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Paratopological Polygroups versus Topological Polygroups

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Abstract. In this paper, we define the notion of paratopological polygroups and find their topological properties. In particular, we find those properties that make a paratopological polygroup a topological polygroup. We also give an example of topological polygroup and obtain some of their properties.

1. Introduction and Preliminaries

We follow the terminology of [1, 2, 6, 7, 11, 13]. The notion of hyperstructure, as a generalization of algebraic structure, was introduced by F. Mary at the 8th congress of Scandinavian Mathematicians in 1934 [19]. One of the most important instances of hyperstructures is hypergroupoid. Let H be a nonempty set and $P^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a mapping $o : H \times H \to P^*(H)$. The pair (H, o) is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H, then we define $AoB = \bigcup_{a\in A, b\in B} aob$. A *semihypergroup* is a hypergroupoid (H, o) such that:

$$\forall(a, b, c) \in H^3 : ao(boc) = (aob)oc.$$

A *hypergroup* is a semihypergroup (H, o) such that:

$$\forall a \in H, aoH = Hoa.$$

This condition is called the *reproduction axiom*.

Hypergroupoid theory has been developed by Koskas [18], Corsini [3, 4], Davvaz [5, 8, 20], Vougioukis [21] and Jafarabadi [15]. Hypergroupoids have many applications in pure and applied mathematics. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets, cryptography, combinatorics, codes and artificial intelligence, see [7].

Polygroups are special subclasses of hypergroups, which were studied in 1981 by Ioulidis in [14]. A *polygroup* is a system $\wp =< P, o, e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unary operation on P (called the *inversion* on the polygroup P), and (P, o) is a semihypergroup satisfying the following axioms:

(i) $\forall x \in P$: $eox = xoe = x$;

(ii) $\forall x, y, z \in P$, $x \in yoz$ implies $y \in xoz^{-1}$ and $z \in y^{-1}ox$.

In any polygroup the following hold: $e \in xox^{-1} \cap x^{-1}ox, e^{-1} = e, (x^{-1})^{-1} = x, (xoy)^{-1} = y^{-1}ox^{-1}$. A nonempty subset N of polygroup P is called a *normal subpolygroup* of P if for any $a, b \in N$, $aob^{-1} \subseteq N$ and $a^{-1}oNoa \subseteq N$.

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Algebraic structures which also have a topology are useful in mathematics. In the same direction, some of mathematicians have studied the properties of hypergroupoids endowed with a topology. Ameri [1] and Hoskova [12] have defined and studied $\tau_u(\tau_\ell)$-topological hypergroupoids. Heidari et al [10] have investigated the properties of these topologies on hypergroups. Now we introduce a topology on $P^*(H)$.

Lemma 1.1. ([13]) *Let (H,τ) be a topological space. Then the family $\mathfrak{I}$ consisting of all sets*

$$S_V = \{U \in P^*(H) : U \subseteq V\}; V \in \tau,$$

is a basis for a topology τ_u on $P^*(H)$.

Definition 1.2. Let (H, o) be a hypergroup and (H,τ) be a topological space. Then, the hyperoperation o is said to be continuous if $o : H \times H \to P^*(H)$ is continuous. Here $H \times H$ and $P^*(H)$ are equipped with the product topology and τ_u, respectively.

Recently, Heidari et al. [11] defined the notion of topological polygroups and proved some properties. They also proved the isomorphism theorem of topological polygroups.

In this paper, we define the notion of paratopological polygroups and find its topological properties. In particular, we find some properties that make a paratopological polygroup into a topological polygroup. In Section 2, we define the concept of τ-complete part and prove some properties of τ-complete part (para)topological polygroups. In Section 3, we introduce a uniformity on polygroups and show that every polygroup whose topology is induced by this uniformity is a topological polygroups. We also obtain some properties of topological polygroups.

2. Continuity of the Inversion on Paratopological Polygroups

A *paratopological group* is a pair (G,τ) consisting of a group G and a topology τ on G making the group operation continuous. A paratopological group G with continuous inversion is called a *topological group*. The continuity of the inversion is one of the important research topics in the theory of paratopological groups. In this section, we define paratopological polygroups and find conditions that make them into topological polygroups.

