in L_j with $a \vee^{L_j} b = j(1)$.

Let (L, τ) be a topoframe. An element $e \in L$ is called *disconnected* if there are disjoint non-zero elements $a, b \in \tau_e$ such that $a \vee b = e$. By the following proposition, we use sometimes the topological term "e is connected (resp. disconnected)" instead of "$(\downarrow e, \tau_e)$ is connected (resp. disconnected)".

Proposition 4.7. *Let (L, τ) be a topoframe and $e \in L$. Then the following statements are equivalent.*

1. *$(\downarrow e, \tau_e)$ is connected.*
2. *e is connected.*
3. *There are no mutually separated elements a and b in L with $e = a \vee b$.*

Proof. By definition, parts (1) and (2) are equivalent. Also, by Remark 4.6, Parts (1) and (3) are equivalent. □

Corollary 4.8. *If a and b are mutually separated in (L, τ) and e is a connected element below $a \vee b$, then $e \leq a$ or $e \leq b$.*

Proof. Since a and b are mutually separated in (L,τ), we have $\overline{e \wedge a} \wedge b \wedge e \leq \overline{a} \wedge b = 0$ and $\overline{e \wedge b} \wedge a \wedge e \leq \overline{b} \wedge a = 0$, and since e is connected, we have $e \wedge a = 0$ or $e \wedge b = 0$, unless $e \wedge a$ and $e \wedge b$ are mutually separated in (L,τ) with $e = e \wedge (a \vee b) = (e \wedge a) \vee (e \wedge b)$. By using Proposition 4.7, we contradict the assumption that e is connected. If $e \wedge a = 0$, then $e = e \wedge (a \vee b) = e \wedge b$, and if $e \wedge b = 0$ then $e = e \wedge (a \vee b) = e \wedge a$. Hence $e \leq b$ or $e \leq a$. □

The last proposition and its corollary provides us with some neat ways of proving a given topoframe (L,τ) is connected. For this purpose consider, first, the following lemma.

Lemma 4.9. *Let (L,τ) be a topoframe, and let a_1 and a_2 be connected elements of L with $a_1 \wedge a_2 \neq 0$. Then $a_1 \vee a_2$ is connected.*

Proof. Let $c := a_1 \vee a_2$. Using Proposition 4.7, we suppose that, to the contrary, $c = a \vee b$, where a and b are mutually separated in (L,τ). Then, since a_i is a connected element below $a \vee b$ for each $i = 1,2$, we have $a_i \leq a$ or $a_i \leq b$, by Corollary 4.8. Since the a_i's are not disjoint, while a and b are, we must have $a_i \leq a$ for $i = 1,2$ or $a_i \leq b$ for $i = 1,2$; say the latter. Then $c \leq b$, so $a = 0$, since $a = a \wedge c = a \wedge b \leq a \wedge \overline{b} = 0$. Thus c can never be the join of two (non-trivial) mutually separated elements in (L,τ), and so c is connected. □

In a topoframe, the meet of any two connected elements need not necessarily be connected. However, by Lemma 4.9, the join of two connected elements a and b is connected if and only if $a \wedge b$ is non-zero.

Proposition 4.10. *Let (L,τ) be a topoframe. Suppose that $\{a_\lambda\}_{\lambda\in\Lambda}$ and $\{b_n\}_{n\in\mathbb{N}}$ are subsets of L. Then*

1. *if $\bigvee a_\lambda = 1$, $\bigwedge a_\lambda \neq 0$, and for each λ the pointfree subspace $(\downarrow a_\lambda, \tau_{a_\lambda})$ is connected, then (L,τ) is connected;*
2. *if $\bigvee_{n=1}^{\infty} b_n = 1$, where for each $n \in \mathbb{N}$, $(\downarrow b_n, \tau_{b_n})$ is connected and $b_{n-1} \wedge b_n \neq 0$ for each $n \geq 2$, then (L,τ) is connected.*

Proof. 1. Suppose a and b are mutually separated in (L,τ) with $1 = a \vee b$. Then, since a_λ is a connected element below $a \vee b$ for each λ, we have $a_\lambda \leq a$ or $a_\lambda \leq b$, by Corollary 4.8. Since the a_λ's are not disjoint, while a and b are, we must have $a_\lambda \leq a$ for all λ or $a_\lambda \leq b$ for all λ; say the latter. Then $1 \leq b$, so that $a = 0$, because $a = a \wedge 1 = a \wedge b \leq a \wedge \overline{b} = 0$. Thus 1 can never be the join of two (non-trivial) mutually separated elements in (L,τ), and so (L,τ) is connected.

