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Beyond Gevrey Regularity: Superposition and Propagation of Singularities

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Abstract. We propose the relaxation of Gevrey regularity condition by using sequences which depend on two parameters, and define spaces of ultradifferentiable functions which contain Gevrey classes. It is shown that such a space is closed under superposition, and therefore inverse closed as well. Furthermore, we study partial differential operators whose coefficients are less regular then Gevrey-type ultradifferentiable functions. To that aim we introduce appropriate wave front sets and prove a theorem on propagation of singularities. This extends related known results in the sense that assumptions on the regularity of the coefficients are weakened.

1. Introduction

Gevrey classes serve as an important reservoir of functions related to different aspects of general theory of linear partial differential operators such as hypoellipticity, local solvability and propagation of singularities, since they describe regularities stronger than smoothness and weaker than analyticity [1, 7, 14]. For example, the Cauchy problem for weakly hyperbolic linear partial differential equations (PDEs) is well-posed for certain values of the Gevrey index t, while it is ill-posed in the class of analytic functions, cf. [3, 21] and the references given there.

Since the union of Gevrey classes is strictly contained in the class of smooth functions, it is of interest to study intermediate spaces of smooth functions by introducing appropriate regularity conditions. This is done in [18] by replacing Gevrey sequences $\{p!^t\}_{p\in\mathbf{N}}$, $t>1$ with two-parameter dependent sequences of the form $M_p^{\tau,\sigma}=p^{\tau p^\sigma}$, $p\in\mathbf{N}$, $\tau>0$, $\sigma>1$. The corresponding spaces of ultradifferentiable functions denoted by $\mathcal{E}_{\tau,\sigma}(U)$ extend Gevrey regularity, see Section 2 for the definition, and [18, 24, 25] for the main properties.

The spaces $\mathcal{E}_{\tau,\sigma}(U)$ can be used e.g. in situations when hypoellipticity of a PDE is better than C^∞ but worse than Gevrey hypoellipticity. In particular, the space $\mathcal{E}_{\{1,2\}}(U)$ is recently explicitly used in the study of strictly hyperbolic equations to capture the regularity of the coefficients in the space variable (with low regularity in time), which ensures that the corresponding Cauchy problem is well posed in appropriate solution spaces. We refer to [4] for details.

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In this paper we give a further insight to the extended Gevrey regularity by proving superposition theorem for $\mathcal{E}_{\tau,\sigma}(U)$, Theorem 2.2, which immediately implies the inverse closedness property. In contrast to the proof of inverse closedness of Carleman classes based on almost increasing property of defining sequences (see [9, 22, 23]), here use a modified Faá di Bruno property of the sequence $M_p^{\tau,\sigma}$, see Lemma 2.3. The proof of Theorem 2.2 holds even if $\sigma = 1$ and $\tau \geq 1$, so that we recover the well known result of superposition in Gevrey classes.

Then we proceed with microlocal analysis related to the extended Gevrey regularity by introducing appropriate wave-front sets $\mathrm{WF}_{\tau,\sigma}(u)$, $u \in \mathcal{D}'(U)$. In particular, in Theorem 3.1 we show that one can use a single cut-off function for the definition of $\mathrm{WF}_{\tau,\sigma}(u)$, $u \in \mathcal{D}'(U)$, instead of admissible sequences of cut-off functions which were used in [18]. We refer to [18] for a discussion on different types of wave-front sets adjusted to the problem under consideration. For example, in time-frequency analysis it is convenient to use wave front sets related to modulation spaces, see [15, 16] and the references given there.

Finally, we prove the propagation of singularities in the case when the coefficients $a_\alpha(x)$ of the partial differential operator $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ belong to $\mathcal{E}_{\tau,\sigma}(U)$, see Theorem 4.1. We recall that analytic coefficients were treated in [6, Theorem 8.6.1], while [18, Theorem 1.1] treats constant coefficients. It turns out that an additional information is needed in the study of operators with variable coefficients, since it is not possible to use the commutativity properties which hold true when the coefficients are constants. To overcome these difficulties in the proof of Theorem 4.1 we use the inverse closedness property, investigation of summands in generalized Faá-di Bruno's formula, and explicit construction of approximate solutions, cf. Subsection 4.1.