Definition 2.1. Let $(P, o, e, ^{-1})$ be a polygroup and (P,τ) be a topological space. The system $(P, o, e, ^{-1}, \tau)$ is called a paratopological polygroup if the operation $o : P \times P \to P^*(P)$ is continuous.

A paratopological polygroup (P,τ) with continuous inversion is called a topological polygroup [10].

Now, we give an example of a paratopological polygroup which is not a topological polygroup.

Example 2.2. Let $(\mathbb{R}, +)$ be the additive group of real numbers. Put $P_\mathbb{R} = \mathbb{R} \cup \{a\}$, where $a \notin \mathbb{R}$. We can define a hyperoperation o on $P_\mathbb{R}$ as follows:
(i) $aoa = 0$,
(ii) $0ox = xo0 = x$, for every $x \in P_\mathbb{R}$,
(iii) $aox = xoa = x$, for every $x \in P_\mathbb{R} - \{0, a\}$,
(iv) $xoy = x + y$, for every $(x, y) \in \mathbb{R}^2$ such that $y \neq -x$,
(v) $xo(-x) = \{0, a\}$, for every $x \in P_\mathbb{R} - \{0, a\}$.

Jafarpour et al. [16] proved that $(P_\mathbb{R}, o, e, ^{-1})$ is a polygroup. Now, let τ be the sorgenfrey topology on $\mathbb{R}$ and $\eta(e)$ be the neighborhood filter at e. The topology τ_a on $P_\mathbb{R}$ is generated by the base $\tau \cup \{\{a\} \cup U : U \in \eta(e)\}$. It is easy to see that $(P_\mathbb{R}, o, e, ^{-1}, \tau_a)$ is a paratopological polygroup. But the inversion in $P_\mathbb{R}$ is not continuous, since $([0,\delta))^{-1} \subseteq [0,\epsilon)$ for all $\epsilon, \delta > 0$.

Lemma 2.3. ([11]) *Let $(P, o, e, ^{-1})$ be a polygroup and τ be a topology on P. Then, the following assertions hold:*

(1) The mapping $o : P \times P \to P^*(P)$ is continuous if and only if for every $x, y \in P$ and $U \in \tau$ such that $xoy \subseteq U$, there exist $V, W \in \tau$ such that $x \in V$, $y \in W$ and $VoW \subseteq U$;

(2) The mapping $inv : P \to P$ is continuous if and only if for every $x \in P$ and $U \in \tau$ such that $x^{-1} \in U$, there exists $V \in \tau$ such that $x \in V, V^{-1} \subseteq U$.

Definition 2.4. If (H, o) is a hypergroupoid and (H, τ) is a topological space, then we say that o is τ-closed when for every $x, y \in H$, xoy is a closed subset of (H, τ).

Lemma 2.5. *Let $(P, o, e,^{-1}, \tau)$ be a regular paratopological polygroup and o be τ-closed. Then the graph of the inversion is a closed subset of $P \times P$.*

Proof. Since the identity of P is unique, $gr_{inv} = \{(x, y) : e \notin xoy \cap yox\}$ is the graph of the inversion on P. It is sufficient to prove that $(gr_{inv})^c = \{(x, y); e \notin xoy \text{ or } e \notin yox\}$ is an open subset in $P \times P$. Let $e \notin xoy$. Since o is τ-closed, from the regularity of P, it follows that open subsets U and V exist such that $e \in U$, $xoy \subseteq V$ and $U \cap V = \emptyset$. Now, since P is a paratopological polygroup, there exists open neighborhoods W_1 and W_2 in P of x and y, respectively, such that $W_1 o W_2 \subseteq V$, hence $(x, y) \in W_1 \times W_2 \subseteq (gr_{inv})^c$. This completes the proof of the lemma. □

A topological space X is called (countably compact) compact if any (countable) open cover of X has a finite subcover.

