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# Canonical Hankel Wavelet Transformation and Calderón's Reproducing Formula

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**Abstract.** In this work, we have discussed some basic properties of canonical Hankel wavelet transformation. Further the Calderón's reproducing formula for linear canonical Hankel wavelet transformation is obtained.

## 1. Introduction

Let $g$ and $h$ be any two functions in $L^2(\mathbb{R})$ which satisfy

$$\int_0^\infty \hat{g}(k\lambda)\hat{h}(k\lambda)\frac{dk}{k} = 1, \qquad \text{for } k > 0, \ \lambda \in \mathbb{R} \setminus \{0\}, \tag{1}$$

where $\hat{}$ denotes the Fourier transform on $\mathbb{R}$. If

$$g_k(x) = \frac{1}{k}g\left(\frac{x}{k}\right) \quad \text{and} \quad h_k(x) = \frac{1}{k}h\left(\frac{x}{k}\right),$$

then the classical Calderón's formula [2] can be formulated as:

$$f = \int_0^\infty f * g_k * h_k \frac{dk}{k}, \tag{2}$$

where $*$ denotes the classical convolution on $\mathbb{R}$. Above formula was earlier used in the Calderón-Zygmund theory of singular integral operators, but afterwards it was carried to various areas of applied mathematics, in particular in wavelet theory [5, 7]. Motivated by [7], Pathak et al. [11] defined the Calderón's formula associated with Hankel convolution. Further extending the theory of [7, 11] Upadhyay et al. [17] introduced the Calderón's reproducing formula associated with Watson convolution.
Now, in this paper we have defined the Calderón's formula associated with linear canonical Hankel wavelet transformation.
The linear canonical transformation (LCT) was first introduced in 1970's as an integral transformation with

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four parameters $a, b, c, d$ [4, 9]. Wolf [18] introduced the canonical Hankel transformation of function $f$ for $n$-dimension and $\nu \geq 1 - n$ as:

$$\tilde{f}(y) = \int_0^\infty x^{n-1}\mathcal{K}(y,x)f(x)dx,$$

where

$$\mathcal{K}(y,x) = b^{-1}e^{-\frac{i\pi}{2}(\frac{n}{2}+\nu)}(xy)^{1-\frac{n}{2}}e^{\frac{i}{2b}(ax^2+dy^2)}J_{\frac{n}{2}+\nu-1}\left(\frac{xy}{b}\right),$$

which reduces to conventional Hankel transformation for $n = 2$, $a = d = 0$ and $b = -c = 1$. Bultheel et al. [1] reduces $\mathcal{K}(y,x)$ to the kernel of fractional Hankel transformation by replacing $a = d = \cos\theta$ and $b = -c = \sin\theta$.
Motivated by Wolf [18] and Bultheel et al. [1], we have introduced the linear canonical Hankel transformation (LCHT) $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}$ of integrable function $f$ on $I = (0,\infty)$ depending on the uni-modular matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^{-1}$, with three more real parameters $\alpha, \beta, \nu$ for two dimension [13]:

$$(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(y) = \int_0^\infty K^A(y,x)f(x)dx, \tag{3}$$

where

$$K^A(y,x) = \nu\beta\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}x^{-1-2\alpha+2\nu}e^{\frac{i\beta}{2b}(ax^{2\nu}+dy^{2\nu})}(xy)^\alpha J_\mu\left(\frac{\beta}{b}(xy)^\nu\right), \quad b \neq 0,$$

for all $\nu\mu - \alpha + 2\nu \geq 1$, and $J_\mu$ is Bessel function of first kind of order $\mu$. The inverse of (3) is given by:

$$f(x) = \left(\mathscr{H}^{A^{-1}}_{\mu,\nu,\alpha,\beta}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(y)\right)(x) = \int_0^\infty K^{A^{-1}}(x,y)(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(y)dy,$$

where $A^{-1}$ is inverse of the matrix $A$. Moreover kernel $K^A$ of linear canonical Hankel transformation satisfy the following properties:

(i) $$\int_0^\infty K^A(y,\xi)K^B(\xi,x)d\xi = K^{AB}(y,x),$$

(ii) $$\int_0^\infty K^A(y,\xi)K^{A^{-1}}(\xi,x)d\xi = \delta(y-x),$$

where

$$\delta(y-x) = \left(\frac{\nu\beta}{b}\right)^2 y^\alpha x^{-1-\alpha+2\nu}\int_0^\infty t^{-1+2\nu}J_\mu\left(\frac{\beta}{b}(yt)^\nu\right)J_\mu\left(\frac{\beta}{b}(xt)^\nu\right)dt.$$

The linear canonical Hankel transformation satisfy the additivity and reversibility conditions as:

$$\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\mathscr{H}^B_{\mu,\nu,\alpha,\beta} = \mathscr{H}^{AB}_{\mu,\nu,\alpha,\beta} \quad \text{and} \quad (\mathscr{H}^A_{\mu,\nu,\alpha,\beta})^{-1} = \mathscr{H}^{A^{-1}}_{\mu,\nu,\alpha,\beta}.$$

The Parseval's identity of operator $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}$ become as [13],

$$\int_0^\infty x^{-1-2\alpha+2\nu}f(x)\overline{g(x)}dx = \int_0^\infty y^{-1-2\alpha+2\nu}\left(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f\right)(y)\overline{\left(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}g\right)(y)}dy.$$

Throughout this study we have used $\mathscr{H}^A_\mu$ as a particular case of $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}$ for $\nu = \beta = 1, \alpha = -\mu$. Therefore the LCHT of a function $f \in L^2_\mu(I)$ of order $\mu \geq -\frac{1}{2}$ reduces as:

$$(\mathscr{H}^A_\mu f)(\omega) = \tilde{f}^A(\omega) = \int_0^\infty \mathbb{K}^A(\omega,t)f(t)dt, \tag{4}$$

where

$$\mathbb{K}^A(\omega,t)=\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}t^{1+2\mu}e^{\frac{i}{2b}(at^2+d\omega^2)}(t\omega)^{-\mu}J_\mu\left(\frac{t\omega}{b}\right),\quad b\neq 0.$$

The inverse of (4) is given as follows:

$$f(t)=((\mathscr{H}_\mu^A)^{-1}\tilde{f}^A)(t)=\int_0^\infty \mathbb{K}^{A^{-1}}(t,\omega)\tilde{f}^A(\omega)d\omega.$$

For the operator $\mathscr{H}_\mu^A$, the Parseval equality becomes

$$\int_0^\infty f(t)\overline{g(t)}t^{1+2\mu}dt=\int_0^\infty \tilde{f}^A(\omega)\overline{\tilde{g}^A(\omega)}\omega^{1+2\mu}d\omega. \tag{5}$$

**Definition 1.1.** *The space $L_\mu^p(I)$, is the space of all those real valued measurable function $\phi$ on $I=(0,\infty)$ such that the norm*

$$\|\phi\|_{L_\mu^p}=\begin{cases}\left(\int_0^\infty|\phi(t)|^p t^{1+2\mu}dt\right)^{\frac{1}{p}}, & 1\le p<\infty,\ \mu\in\mathbb{R},\\ ess\sup\limits_{t\in I}|\phi(t)|, & p=\infty,\end{cases}$$