2. b_1 is connected, and if $b_1 \vee ... \vee b_{n-1}$ is connected, so is $a_n = b_1 \vee ... \vee b_n$ by Lemma 4.9. Thus, by Proposition 4.7, $(\downarrow a_n, \tau_{a_n})$ is connected, for $n = 1,2,...$. Since $\bigwedge a_n = b_1$ is non-zero and $\bigvee a_n = 1$, (L,τ) is connected, by part (1). □

A sub-topolocale $(L_j, j(\tau))$ of a topolocale (L,τ) is called *dense* if and only if j is a dense map, and the element e is dense in (L,τ) if and only if $e^\perp = 0$, or equivalently, $\overline{e} = 1$.

Remark 4.11. In a topoframe (L,τ), if $e \in L$ is a dense element, then the pointfree subspace $(\downarrow e, \tau_e)$ is dense, because whenever $j(t) = 0$, for some $t \in \tau$, then $t \wedge e = 0$ and hence $1 = \overline{e} \leq \overline{t'} = t'$, that is $t = 0$. Conversely, let $(\downarrow e, \tau_e)$ is a dense pointfree subspace. Then, by definition, $e^\perp \wedge e = 0$, i.e., $j(e^\perp) = 0$. Consequently, $e^\perp = 0$, since j is a dense map and $e^\perp \in \tau$. Thus e is a dense element, as desired.

The following proposition states that if a topolocale (L,τ) has at least a connected dense $\searrow$ sub-topolocale, then (L,τ) is connected.

Proposition 4.12. *Every dense $\searrow$ sub-topolocale of a disconnected topolocale is disconnected. In particular, every dense pointfree subspace of a disconnected topoframe (topolocale) is disconnected.*

Proof. Suppose (L,τ) is disconnected and $(L_j, j(\tau))$ a dense $\searrow$ sub-topolocale of (L,τ). Assume that $1 = c \vee d$, where c and d are disjoint, (non-trivial) clopen elements in L. Then $j(1) = j(c) \vee^{L_j} j(d)$. Also $j(c)$ and $j(d)$ are disjoint since $j(c) \wedge j(d) = j(c \wedge d) = j(0) = 0$, and clopen elements in L_j, by parts (1) and (2) of Proposition 3.10. To show that $j(c)$ and $j(d)$ are non-trivial, assume, to the contrary, that $j(c) = 0$, say. Then $c = 0$, by the density of j. This contradicts the fact that c is non-zero. Thus $j(c)$ and $j(d)$ are non-zero and so $(L_j, j(\tau))$ is disconnected. □

Corollary 4.13. *Let (L, τ) be a topoframe . If e is a connected element of L and $e \leq a \leq \overline{e}$, then a is connected.*

Proof. Suppose, to the contrary, that a is disconnected. To derive a contradiction, by using Proposition 4.12 and Remark 4.11, it is enough to show e is dense in $(\downarrow a, \tau_a)$. Since $a \leq \overline{e}$, we have $a \leq \overline{e} \wedge a$, and, by part (6) of Proposition 3.11, $a \leq Cl_{(\downarrow a, \tau_a)}(e)$. On the other hand, since $e \leq a$, we infer $Cl_{(\downarrow a, \tau_a)}(e) \leq a$. So $Cl_{(\downarrow a, \tau_a)}(e) = a$, that is e is disconnected, by definition. □

If one needs connectedness of a topoframe which is not itself connected, one can usually just look at the individual "components" (maximal connected pieces) of 1, as described now.