Theorem 2.6. *Let $(P, o, e,^{-1}, \tau)$ be a regular compact paratopological polygroup and o be τ-closed. Then the inversion on P is continuous.*

Proof. Let F be a closed subset of P. Then $P \times F$ is closed in $P \times P$. By Lemma 2.5, gr_{inv} is a closed subset of $P \times P$. Hence $gr_{inv} \cap (P \times F)$ is a closed subset of $P \times P$. On the other hand, since P is a compact space, we can apply Theorem 3.1.6 from [9] to conclude that the natural projection $\pi_1 : P \times P \to P$ onto the first coordinate is closed. Therefore $\pi_1(gr_{inv} \cap (P \times F)) = F^{-1}$ is closed. This implies the assertion of the theorem. □

Theorem 2.7. *Let $(P, o, e,^{-1}, \tau)$be a regular sequential countably compact paratopological polygroup and o be a τ-closed. Then the inversion on P is continuous.*

Proof. Since P is a sequential space, by Proposition 1.6.15 from [9], it is sufficient to show that the inversion of P is sequentially continuous. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence which convergences to x. We put $C = \{x_n\}_{n=1}^{\infty} \cup \{x\}$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x, the set C with the topology induced from P is a compact space. Since P is countably compact, by Corollary 3.10.14 from [9], $C \times P$ is a countably compact space. Then the closedness of gr_{inv} in the space $S \times S$ implies that $G = (C \times P) \cap gr_{inv}$ is countably compact and being countable, is compact. It follows from compactness of G that $\pi_1 : G \to C$ and $\pi_2 : G \to C^{-1}$ are homeomorphisms. Hence $inv|_C = \pi_2 o \pi_1^{-1}|_C : C \to C^{-1}$ is continuous. □

Example 2.8. Let (G, τ) be a topological group and H be a compact subgroup of G. For every $g_1, g_2 \in G$ we define $g_1 o_H g_2 = \{g_1 h g_2 | h \in H\} = g_1 H g_2$. Then (G, o_H) is a hypergroupoid. Suppose that U is an open set G such that $g_1 o_H g_2 \subseteq U$. So, for any $h \in H$ we have $g_1 h g_2 \in U$. Since G is a topological group, there exists open subsets $U_h(g_1), U_h, U_h(g_2)$ in G such that $g_1 \in U_h(g_1)$, $g_2 \in U_h(g_2)$, $h \in U_h$ and $U_h(g_1).U_h.U_h(g_2) \subseteq U$. On the other hand, since H is a compact subset of G, there exists an integer $n \geq 1$ such that $H \subseteq \bigcup_{i=1}^{n} U_{h_i}$. Now, we put $U_{g_1} = \bigcap_{i=1}^{n} U_{h_i}(g_1)$, $V = \bigcup_{i=1}^{n} U_{h_i}$ and $U_{g_2} = \bigcap_{i=1}^{n} U_{h_i}(g_2)$, and we have $U_{g_1} V U_{g_2} \subseteq U$. It means that U_{g_1}, U_{g_2} are open neighborhoods of g_1 and g_2 respectively such that

$$o_H(U_{g_1} \times U_{g_2}) = U_{g_1}.H.U_{g_2} \subseteq U_{g_1}.V.U_{g_2} \subseteq U.$$

Therefore o_H is continuous.