*is finite.*

As per [10, 14, 15], we have given the canonical Hankel convolution of $\phi$ and $\psi\in L_\mu^1(I)$ as [8]:

$$(\psi\#_A\phi)(t)=\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}\int_0^\infty \phi(\omega)(\tau_t^A\psi)(\omega)e^{\frac{ia}{2b}\omega^2}\omega^{1+2\mu}d\omega, \tag{6}$$

where the canonical Hankel translation $\tau_t^A$ is given as:

$$\begin{aligned}(\tau_t^A\psi)(\omega) &= \psi^A(t,\omega)\\ &= \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b}\int_0^\infty \psi(z)D_\mu^A(t,\omega,z)e^{\frac{ia}{2b}z^2}z^{1+2\mu}dz,\end{aligned}$$

where

$$D_\mu^A(t,\omega,z) = \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b}\int_0^\infty (ts)^{-\mu}J_\mu\left(\frac{ts}{b}\right)(\omega s)^{-\mu}J_\mu\left(\frac{\omega s}{b}\right)(zs)^{-\mu}J_\mu\left(\frac{zs}{b}\right)e^{-\frac{ia}{2b}(t^2+\omega^2+z^2)}s^{1+2\mu}ds.$$

For $0<t,\omega,z<\infty$, we have

$$\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}\int_0^\infty (zs)^{-\mu}J_\mu\left(\frac{zs}{b}\right)e^{\frac{i}{2b}(az^2+ds^2)}z^{1+2\mu}D_\mu^A(t,\omega,z)dz=(ts)^{-\mu}J_\mu\left(\frac{ts}{b}\right)(\omega s)^{-\mu}J_\mu\left(\frac{\omega s}{b}\right)e^{-\frac{ia}{2b}(t^2+\omega^2)}e^{\frac{id}{2a}s^2},$$

and

$$\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}\int_0^\infty D_\mu^A(t,\omega,z)e^{\frac{ia}{2b}z^2}z^{1+2\mu}dz=\frac{e^{-\frac{ia}{2b}(t^2+\omega^2)}}{(2b)^\mu\Gamma(\mu+1)}.$$

As per [3, 6, 10, 14, 15], we have defined the canonical Hankel wavelet $\psi_{n,m,A}$ of a function $\psi\in L_\mu^2(I)$ with dilation and translation parameters $m>0,\ n\ge 0$ respectively, as:

$$\begin{aligned}\psi_{n,m,A} &= \mathcal{D}_m(\tau_n^A\psi)(t)=\mathcal{D}_m\psi^A(n,t)=\psi_m^A(n,t) &(7)\\ &= m^{-\frac{5}{2}-2\mu}e^{\frac{ia}{2b}(\frac{1}{m^2}-1)(t^2+n^2)}\psi^A\left(\frac{n}{m},\frac{t}{m}\right)\\ &= m^{-\frac{5}{2}-2\mu}e^{\frac{ia}{2b}(\frac{1}{m^2}-1)(t^2+n^2)}\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}\int_0^\infty \psi(z)D_\mu^A\left(\frac{n}{m},\frac{t}{m},z\right)e^{\frac{ia}{2b}z^2}z^{1+2\mu}dz, &(8)\end{aligned}$$

where $\mathcal{D}_m$ denotes the canonical dilation operator.
We define the wavelet transformation involving canonical Hankel wavelet as:

$$W_\psi^A f(n,m) = \int_0^\infty f(t)\overline{\psi_{n,m,A}(t)}t^{1+2\mu}dt, \tag{9}$$

and the admissibility condition of the canonical Hankel wavelet is given by:

$$C_\psi^{\mu,A} = \int_0^\infty \omega^{-1}|(\mathscr{H}_\mu^A e^{-\frac{ia}{2b}(\cdot)^2}\psi)(\omega)|^2 d\omega < \infty. \tag{10}$$

The inversion formula for (9), is given as:

$$f(t) = \frac{1}{b^2 C_{\psi,\phi}^{\mu,A}}\int_0^\infty\int_0^\infty (W_\psi^A f)(n,m)\phi_{n,m,A}(t)n^{1+2\mu}dn\,dm. \tag{11}$$

The whole paper consists of four sections. In Section 1, brief introduction about LCHT, wavelet transformation associated with the particular case of LCHT and Calderón's formula are given. Section 2 contains the preliminary results for canonical translation, canonical convolution and canonical Hankel wavelet. Section 3 is devoted to study some properties for canonical Hankel wavelet transformation. In the last section, Calderón's formula associated with the canonical Hankel wavelet transformation is obtained.