We summarize the paper as follows. In Section 2 we discuss regularity conditions related to the sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, $p \in \mathbf{N}$ (cf. [17, 18]), and introduce the spaces of ultradifferentiable functions $\mathcal{E}_{\tau,\sigma}(U)$. We prove the superposition theorem and inverse closedness in $\mathcal{E}_{\tau,\sigma}(U)$ which will be used in Section 4. In Section 3 we introduce wave front sets $\mathrm{WF}_{\tau,\sigma}(u)$, $u \in \mathcal{D}'(U)$, in the context of extended Gevrey regularity and explain enumeration, an important technical tool in our analysis. The main result there is Theorem 3.1 which gives a convenient equivalent definition of the wave front set $\mathrm{WF}_{\tau,\sigma}(u)$. Finally, in Section 4 we prove the propagation of singularities, Theorem 4.1. The proof is given in details since it contains new nontrivial observations and facts in comparison with the proof of [18, Theorem 1.1].

1.1. Notation

Throughout the paper the notation is standard. For example, $\mathbf{N}$,$\mathbf{Z}_+$, $\mathbf{R}_+$ denote the sets of nonnegative integers, positive integers, and positive real numbers, respectively, and Lebesgue spaces over an open set $\Omega \subset \mathbf{R}^d$ are denoted by $L^p(\Omega)$, $1 \leq p < \infty$. For $x \in \mathbf{R}^d$ we put $\langle x \rangle = (1+|x|^2)^{1/2}$. The integer parts (the floor and the ceiling functions) of $x \in \mathbf{R}_+$ are denoted by $\lfloor x \rfloor := \max\{m \in \mathbf{N} : m \leq x\}$ and $\lceil x \rceil := \min\{m \in \mathbf{N} : m \geq x\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$ we write $\partial^\alpha = \partial^{\alpha_1} \dots \partial^{\alpha_d}$, $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$, and $|\alpha| = |\alpha_1| + \dots |\alpha_d|$. Open ball of radius $r > 0$ centered at $x_0 \in \mathbf{R}^d$ is denoted by $B_r(x_0)$, and $\operatorname{card} A$ denotes the cardinal number of A. The closure of the open set U in $\mathbf{R}^d$ is denoted by $\overline{U}$. The Fourier transform of $u \in L^1(\mathbf{R}^d)$ is normalized as

$$\mathcal{F}_{x\to\xi} u(x) = \widehat{u}(\xi) = \int_{\mathbf{R}^d} u(x) e^{-2\pi i \langle x,\xi\rangle}\, dx = \int_{\mathbf{R}^d} u(x) e^{-2\pi i x\xi}\, dx, \quad \xi \in \mathbf{R}^d,$$

and the convolution of $f, g \in L^1(\mathbf{R}^d)$ is given by $f * g(x) = \int_{\mathbf{R}^d} f(x-y)g(y)dy$. Both transforms can be extended in different ways.

We denote by $C^\infty(K)$ the set of smooth functions on a regular compact set K, and $\mathcal{D}(U)$ and $\mathcal{E}(U)$ denote test function spaces for the space of Schwartz distributions $\mathcal{D}'(U)$, and for the space of compactly supported distributions $\mathcal{E}'(U)$, respectively.

We will use the Stirling formula: $N! = N^N e^{-N} \sqrt{2\pi N} e^{\frac{\theta_N}{12N}}$, for some $0 < \theta_N < 1$, $N \in \mathbf{Z}_+$, and formulas for multinomial coefficients:

$$\binom{|a|}{a_1, a_2, \dots a_m} := \binom{|a|}{a_1}\binom{|a| - a_1}{a_2} \dots \binom{|a| - a_1 - \dots - a_{m-2}}{a_{m-1}} = \frac{|a|!}{a_1! a_2! \dots a_m!} = \sum_{k=1}^{m} \binom{|a| - 1}{a_1, \dots, a_k - 1, \dots a_m},$$

where $|a| = a_1 + a_2 + \dots + a_m$, $a_k \in \mathbf{N}$, $k \leq m$.

2. Classes $\mathcal{E}_{\tau,\sigma}(U)$ and their superposition property

In this section we introduce test function spaces denoted by $\mathcal{E}_{\tau,\sigma}(U)$ via defining sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, depending on parameters $\tau > 0$ and $\sigma > 1$. The flexibility obtained by introducing the two-parameter dependence enables the study of smooth functions which are less regular than the Gevrey functions. When $\tau > 1$ and $\sigma = 1$, we recapture the Gevrey classes.