Example 2.9. Suppose that H is a compact subgroup of a topological group (G, τ). Then the system $G//H = (\{HgH, g \in G\}, *, H,^{-I})$, where $(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H | h \in H\}$ and $(HgH)^{-I} = Hg^{-1}H$, is a polygroup [7]. Let $\pi : G \to G//H$, where $\pi(g) = \{HgH\}$. Then we define a topology $\tau_{G//H}$ on $G//H$ as follows: A subset U of $G//H$ is open if $\pi^{-1}(U)$ is an open subset of G. It is easy to show that π is open and $\pi o(o_H) = *o(\pi \times \pi)$. Now, we prove that $(G//H, \tau_{G//H})$ is a topological polygroup. Let U be an open subset in $G//H$ such that $(Hg_1H) * (Hg_2H) \subseteq U$, i.e., $\pi(g_1) * \pi(g_2) \subseteq U$. Since $\pi o(o_H) = *o(\pi \times \pi)$, we have $g_1 o_H g_2 \subseteq \pi^{-1}(U)$. On the other hand, since $\pi^{-1}(U)$ is open in $P^*(H)$, the continuity of o_H implies that there exist open neighborhoods W_1, W_2 of g_1, g_2 in G such that $W_1 o_H W_2 \subseteq \pi^{-1}(U)$. Thus $\pi(W_1) * \pi(W_2) \subseteq U$ which implies that $\pi(W_1), \pi(W_2)$ are open neighborhoods of Hg_1H, Hg_2H in $G//H$. Therefore $*$ is continuous. Now, we prove that the inversion on $G//H$ is continuous. Suppose that U is an open subset of $Hg^{-1}H$ in $G//H$, i.e., $\pi(g^{-1}) \subseteq U$. Since the inversion is continuous in G, there exists an open neighborhood V of g in G such that $V^{-1} \subseteq \pi^{-1}(U)$. Thus $(\pi(V))^{-1} \subseteq U$, which implies that $\pi(V)$ is the open neighborhood of HgH in $G//H$, hence the inversion on $G//H$ is continuous. Therefore $(G//H, \tau)$ is a topological polygroup.

3. Complete Part on Open Subsets of (Para)topological Polygroup and Results

Complete parts were introduced for the first time by Koskas [18] on hypergroups. Many studies have been done on this concept in semihypergroups, for example see [3, 4, 6, 14, 20]. Let (H, o) be a semihypergroup. A subset A of H is called a *complete part* if for all elements $a_1, \ldots, a_n$ of H, $\Pi_{i=1}^n a_i \cap A \neq \emptyset$ implies $\Pi_{i=1}^n a_i \subseteq A$.

Definition 3.1. A polygroup (H, o) with a topology τ on H is called a τ-complete part if every $U \in \tau$ is a complete part.

This concept was defined by Heidari et al. [10] on topological semihypergroups. In [11], they studied topological polygroups P with this property .

In this section, we obtain some results for the (para)topological polygroups (P, τ) that are τ-complete parts.

Lemma 3.2. *For every τ-complete part paratopological polygroup $(P, o, ^{-1}, \tau)$, the inversion is continuous if and only if it is continuous at e.*

Proof. Suppose U is an open neighborhood of x^{-1}. Since P is a τ-complete part, Lemma 2.13 from [11] implies that xoU is an open neighborhood of e. Continuity of the inversion at e implies that there exists an open neighborhood V of e such that $V^{-1} = inv(V) \subseteq U$. We have $(Vox)^{-1} = x^{-1}oV^{-1} \subseteq x^{-1}o(xoU)$. On the other hand, since U is a complete part and $x^{-1}oxoU \cap U \neq \emptyset$, $x^{-1}oxoU \subseteq U$. Since Vox is an open neighborhood of x, $inv : P \to P$ is continuous at x. □

Lemma 3.3. *Let $(P, o, e, ^{-1}, \tau)$ be a τ-complete part paratopological polygroup. If U, V are open neighborhoods of e, where $V^2 \subseteq U$, then $(\overline{V^{-1}})^{-1} \subseteq U$.*

Proof. It is sufficient to show that $\overline{V^{-1}} \subseteq U^{-1}$. Let $x \in \overline{V^{-1}}$. Since P is a τ-complete part, Lemma 2.13 from [10] implies that xoV is an open neighborhood of x. Then $xoV \cap V^{-1} \neq \emptyset$, so there exist $v_1, v_2 \in V$ such that $v_2^{-1} \in xov_1$. Now, we have $x \in v_2^{-1}ov_1^{-1} \subseteq (V^{-1})^2 \subseteq U^{-1}$. □

Let $(P, o, e, ^{-1}, \tau)$ be a paratopological polygroup. We say that P is *topologically periodic* if for each $x \in P$ and every open neighborhood of e, there exists an integer $n > 0$ such that $x^n = xox \ldots ox \subseteq U$.