## 2. Preliminaries

In this section, we enlisted some basic results:

**Lemma 2.1.** *Let $f$ and $g \in L_\mu^2(I)$ and $\frac{1}{p}+\frac{1}{q}=1,\ 1<p,q<\infty$. Then*

(i) $\|\tau_\omega^A g(t)\|_{L_\mu^p} \le \frac{1}{(2b)^\mu\Gamma(\mu+1)}\|g\|_{L_\mu^p}$,

(ii) $\|e^{-\frac{ia}{2b}(\cdot)^2} f\,\#_A\, g\|_{L_\mu^\infty} \le \frac{1}{2^\mu b^{\mu+1}\Gamma(\mu+1)}\|f\|_{L_\mu^p}\|g\|_{L_\mu^q}$.

*Proof.* Proof is straight forward as [10]. □

**Lemma 2.2.** *If $\psi \in L_\mu^2(I)$, and $\psi_m^A$ is canonical dilation of $\psi$ for $m>0$ given by*

$$\psi_m^A = m^{-\frac{5}{2}-2\mu}e^{\frac{ia}{2b}(\frac{1}{m^2}-1)t^2}\psi\left(\frac{t}{m}\right), \tag{12}$$

*then*

$$\|\psi_m^A\|_{L_\mu^p} \le m^{-\frac{5}{2}-2\mu+\frac{2+2\mu}{p}}\|\psi\|_{L_\mu^p}.$$

*Proof.* Let $\psi \in L_\mu^2(I)$, and $\psi_m^A$ is canonical dilation of $\psi$ for $m>0$. Then

$$|\psi_m^A(t)| \le m^{-\frac{5}{2}-2\mu}\left|\psi\left(\frac{t}{m}\right)\right|.$$

Therefore

$$\int_0^\infty |\psi_m^A(t)|^p t^{1+2\mu}dt \le (m^{-\frac{5}{2}-2\mu})^p\int_0^\infty\left|\psi\left(\frac{t}{m}\right)\right|^p t^{1+2\mu}dt.$$

Hence

$$\|\psi_m^A\|_{L_\mu^p} \le m^{-\frac{5}{2}-2\mu+\frac{2+2\mu}{p}}\|\psi\|_{L_\mu^p}.$$

□

**Lemma 2.3.** *For $\psi, \phi \in L^2_\mu(I)$, canonical Hankel transformation of some functions are given as:*

$$(i)\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\frac{\cdot}{m})^2}\psi^A_m)(\omega) = \frac{1}{\sqrt{m}}e^{-\frac{id}{2b}(m^2-1)\omega^2}\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2}\psi)(m\omega), \tag{13}$$

$$(ii)\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2}\overline{\psi^A_m})(\omega) = \frac{e^{i\pi(1+\mu)}}{\sqrt{m}}e^{\frac{id}{2b}(m^2+1)\omega^2}\overline{\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}m^2(\cdot)^2}\psi)(m\omega)}, \tag{14}$$

$$(iii)\mathscr{H}^A_\mu(\phi\#_A e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi)(\omega) = e^{-\frac{id}{2b}\omega^2}\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2}\phi)(\omega)\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\frac{\cdot}{m})^2}\psi)(\omega), \tag{15}$$

*where $\psi^A_m$ is given by* (12).

## 3. Normalized wavelet transformation involving canonical Hankel wavelet

As per [12, 16], we define normalized continuous canonical Hankel wavelet as follows:

**Definition 3.1.** *A function $\psi \in L^2_\mu(I)$ is a normalized continuous canonical Hankel wavelet if $\|\psi\|_{L^2_\mu} = 1$ and it satisfy the admissibility condition as* (10).