The spaces $\mathcal{E}_{\tau,\sigma}(U)$ are recently introduced and studied in [17, 18, 24, 25]. Here we recall their basic properties which are used in the rest of the paper, and collect new results in Subsection 2.1. We employ Komatsu's approach [10] to spaces of ultradifferentiable functions. Another widely used approach is that of Braun, Meise, Taylor, Vogt and their collaborators, see e.g. [2] and the recent contribution [19]. These two approaches are equivalent in many interesting situations, cf. [12] for more details.

Essential properties of the defining sequences are given in the following lemma. We refer to [17] for the proof.

Lemma 2.1. *Let $\tau > 0$, $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{Z}_+$, $M_0^{\tau,\sigma} = 1$. Then there exists an increasing sequence of positive numbers C_q, $q \in \mathbf{N}$, and a constant $C > 0$ such that:*

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}$, $p \in \mathbf{Z}_+$

$\overline{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma + q^\sigma} M_p^{\tau 2^{\sigma-1},\sigma} M_q^{\tau 2^{\sigma-1},\sigma}$, $p, q \in \mathbf{N}$,

$\overline{(M.2)'}$ $M_{p+q}^{\tau,\sigma} \leq C_q^{p^\sigma} M_p^{\tau,\sigma}$, $p, q \in \mathbf{N}$,

$(M.3)'$ $\displaystyle\sum_{p=1}^{\infty} \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty$*. Moreover,* $\displaystyle\frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} \leq \frac{1}{(2p)^{\tau(p-1)^{\sigma-1}}}$, $p \in \mathbf{N}$.

Let $\tau, h > 0$, $\sigma > 1$ and $K \subset\subset \mathbf{R}^d$ a regular compact set. By $\mathcal{E}_{\tau,\sigma,h}(K)$ we denote the Banach space of functions $\phi \in C^\infty(K)$ such that

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha \in \mathbf{N}^d} \sup_{x \in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} M_{|\alpha|}^{\tau,\sigma}} < \infty, \tag{2.1}$$

and obviously,

$$\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K), \quad 0 < h_1 \leq h_2,\ 0 < \tau_1 \leq \tau_2,\ 1 < \sigma_1 \leq \sigma_2,$$

where $\hookrightarrow$ denotes the strict and dense inclusion.

The set of functions from $\mathcal{E}_{\tau,\sigma,h}(K)$ with support contained in K is denoted by $\mathcal{D}^K_{\tau,\sigma,h}$. If U is an open set $\mathbf{R}^d$ and $K \subset\subset U$ then we define families of spaces by introducing the following projective and inductive limit topologies,

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \to \infty} \mathcal{E}_{\tau,\sigma,h}(K),$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \to 0} \mathcal{E}_{\tau,\sigma,h}(K),$$

$$\mathcal{D}_{\{\tau,\sigma\}}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}^K_{\{\tau,\sigma\}} = \varinjlim_{K \subset\subset U} (\varinjlim_{h \to \infty} \mathcal{D}^K_{\tau,\sigma,h}),$$

$$\mathcal{D}_{(\tau,\sigma)}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}^K_{(\tau,\sigma)} = \varinjlim_{K \subset\subset U} (\varprojlim_{h \to 0} \mathcal{D}^K_{\tau,\sigma,h}).$$

We will use abbreviated notation τ, σ for $\{\tau, \sigma\}$ or (τ, σ). The spaces $\mathcal{E}_{\tau,\sigma}(U)$, $\mathcal{D}^K_{\tau,\sigma}$ and $\mathcal{D}_{\tau,\sigma}(U)$ are nuclear, cf. [17].

If $\tau > 1$ and $\sigma = 1$, then $\mathcal{E}_{\{\tau,1\}}(U) = \mathcal{E}_{\{\tau\}}(U)$ is the Gevrey class, and $\mathcal{D}_{\{\tau,1\}}(U) = \mathcal{D}_{\{\tau\}}(U)$ is its subspace of compactly supported functions in $\mathcal{E}_{\{\tau\}}(U)$. If $0 < \tau \leq 1$ then $\mathcal{E}_{\tau,1}(U)$ consists of quasianalytic functions. In particular, $\mathcal{D}_{\tau,1}(U) = \{0\}$ when $0 < \tau \leq 1$, and $\mathcal{E}_{\{1,1\}}(U) = \mathcal{E}_{\{1\}}(U)$ is the space of analytic functions on U.