Theorem 3.4. *Let P be a τ-complete part countably compact paratopological polygroup. Then P is a topological polygroup.*

Proof. According to Lemma 3.2, it suffices to verify the continuity of the inversion at identity e. Let U be an open neighborhood of the identity e in P and $\{V_i | i \in \omega\}$ be a family of open neighborhoods of e such that $V_0 = U$, $V_{i+1}^2 \subseteq V_i$ for each $i \geq 1$. By Lemma 3.3 we have $(\overline{V_{i+1}^{-1}})^{-1} \subseteq V_i$. Now, we will show that $F = \bigcap \overline{V_i^{-1}} | i \in \omega\} \subseteq U$. If $x \in F$, so $x \in V_i^{-1}$ for each $i \in \omega$. Therefore $(x^{-1})^{n-1} \in ((\overline{V_{i+1}^{-1}})^{-1})^{n-1} \subseteq V_i^{n-1}$ for each $i \in \omega$. Since P is topologically periodic, there exists $n \in \omega - \{0\}$ such that $V_{i_0}^{n-1} \subseteq V_1$. Then

$x \in x^n o(x^{-1})^{n-1} \subseteq V_1^2 \subseteq U$. Since P is countably compact, there exist $i_1, \dots, i_n \in \omega$ such that $\bigcap_{i=1}^n \overline{V_{i_j}^{-1}} \subseteq U$. Hence $V = \bigcap_{i=1}^n V_{i_j}$ is an open neighborhood of e such that $inv(V) \subseteq \bigcap_{i=1}^n \overline{V_{i_j}^{-1}} \subseteq U$. This means that inv is continuous. □

Proposition 3.5. *Let P be a τ-complete part countably compact topological polygroup. Then P is regular.*

Proof. By Theorem 1.5.5 of [9], it is enough to show that for any $x \in P$ and every open neighborhood U of x, there exists an open neighborhood W of x such that $x \in W \subseteq \overline{W} \subseteq U$. Let $x \in P$. Suppose U is an open neighborhood of x. Since U is a complete part, Lemma 2.13 from [11] shows that $x^{-1}oU$ is an open neighborhood of e. By Theorem 2.18 of [10], there exists a neighborhood V of e such that $e \in V \subseteq \overline{V} \subseteq x^{-1}oU$. Hence $x \in xoV \subseteq xo\overline{V} \subseteq x^{-1}oxoU$. But we have $xo\overline{V} = \overline{xoV}$. If we let $W = xoV$, the proof is complete. □

4. Uniformity on a Polygroup and its Topology

In this section, we introduce a uniformity on an arbitrary polygroup P. We obtain a topology on P and prove that with this topology, P is a topological polygroup. Let X be a non-empty set and A, B be subsets of $X \times X$. Define

1. $\Delta = \{(x, x) : x \in X\}$,
2. $A^{-1} = \{(x, y) : (y, x) \in A\}$,
3. $A + B = \{(x, z) \in X \times X : (x, y) \in A \text{ and } (y, z) \in B \text{ for some } y \in X\}$.

A *uniformity* on a nonempty set X is a subfamily $\mathcal{U}$ of the set of all subsets of $X \times X$ which satisfies the following conditions:

(U_1) $\Delta \subseteq U$ for every $U \in \mathcal{U}$,
(U_2) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,
(U_3) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,
(U_4) For every $U \in F$, there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$,
(U_5) If $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{U}$.

For every uniformity $\mathcal{U}$ on a set X, the family

$$\tau_{\mathcal{U}} = \{V \subseteq X, \text{ for any } x \in V, \text{ there exists } U \in \mathcal{U} \text{ such that } U[x] \subseteq V\}$$

is a topology on X which $U[x] = \{y | (x, y) \in U\}$, for every $U \in \mathcal{U}$.

Now, we define a uniformity on polygroups.