Let (L, τ) be a topoframe. For any $a \in L$ and a connected element $0 < x \leq a$, if it exists, the largest connected element c of L with $x \leq c \leq a$ is called a *component* of a dominating x. In particular, for any coprime element $p \in L$ as a connected element below a member a, if it exists, the largest connected element c of L with $p \leq c \leq a$ is a component of a dominating p. In a topoframe (L, τ) a component of 1_L dominating a coprime element p, if it exists, is the supremum of all connected elements dominating p. If c_1 and c_2 are two different components of a member a, whenever they exist, then $c_1 \wedge c_2 = 0$; otherwise, by Lemma 4.9, $c_1 \vee c_2$ would be a connected element greater than c_1 and c_2 , which is impossible. Applying this argument, we infer that if $\{a_\lambda\}_{\lambda \in \Lambda}$ is the family of all components of 1_L, whenever this exists, then $a_{\lambda_1} \wedge a_{\lambda_2} = 0$ for each $\lambda_1 \neq \lambda_2$ in Λ.

Remark 4.14. Let (L, τ) be a topoframe. The components of 1_L (when they exist) are closed elements. For this, let c be a component of 1_L in L. Then, by Corollary 4.13, $\overline{c}$ is a connected element which dominates c. Since c is a component of 1_L, this implies $c = \overline{c}$. Hence c is closed.

5. Special Sub-topolocales

In this section, we show that any sub-topolocale of a regular (resp. completely regular) topolocale is also regular (resp. completely regular). Then we show that some special kind of sub-topolocales of normal topolocales are normal.

As is defined in [27], a topoframe (topolocale) L_τ is said to be *regular* if and only if for every $t \in \tau$,

$$t = \bigvee \{x \in \tau \mid \overline{x} \leq t\},$$

and L_τ is said to be *completely regular* if and only if for every $t \in \tau$, there exists $\{f_i\}_{i \in I} \subseteq \mathcal{R}(L_\tau)$ such that

$$t = \bigvee_{coz(f_i) \leq t} coz(f_i).$$

Proposition 5.1. *Each sub-topolocale of a regular topolocale is regular.*

Proof. Let (L, τ) be a regular topolocale and j a topo-nucleus on (L, τ). Let $j(t) \in j(\tau)$ for some $t \in \tau$. Then for such t, we have

$$t = \bigvee \{x \in \tau \mid x \prec t \text{ in } \tau\},$$

by Lemma 2.8. Hence

$$\begin{aligned} j(t) &= j(\bigvee\{x \in \tau \mid x \prec t\}) \\ &= \bigvee\nolimits^{L_j}\{j(x) \mid x \in \tau, x \prec t \text{ in } \tau\} && \text{by definition} \\ &\leq \bigvee\nolimits^{L_j}\{j(x) \mid x \in \tau, j(x) \prec j(t) \text{ in } j(\tau)\} && \text{by Lemma 2.6} \\ &= \bigvee\nolimits^{L_j}\{j(x) \mid x \in \tau, Cl_{(L_j, j(\tau))} j(x) \leq j(t)\} && \text{by Lemma 2.8} \\ &\leq j(t). \end{aligned}$$

Hence the sub-topolocale $(L_j, j(\tau))$ is a regular topolocale. □

Consider now the following assertion.

Proposition 5.2. *Each sub-topolocale of a completely regular topolocale is completely regular.*

Proof. Let (L, τ) be a completely regular topolocale and j a topo-nucleus on (L, τ) and let $j(t) \in j(\tau)$ for some $t \in \tau$. Since L_τ is completely regular, there exists $\{f_i\}_i \subseteq \mathcal{R}(L_\tau)$ such that

$$t = \bigvee_i \{coz(f_i) \mid coz(f_i) \leq t\}.$$

Note that for any i, the composite $j \circ f_i$ of two topoframe map j and f_i is a topoframe map belonging to $\mathcal{R}((L_j)_{j(\tau)})$. Hence

$$\begin{aligned} j(t) &= j(\textstyle\bigvee_i \{coz(f_i) \mid coz(f_i) \leq t\}) \\ &= \textstyle\bigvee^{L_j} \{j(coz(f_i)) \mid coz(f_i) \leq t\} \\ &\leq \textstyle\bigvee^{L_j} \{j(coz(f_i)) \mid j(coz(f_i)) \leq j(t)\} \\ &= \textstyle\bigvee^{L_j} \{coz(j \circ f_i) \mid j \circ f_i \in \mathcal{R}((L_j)_{j(\tau)}), coz(j \circ f_i) \leq j(t)\} \\ &\leq j(t) \end{aligned}$$

and so the sub-topolocale $(L_j, j(\tau))$ is completely regular. □

Finally, we show that certain types of sub-topolocales of a normal topolocale are normal. A topoframe (topolocale) L_τ is *normal* if and only if whenever a and b are disjoint closed elements in L, there are disjoint open elements u and v with $a \leq u$ and $b \leq v$. The next lemma presents an equivalence statement for normality that will be needed later.