The space $\mathcal{E}_{\{1,2\}}(U)$ appears in [4] in the study of strictly hyperbolic equations to describe the regularity of coefficients in the space variable (with low regularity in time), which is sufficient to ensure that the corresponding Cauchy problem is well posed in appropriate solution spaces.

In the following Proposition, we capture the main embedding properties between the above introduced families.

Proposition 2.1. *[18] Let $\sigma_1 \geq 1$. Then for every $\sigma_2 > \sigma_1$ and $\tau > 0$*

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \varprojlim_{\tau\to 0^+} \mathcal{E}_{\tau,\sigma_2}(U).$$

Moreover, if $0 < \tau_1 < \tau_2$, then

$$\mathcal{E}_{\{\tau_1,\sigma\}}(U) \hookrightarrow \mathcal{E}_{(\tau_2,\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\tau_2,\sigma\}}(U), \quad \sigma \geq 1,$$

and

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{E}_{(\tau,\sigma)}(U),$$
$$\varprojlim_{\tau\to 0^+} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{\tau\to 0^+} \mathcal{E}_{(\tau,\sigma)}(U), \quad \sigma \geq 1.$$

We conclude that

$$\mathcal{E}_{\tau_0,\sigma_1}(U) \hookrightarrow \bigcap_{\tau>\tau_0} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \mathcal{E}_{\tau_0,\sigma_2}(U),$$

for any $\tau_0 > 0$ whenever $\sigma_2 > \sigma_1 \geq 1$, and in particular,

$$\varinjlim_{t\to\infty} \mathcal{E}_{\{t\}}(U) \hookrightarrow \mathcal{E}_{\tau,\sigma}(U) \hookrightarrow C^{\infty}(U), \quad \tau > 0, \ \sigma > 1,$$

so that the regularity in $\mathcal{E}_{\tau,\sigma}(U)$ can be thought of as an extended Gevrey regularity.

Non-quasianalyticity condition $(M.3)'$ provides the existence of partitions of unity in $\mathcal{E}_{\{\tau,\sigma\}}(U)$ which we formulate in the next Lemma.

Lemma 2.2. *Let $\tau > 0$ and $\sigma > 1$. Then there exists a compactly supported function $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ such that $0 \leq \phi \leq 1$ and $\int_{\mathbf{R}^d} \phi\, dx = 1$.*

A compactly supported Gevrey function from $\mathcal{E}_{\{\tau\}}(U)$ belongs to $\mathcal{D}_{\{\tau,\sigma\}}(U)$. However, in the proof of Lemma 2.2 given in [17] we constructed a compactly supported function in $\mathcal{D}_{\{\tau,\sigma\}}(U)$ which does not belong to $\mathcal{D}_{\{t\}}(U)$, for any $t > 1$.

Note that the additional exponent σ which appears in the power of term h in (2.1) makes the definition of $\mathcal{E}_{\{\tau,\sigma\}}(U)$ different from the definition of Carleman class C^L, cf. [6]. This difference is essential for many calculations. For example, defining sequences for Carleman classes satisfy Komatsu's condition (M.2)′ known as "stability under differential operators", while $M_p^{\tau,\sigma}$ do not satisfy (M.2)′ for $\tau > 0$ and $\sigma > 1$. However, we have the following "stability properties".

If $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ is a partial differential operator of order m with $a_\alpha \in \mathcal{E}_{\tau,\sigma}(U)$, then $P : \mathcal{E}_{\tau,\sigma}(U) \to \mathcal{E}_{\tau,\sigma}(U)$ is a continuous linear map with respect to the topology of $\mathcal{E}_{\tau,\sigma}(U)$. In particular, $\mathcal{E}_{\tau,\sigma}(U)$ is closed under pointwise multiplications and finite order differentiation, see [24, Theorem 2.1]. For operators of "infinite order" continuity properties are slightly different.