Definition 4.1. Let P be a polygroup and N be a normal subpolygroup of P. We denote $U_N = \{(x, y) \in P \times P : xN = yN\}$.

Theorem 4.2. *Let $\aleph$ be an arbitrary family of normal subpolygroups of P which is closed under intersection. Then*

$$\mathcal{U}_{\aleph} = \{U \subseteq P \times P : U_N \subseteq U \text{ for some } N \in \aleph\}$$

is a uniformity on P.

Proof. (U_1) For every $U \in \mathcal{U}_{\aleph}$, there exists $N \in \aleph$ such that $U_N \subseteq U$. Since $xN = xN$ for any $x \in P$, $\Delta \subseteq U_N \subseteq U$.

(U_2) Let $U \in \mathcal{U}$. There exists $N \in \aleph$ such that $U_N \subseteq U$. It is easy to show that $U_N = U_N^{-1}$, hence $U_N = U_N^{-1} \subseteq U^{-1}$. Therefore $U^{-1} \in \mathcal{U}_{\aleph}$.

(U_3) For every $U, V \in \mathcal{U}$, there exist $N_1, N_2 \in \aleph$ such that $U_{N_1} \subseteq U$, $U_{N_2} \subseteq V$. Let $(x, y) \in U_{N_1} \cap U_{N_2}$. Then $xoN_1 = yoN_1$ and $xoN_2 = yoN_2$, so $xo(N_1 \cap N_2) = yo(N_1 \cap N_2)$. Hence $(x, y) \in U_{N_1 \cap N_2}$. Conversely, let $(x, y) \in U_{N_1 \cap N_2}$. Then $xo(N_1 \cap N_2) = yo(N_1 \cap N_2)$, hence $y^{-1}ox \subseteq N_1$ and $y^{-1}ox \subseteq N_2$. So $(x, y) \in U_{N_1}$, $(x, y) \in U_{N_2}$. Therefore $U_{N_1 \cap N_2} = U_{N_1} \cap U_{N_2}$. Since $\aleph$ is closed under intersection, $U \cap V \in \mathcal{U}_{\aleph}$.

(U_4) For any $U \in \mathcal{U}_{\aleph}$, there exists $N \in \aleph$ such that $U_N \subseteq U$. We have $U_N + U_N \subseteq U_N$. Since $V = U_N \in \mathcal{U}_{\aleph}$, this completes ($U_4$).

(U_5) Let $U \in \mathcal{U}_{\aleph}$ and $U \subseteq V \subseteq X \times X$. Then there exists $N \in \aleph$ such that $U_N \subseteq U \subseteq V$, which means that $V \in \mathcal{U}_{\aleph}$. □

For the uniformity $\mathcal{U}_{\aleph}$ on the polygroup P, the set $\beta = \{xoN | x \in P, N \in \aleph\}$ is a base for the topology $\tau_{\aleph}$.

Theorem 4.3. *The system $(P, o, e, ^{-1}, \tau_{\aleph})$ ($\aleph$ is a family of normal subpolygroups of P which is closed under intersection) is a topological polygroup.*

Proof. Suppose $xoy \subseteq O$, where $x, y \in P$ and O is an open subset of P. By the definition of the topology $\tau_{\aleph}$, there exists a normal subgroup N_t such that $U_N(t) = tN_t \subseteq O$, for any $t \in xoy$. Since $\aleph$ is closed under intersection, we have $N = \bigcap_{t \in xoy} N_t \in \aleph$. Therefore xoN, yoN are neighborhoods of x, y such that $(xoN)o(yoN) \subseteq xoyoN \subseteq O$. This proves that o is continuous on P. Now, let O be an open neighborhood of x^{-1}. By the definition of the topology $\tau_{\aleph}$, there exists a normal subgroup N such that $x^{-1}oN \subseteq O$. Since Nox is an open neighborhood of x, $(Nox)^{-1} = x^{-1}oN^{-1} = x^{-1}oN \subseteq O$. Therefore *inv* is continuous, this completes the proof. □