Lemma 5.3. *A topoframe (topolocale) L_τ is normal if and only if for any closed element k and open element t in L with $k \leq t$, there exists an open element v such that*

$$k \leq v \leq \overline{v} \leq t.$$

Proof. Let $k \in \tau'$ and $t \in \tau$ with $k \leq t$. Then $k \wedge t' = 0$ in τ' and hence there are $u, v \in \tau$ with $u \wedge v = 0$, $t' \leq u$ and $k \leq v$. Therefore $k \leq v \leq u' \leq t$. Consequently,

$$k \leq v \leq \overline{v} \leq u' \leq t,$$

since u' is closed.

For the converse, let $a, b \in \tau'$ with $a \wedge b = 0$. Then $a \leq b'$ and hence there exists an open element v such that

$$a \leq v \leq \overline{v} \leq b'.$$

So that $a \leq v$, $b \leq \overline{v}'$ and $v \wedge \overline{v}' \leq \overline{v} \wedge \overline{v}' = 0$. □

Theorem 5.4. *Let (L, τ) be a normal topolocale and let j be a topo-nucleus on L. Then the following statements hold.*

1. *If j is a $\searrow$ topo-nucleus with $j(\tau') \subseteq \tau'$, then $(L_j, j(\tau))$ is a normal topolocale.*
2. *If j is a $\nearrow$topo-nucleus with $j(\tau) \subseteq \tau$, then $(L_j, j(\tau))$ is a normal topolocale.*

Proof. 1. Let $j(k)$, for some $k \in \tau'$, be a closed element in L_j and let $j(t)$, for some $t \in \tau$, be an open element in L_j with $j(k) \leq j(t)$. Then $j(k) \leq t$, since $j(t) \leq t$, and $j(k)$ is closed, since $j(\tau') \subseteq \tau'$. Using Lemma 5.3, for closed element $j(k)$ in L, and open element t in L with $j(k) \leq t$, there exists an open element v in L such that $j(k) \leq v \leq \overline{v} \leq t$. Now, using part (2) of Lemma 2.8, we have the equivalence

$$\overline{v} \leq t \quad \text{if and only if} \quad v \prec t \text{ in } \tau\,,$$

and, by Lemma 2.6, we have the implication

$$v \prec t \text{ in } \tau \quad \text{implies} \quad j(v) \prec j(t) \text{ in } j(\tau)\,.$$

Again, using part (2) of Lemma 2.8, we have

$$j(v) \prec j(t) \text{ in } j(\tau) \quad \text{implies} \quad Cl_{(L_j, j(\tau))} j(v) \leq j(t)\,.$$

On the other hand, $j(k) \leq v$ implies $j(k) \leq j(v)$, so that

$$j(k) \leq j(v) \leq Cl_{(L_j, j(\tau))} j(v) \leq j(t)\,,$$

as desired.

2. The proof of the second assertion is similar. □

Consider, finally, the following result.

Corollary 5.5. *Let (L, τ) be a normal topolocale. Then the following statements hold.*

1. *If e is a closed element in L, then $(\downarrow e, \tau_e)$ is a normal topolocale, and*
2. *If e is an open element in L, then $(\uparrow e, \tau^e)$ is a normal topolocale.*

Proof. 1. Since e is a closed element in L, for any closed element $a \in L$, $a \wedge e$ is also closed in L. So the $\searrow$ topo-nucleus $j := e \wedge (-)$ restricted to τ' takes value in τ' and thus, by part (1) of Theorem 5.4, $(\downarrow e, \tau_e)$ is a normal topolocale.

2. Since e is an open element in L, for each open element $a \in L$, $a \vee e$ is also open in L. So the closed topo-nucleus $j := e \vee (-)$ restricted to τ takes value in τ and thus, by part (2) of Theorem 5.4, $(\uparrow e, \tau^e)$ is a normal topolocale. □

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