Let $\tau > 0$, $\sigma > 1$, and let $a_\alpha \in \mathcal{E}_{(\tau,\sigma)}(U)$ (resp. $a_\alpha \in \mathcal{E}_{\{\tau,\sigma\}}(U)$) where U is an open set in $\mathbf{R}^d$. Then

$$P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)\partial^\alpha$$

is of class (τ,σ) (resp. $\{\tau,\sigma\}$) on U if for every $K \subset\subset U$ there exists constant $L>0$ such that for any $h>0$ there exists $A>0$ (resp. for every $K\subset\subset U$ there exists $h>0$ such that for any $L>0$ there exists $A>0$) such that,

$$\sup_{x\in K}|\partial^\beta a_\alpha(x)| \leq A h^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}\frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad \alpha,\beta\in\mathbf{N}^d.$$

If $\tau>1$ and $\sigma=1$, then $P(x,\partial)$ of class $(\tau,1)$ (resp. $\{\tau,1\}$) is Komatsu's ultradifferentiable operator of class $(p!^\tau)$ (resp. $\{p!^\tau\}$), see [11].

The following theorem gives the continuity properties of such differential operators on $\mathcal{E}_{\tau,\sigma}(U)$, cf. [18, Theorem 2.1] for the proof.

Theorem 2.1. *Let $P(x,\partial)$ be a differential operator of class (τ,σ) (resp. $\{\tau,\sigma\}$). Then*

$$P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U)\longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)$$

is a continuous linear mapping, and the same holds for

$$P(x,\partial): \quad \varinjlim_{\tau\to\infty}\mathcal{E}_{\tau,\sigma}(U)\longrightarrow \varinjlim_{\tau\to\infty}\mathcal{E}_{\tau,\sigma}(U).$$

2.1. Superposition in $\mathcal{E}_{\tau,\sigma}(U)$

We prove in this subsection that the classes $\mathcal{E}_{\tau,\sigma}(U)$, $\tau>0$, $\sigma>1$, are stable under *superposition*, and conclude that they are inverse closed. We refer to [5, 8, 19] for related results. We emphasize here that the inverse-closedness of $\mathcal{E}_{\tau,\sigma}(U)$ plays an essential role in the proof our main result, Theorem 4.1.

Recall, an algebra $\mathcal{A}$ is *inverse-closed* in $C^\infty(U)$ if for any $\varphi\in\mathcal{A}$ for which $\varphi(x)\neq 0$ on U it follows that $\varphi^{-1}\in\mathcal{A}$. It is proved in [23] that a Carleman class defined by a sequence M_p is inverse closed in $C^\infty(U)$ if there exists $C>0$ such that

$$\left(\frac{M_p}{p!}\right)^{1/p} \leq C\left(\frac{M_q}{q!}\right)^{1/q}, \quad p\leq q, \quad \text{and} \quad \lim_{p\to\infty} M_p^{1/p}=\infty, \tag{2.2}$$

where the condition on the left hand side of (2.2) is equivalent to the statement that $(M_p/p!)^{1/p}$ is an almost increasing sequence, cf. [9, 22, 23].

By the Stirling formula $(M_p/p!)^{1/p}$ is an almost increasing sequence if and only if

$$\frac{M_p^{1/p}}{p}\leq C\frac{M_q^{1/q}}{q}, \quad p\leq q.$$

For example, Gevrey classes $\mathcal{E}_{\{\tau\}}(U)$, $\tau\geq 1$ are inverse-closed algebras.

Since $\Big(\frac{M_p^{\tau,\sigma}}{p^p}\Big)^{1/p} = p^{\tau p^{\sigma-1}-1}$ when $M_p^{\tau,\sigma}=p^{\tau p^\sigma}$, $\tau>0$, $\sigma>1$, and

$$p^{\tau p^{\sigma-1}-1} < q^{\tau q^{\sigma-1}-1}, \quad \lceil (1/\tau)^{1/(\sigma-1)}\rceil < p < q,$$

we conclude that $\Big(\frac{M_p^{\tau,\sigma}}{p^p}\Big)^{1/p}$ is an almost increasing sequence and for any choice of indices k_i, $i=1,\dots,j$, and $k=\sum_{i=1}^j k_i$, we have

$$\frac{M_{k_i}^{\tau,\sigma}}{k_i!}\leq C^{k_i}\left(\frac{M_k^{\tau,\sigma}}{k!}\right)^{k_i/k}, \quad \text{so that} \quad \prod_{i=1}^j \frac{M_{k_i}^{\tau,\sigma}}{k_i!}\leq C^k\frac{M_k^{\tau,\sigma}}{k!}. \tag{2.3}$$