If in addition, $\psi \in L^2_\mu$, then a normalized continuum canonical Hankel wavelet must also satisfy $\mathscr{H}^A_\mu\psi(0) = 0$ because $\mathscr{H}^A_\mu\psi$ is continuous and $\mathscr{H}^A_\mu\psi(0) \neq 0$ would contradict the convergence of the integral (10). By rescaling the spatial coordinate, we may assume that both $\|\psi\|_{L^2_\mu} = 1$ and $C^{\mu,A}_\psi = 1$.

The wavelet transformation (9) can be easily expressed in the form of canonical Hankel convolution as

$$(W^A_\psi f)(n,m) = be^{i\frac{\pi}{2}(1+\mu)}\Big(e^{-\frac{ia}{2b}(\cdot)^2}f\,\#_A\,\overline{\psi^A_m}\Big)(n), \tag{16}$$

where $\psi^A_m$ is defined as (12).
We define

$$N^A_\mu(m) = \left(\int_0^\infty |(W^A_\psi f)(n,m)|^2 n^{1+2\mu}dn\right)^{\frac{1}{2}}. \tag{17}$$

**Lemma 3.2.** *Let $\psi \in L^2_\mu(I)$ be a normalized continuous canonical Hankel wavelet and $f \in L^2_\mu(I)$. Then*

(i) $|W^A_\psi f(n,m)| \leq \dfrac{m^{\frac{-3-2\mu}{2}}}{(2b)^\mu\Gamma(\mu+1)}\|f\|_{L^2_\mu}.$

(ii) *For $m > 0$, $n \to W^A_\psi f(n,m)$ is in $L^2_\mu(I)$ and the norm $N^A_\mu(m)$ satisfies*

$$\int_0^\infty [N^A_\mu(m)]^2 dm = b^2\|f\|^2_{L^2_\mu}. \tag{18}$$

*Proof.* (i) From (16)

$$(W^A_\psi f)(n,m) = be^{i\frac{\pi}{2}(1+\mu)}\Big(e^{-\frac{ia}{2b}(\cdot)^2}f\,\#_A\,\overline{\psi^A_m}\Big)(n).$$

Using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}|(W^A_\psi f)(n,m)| &\leq \|f\|_{L^2_\mu}\|\psi^A_m\|_{L^2_\mu}\\ &\leq \frac{m^{\frac{-3-2\mu}{2}}}{(2b)^\mu\Gamma(\mu+1)}\|f\|_{L^2_\mu}\|\psi\|_{L^2_\mu}\\ &= \frac{m^{\frac{-3-2\mu}{2}}}{(2b)^\mu\Gamma(\mu+1)}\|f\|_{L^2_\mu}.\end{aligned}$$

(ii) Let $f \in L^1_\mu(I) \cap L^2_\mu(I)$. Then using (16) and (5),

$$\begin{aligned}[N^A_\mu(m)]^2 &= \int_0^\infty |(W^A_\psi f)(n,m)|^2 n^{1+2\mu} dn \\ &= b^2 \int_0^\infty \left|\mathscr{H}^A_\mu\left[e^{-\frac{ia}{2b}(\cdot)^2} f \,\#_A\, \overline{\psi^A_m}\right](\omega)\right|^2 \omega^{1+2\mu} d\omega \\ &= b^2 \int_0^\infty \left|\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2} f)(\omega)\right|^2 \left|\mathscr{H}^A_\mu \overline{(e^{\frac{ia}{2b}(\cdot)^2}\psi^A_m)}(\omega)\right|^2 \omega^{1+2\mu} d\omega.\end{aligned}$$

In particular, $n \to W^A_\psi f(n,m) \in L^2_\mu(I)$ for every $m > 0$. Using (14)

$$\begin{aligned}\int_0^\infty [N^A_\mu(m)]^2 dm &= b^2 \int_0^\infty \int_0^\infty \left|\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2} f)(\omega)\right|^2 \left|\frac{e^{i\pi(1+\mu)}}{\sqrt{m}} e^{\frac{id}{2b}(1+m^2)\omega^2} \overline{\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot m)^2}\psi)(m\omega)}\right|^2 \\ &\quad \times \omega^{1+2\mu} d\omega dm \\ &= b^2 \int_0^\infty \left|\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2} f)(\omega)\right|^2 \int_0^\infty \frac{\left|\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot m)^2}\psi)(m\omega)\right|^2}{m} dm\, \omega^{1+2\mu} d\omega \\ &= b^2 \int_0^\infty \left|\mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2} f)(\omega)\right|^2 \omega^{1+2\mu} d\omega \\ &= b^2 \|f\|^2_{L^2_\mu}.\end{aligned}$$