Theorem 4.4. *Any normal subpolygroup in the collection $\aleph$ is a closed subpolygroup of P.*

Proof. Let N be a normal *subpolygroup* in $\aleph$ and $y \in N^c$. Then $yoN \cap N = \emptyset$, by the definition of topology $\tau_{\aleph}$, yoN is an open neighborhood of y which $yoN \subseteq N^c$. This means the N^c is the open subset of P. □

Theorem 4.5. *$(P, \tau_{\aleph})$ is a Hausdorff space if and only if $\bigcap_{N \in \aleph} N = \{e\}$.*

Proof. Let $(P, \tau_{\aleph})$ be a Hausdorff space. Since $\aleph$ is a base of e in $(P, \tau_{\aleph})$, Proposition 1.5.2 of [9] implies $\bigcap_{N \in \aleph} \overline{N} = \{e\}$. But $N \subseteq \overline{N}$, so $\bigcap_{N \in \aleph} N = \{e\}$.

Conversely, let $\bigcap_{N \in \aleph} N = \{e\}$ and x, y be distinct elements of P. Then $e \in x^{-1}oy$. By the assumption, there exists $N \in \aleph$ such that $x^{-1}oy \cap N = \emptyset$. Hence xoN, yoN are two open neighborhoods of x, y such that $xoN \cap yoN = \emptyset$, then we conclude that $(P, \tau_{\aleph})$ is a Housdorff space. □

Theorem 4.6. *Consider the topological polygroup $(P, \tau_{\aleph})$. The following statements are equivalent:*
(*i*) *$(P, \tau_{\aleph})$ is a $T_{3\frac{1}{2}}$-space;*
(*ii*) *$(P, \tau_{\aleph})$ is a T_3-space;*
(*iii*) *$(P, \tau_{\aleph})$ is a T_2-space;*
(*iv*) *$(P, \tau_{\aleph})$ is a T_1-space.*

Proof. The proofs of $(i) \Rightarrow (ii), (ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are clear. Let $(P, \tau_{\aleph})$ be a T_1-space. Since $\tau_{\aleph}$ is a topology induced by uniformity, by Theorem 4.2.9 of [9], $(P, \tau_{\aleph})$ is completely regular and so $(P, \tau_{\aleph})$ is a $T_{3\frac{1}{2}}$-space. □

Recall that a uniform space $(X, \mathcal{U})$ is totally bounded if for each $U \in \mathcal{U}$, there exist $x_1, x_2, \ldots\ldots x_n \in X$ such that $X = \bigcup_{i=1}^{n} U[x_i]$.

Theorem 4.7. *Let $\aleph$ be a family of normal subgroups of a polygroup P. Then the following conditions are equivalent:*
(*i*) *The topological polygroup $(P, \tau_{\aleph})$ is compact;*
(*ii*) *The topological polygroup $(P, \mathcal{U}_{\aleph})$ is totally bounded.*

Proof. $(i) \Rightarrow (ii)$ This is clear by Theorem 14.3.8 from [17].

$(ii) \Rightarrow (i)$ Since $\aleph$ is closed under intersection, so $\bigcap_{N \in \aleph} N \in \aleph$; we put $N_0 = \bigcap_{N \in \aleph} N$. By the assumption, there exist $x_1, x_2, \ldots x_n \in P$ such that $P = \bigcup_{i=1}^{n} U_{N_0}[x_i]$. Now suppose $P = \bigcup_{\alpha \in I} O_{\alpha}$, where each O_{α} is an open subset of P. Then for any $x_i \in P$ there exists $\alpha_i \in I$ such that $U_{N_0}[x_i] \subseteq O_{\alpha_i}$. So $P = \bigcup_{i=1}^{n} U_{N_0}[x_i] \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$.Therefore $P = \bigcup_{i=1}^{n} O_{\alpha_i}$, which means that $(P, \tau_{\aleph})$ is compact. □

Corollary 4.8. *$(P, \tau_{\aleph})$ is a complete space.*

Proof. The proof is clear from Theorem 14.3.8 of [17]. □

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