This completes the proof. □

**Definition 3.3.** *If $f \in L^2_\mu(I)$, then the partial inverse transformation of $f$ is defined as*

$$S^A_\epsilon f(x) = \int_{m>\epsilon} \left( \int_0^\infty W^A_\psi f(n,m) \psi_{n,m,A}(x) n^{1+2\mu} dn \right) dm, \qquad \text{for } \epsilon > 0. \tag{19}$$

**Theorem 3.4.** *The partial inverse transformation of $f \in L^2_\mu(I)$ can be expressed as*

$$S^A_\epsilon f(x) = b e^{i\frac{\pi}{2}(1+\mu)} \int_{m>\epsilon} (W^A_\psi f(n,m) \#_A\, e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)} \psi^A_m)(x) dm. \tag{20}$$

*Proof.* From (8), we have

$$\begin{aligned}&\int_0^\infty W^A_\psi f(n,m) \psi_{n,m,A}(x) n^{1+2\mu} dn \\ &= m^{-\frac{5}{2}-2\mu} \int_0^\infty e^{\frac{ia}{2b}(\frac{1}{m^2}-1)(x^2+n^2)} \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \psi(z) D^A_\mu\left(\frac{n}{m}, \frac{x}{m}, z\right) e^{\frac{ia}{2b}z^2} z^{1+2\mu} dz\; W^A_\psi f(n,m) n^{1+2\mu} dn \\ &= m^{-\frac{5}{2}-2\mu} \int_0^\infty e^{\frac{ia}{2b}(\frac{1}{m^2}-1)(x^2+n^2)} W^A_\psi f(n,m) \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \left( \int_0^\infty \left(\frac{n}{m}\xi\right)^{-\mu} J_\mu\left(\frac{n}{bm}\xi\right) \left(\frac{x}{m}\xi\right)^{-\mu} J_\mu\left(\frac{x}{bm}\xi\right) \xi^{1+2\mu} \right. \\ &\quad \times \left. \left( \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty e^{\frac{i}{2b}(az^2+d\xi^2)} (z\xi)^{-\mu} J_\mu\left(\frac{z\xi}{b}\right) z^{1+2\mu} e^{-\frac{ia}{2b}z^2} \psi(z) dz \right) e^{-\frac{id}{2b}\xi^2} d\xi \right) e^{-\frac{ia}{2b}(\frac{n^2}{m^2}+\frac{x^2}{m^2})} n^{1+2\mu} dn \\ &= m^{-\frac{5}{2}-2\mu} \int_0^\infty e^{-\frac{ia}{2b}x^2} \mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2}\psi)(\xi) e^{-\frac{id}{2b}\xi^2} \xi^{1+2\mu} \left(\frac{x}{m}\xi\right)^{-\mu} J_\mu\left(\frac{x}{mb}\xi\right) e^{i\pi(1+\mu)} e^{-\frac{id}{2b}\frac{\xi^2}{m^2}} \\ &\quad \times \mathscr{H}^A_\mu(e^{-\frac{ia}{2b}(\cdot)^2} W^A_\psi f(\cdot,m)) \left(\frac{\xi}{m}\right) d\xi.\end{aligned}$$

Now setting $\frac{\xi}{m} = \omega$ and using (13), (15), we get

$$\int_0^\infty W_\psi^A f(n,m)\psi_{n,m,A}(x)n^{1+2\mu}dn$$

$$= m^{-\frac{3}{2}-2\mu}e^{i\pi(1+\mu)}\int_0^\infty e^{-\frac{ia}{2b}x^2}\mathscr{H}_\mu^A(e^{-\frac{ia}{2b}(\cdot)^2}\psi)(m\omega)e^{-\frac{id}{2b}(m\omega)^2}(m\omega)^{1+2\mu}(x\omega)^{-\mu}J_\mu\left(\frac{x\omega}{b}\right)e^{-\frac{id}{2b}\omega^2}$$

$$\times \mathscr{H}_\mu^A(e^{-\frac{ia}{2b}(\cdot)^2}W_\psi^A f(\cdot,m))(\omega)d\omega$$

$$= e^{i\pi(1+\mu)}\int_0^\infty e^{-\frac{i}{2b}(ax^2+d\omega^2)}\mathscr{H}_\mu^A(e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi_m^A\#_A\, W_\psi^A f(\cdot,m))(\omega)(x\omega)^{-\mu}J_\mu\left(\frac{x\omega}{b}\right)\omega^{1+2\mu}d\omega$$

$$= be^{i\frac{\pi}{2}(1+\mu)}(W_\psi^A f(n,m)\#_A\, e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi_m^A)(x).$$

Thus we have

$$S_\epsilon^A f(x) = be^{i\frac{\pi}{2}(1+\mu)}\int_{m>\epsilon}(W_\psi^A f(n,m)\#_A\, e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi_m^A)(x)dm.$$

□

**Theorem 3.5.** *Let $\psi \in L_\mu^2(I)$ be a normalized continuous wavelet, $\epsilon > 0$, $f \in L_\mu^2(I)$ and $x \in I$. Then $S_\epsilon^A f(x)$ has a pointwise bound*

$$|S_\epsilon^A f(x)| \le \frac{b\|f\|_{L_\mu^2}}{(2b)^\mu\Gamma(\mu+1)}C_\epsilon,$$

*where* $C_\epsilon = \left(\int_{m>\epsilon}\frac{1}{m^{3+2\mu}}dm\right)^{\frac{1}{2}}$.

*Proof.* From (19) and (20), we get

$$\begin{aligned} S_\epsilon^A f(x) &= \int_{m>\epsilon}\left(\int_0^\infty W_\psi^A f(n,m)\psi_{n,m,A}(x)n^{1+2\mu}dn\right)dm \\ &= be^{i\frac{\pi}{2}(1+\mu)}\int_{m>\epsilon}(W_\psi^A f(n,m)\#_A\, e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi_m^A)(x)dm. \end{aligned}$$

By using Lemma 2.1, we have

$$\begin{aligned} |W_\psi^A f(n,m)\#_A\, e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi_m^A| &\le \frac{1}{|b|}\|W_\psi^A f(n,m)\|_{L_\mu^2}\|\tau_t(e^{-\frac{ia}{2b}(\cdot)^2(\frac{1}{m^2}-1)}\psi_m^A)\|_{L_\mu^2} \\ &\le \frac{1}{|b|}\|W_\psi^A f(n,m)\|_{L_\mu^2}\|\psi\|_{L_\mu^2}\frac{m^{-\frac{3}{2}-\mu}}{(2b)^\mu\Gamma(\mu+1)}. \end{aligned}$$

Now using (17) and (18)

$$\begin{aligned} |S_\epsilon^A f(x)| &\le \int_{m>\epsilon}\|W_\psi^A f(n,m)\|_{L_\mu^2}\frac{m^{-\frac{3}{2}-\mu}}{(2b)^\mu\Gamma(\mu+1)}dm \\ &= \frac{1}{(2b)^\mu\Gamma(\mu+1)}\int_{m>\epsilon}[N_\mu^A(m)]m^{-\frac{3}{2}-\mu}dm \\ &\le \frac{1}{(2b)^\mu\Gamma(\mu+1)}\left(\int_{m>\epsilon}[N_\mu^A(m)]^2dm\right)^{\frac{1}{2}}\left(\int_{m>\epsilon}(m^{-\frac{3}{2}-\mu})^2dm\right)^{\frac{1}{2}} \\ &\le \frac{b\|f\|_{L_\mu^2}}{(2b)^\mu\Gamma(\mu+1)}\left(\int_{m>\epsilon}\frac{1}{m^{3+2\mu}}dm\right)^{\frac{1}{2}} \end{aligned}$$

$$= \frac{b\|f\|_{L^2_\mu}}{(2b)^\mu\Gamma(\mu+1)} C_\epsilon,$$

where for each $\epsilon > 0$, $C_\epsilon = \Big(\int_{m>\epsilon} \frac{1}{m^{3+2\mu}} dm\Big)^{\frac{1}{2}}$ is convergent for $\mu > -1$. □

## 4. Calderón's reproducing formula

In this section, we obtain Calderón's reproducing formula by using the properties of canonical Hankel transformation and canonical Hankel convolution.

**Theorem 4.1.** *Let $\psi, \phi \in L^2_\mu(I)$ be a basic linear canonical Hankel wavelets which satisfies the following admissibility condition*

$$C^{\mu,A}_{\psi,\phi} = \int_0^\infty \frac{|(\mathscr{H}^A_\mu e^{-\frac{ia}{2b}(\cdot)^2}\psi)(\omega)|\overline{|(\mathscr{H}^A_\mu e^{-\frac{ia}{2b}(\cdot)^2}\phi)(\omega)|}}{\omega} d\omega = 1.$$

*Then, the following Calderón's reproducing identity holds,*

$$f(t) = e^{i\pi(1+\mu)} \int_0^\infty \Big((e^{-\frac{ia}{2b}n^2}(e^{-\frac{ia}{2b}(\cdot)^2} f(\cdot)\#_A \overline{\psi^A_m(\cdot)})(n))\#_A \phi^A_m\Big)(t) dm.$$

*Proof.* From (11), we have

$$f(t) = \frac{1}{b^2 C^{\mu,A}_{\psi,\phi}} \int_0^\infty \int_0^\infty (W^A_\psi f)(n,m)\phi_{n,m,A}(t) n^{1+2\mu} dn\, dm.$$

Using (16) and (7), we have

$$\begin{aligned} f(t) &= \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \int_0^\infty (e^{-\frac{ia}{2b}(\cdot)^2} f(\cdot)\#_A \overline{\psi^A_m(\cdot)})(b)\phi^A_m(n,t) n^{1+2\mu} dn\, dm \\ &= \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \int_0^\infty e^{-\frac{ia}{2b}n^2}(e^{-\frac{ia}{2b}(\cdot)^2} f(\cdot)\#_A \overline{\psi^A_m(\cdot)})(n)\phi^A_m(n,t) e^{\frac{ia}{2b}n^2} n^{1+2\mu} dn\, dm. \end{aligned}$$

Using (6), we get the required result as

$$f(t) = e^{i\pi(1+\mu)} \int_0^\infty \Big((e^{-\frac{ia}{2b}n^2}(e^{-\frac{ia}{2b}(\cdot)^2} f(\cdot)\#_A \overline{\psi^A_m(\cdot)})(n))\#_A \phi^A_m\Big)(t) dm.$$

□

**Remark 4.2.** *If $\psi = \phi$, then for*

$$\int_0^\infty \frac{|(\mathscr{H}^A_\mu e^{-\frac{ia}{2b}(\cdot)^2}\psi)(\omega)|^2}{\omega} d\omega = 1.$$

*The Calderón's reproducing identity is given as*

$$f(t) = e^{i\pi(1+\mu)} \int_0^\infty \Big((e^{-\frac{ia}{2b}n^2}(e^{-\frac{ia}{2b}(\cdot)^2} f(\cdot)\#_A \overline{\psi^A_m(\cdot)})(n))\#_A \psi^A_m\Big)(t) dm.$$

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