In other words

$$\prod_{i=1}^{j} k_i^{\tau k_i^{\sigma}} \le C^k \frac{k_1! \cdots k_j!}{k!} k^{\tau k^{\sigma}}, \quad k = \sum_{i=1}^{j} k_i.$$

We will use Faá di Bruno formula as presented in [13]. Let us first fix the notation. A multiindex $\alpha \in \mathbf{N}^d$ is said to be *decomposed into parts* $p_1, \dots, p_s \in \mathbf{N}^d$ with *multiplicities* $m_1, \dots, m_s \in \mathbf{N}$, respectively, if

$$\alpha = m_1 p_1 + m_2 p_2 + \cdots + m_s p_s, \quad s \le |\alpha|, \tag{2.4}$$

where $m_i \in \{0, 1, \dots, |\alpha|\}$, $|p_i| \in \{1, \dots, |\alpha|\}$, $i = 1, \dots, s$.

If $p_i = (p_{i_1}, \dots, p_{i_d})$, $i \in \{1, \dots, s\}$, we put $p_i < p_j$ when $i < j$, that is when there exists $k \in \{1, \dots, d\}$ such that $p_{i_l} = p_{j_l}$, $l \in \{1, \dots, k-1\}$, and $p_{i_k} < p_{j_k}$. With $m = m_1 + \cdots + m_s$ we denote the *total multiplicity* and note that $m \le |\alpha|$. Therefore any decomposition of α can be identified with the triple (s, p, m), and the set of all decompositions of the form (2.4) is denoted by π.

Let $f : U \to \mathbf{C}$ and $g : V \to U$ be smooth functions, where U, V are open in $\mathbf{R}$ and $\mathbf{R}^d$, respectively. The generalized Faa di Bruno formula is given by

$$\partial^{\alpha}(f(g)) = \alpha! \sum_{(s,p,m)\in\pi} f^{(m)}(g) \prod_{k=1}^{s} \frac{1}{m_k!} \Big(\frac{1}{p_k!} \partial^{p_k} g\Big)^{m_k}. \tag{2.5}$$

A sequence M_p, $p \in \mathbf{N}$ of positive numbers satisfies *the Faá di Bruno property* if there exist a constant $C > 0$ such that for every $j \in \mathbf{Z}_+$ and $k_i \in \mathbf{Z}_+$ we have

$$M_j \prod_{i=1}^{j} M_{k_i} \le C^{\sum_{i=1}^{j} k_i} M_{\sum_{i=1}^{j} k_i}. \tag{2.6}$$

By [19, Lemma 2.2] if M_p satisfies $(M.2)'$ and if $M_p^{1/p}$ is an almost increasing sequence, then M_p satisfies the Faá di Bruno property. Since $M_p^{\tau,\sigma} = p^{\tau p^{\sigma}}$, $\tau > 0$, $\sigma > 1$, does not satisfy $(M.2)'$ we first prove a modified version of the Faá di Bruno property.

Lemma 2.3. *Let $\tau > 0$, $\sigma > 1$ and let $M_p^{\tau,\sigma} = p^{\tau p^{\sigma}}$, $p \in \mathbf{N}$. Then there exist a constant $C > 0$ such that for every $j \in \mathbf{Z}_+$ and $k_i \in \mathbf{Z}_+$, $i = 1, \dots, j$, we have*

$$\frac{M_j^{\tau,\sigma}}{j!} \prod_{i=1}^{j} \frac{M_{k_i}^{\tau,\sigma}}{k_i!} \le C^{k^{\sigma}} \frac{M_k^{\tau,\sigma}}{k!}, \tag{2.7}$$

where $\sum_{i=1}^{j} k_i = k$.

Proof. We follow the ideas from the proof of [19, Theorem 4.11.].

The assertion is trivial when $j = k$ since then $k_i = 1$ for all $1 \le i \le j$ and therefore

$$\frac{M_j^{\tau,\sigma}}{j!} \prod_{i=1}^{j} \frac{M_{k_i}^{\tau,\sigma}}{k_i!} = \frac{M_k^{\tau,\sigma}}{k!} \Big(\frac{M_1^{\tau,\sigma}}{1!}\Big)^k = \frac{M_k^{\tau,\sigma}}{k!}.$$

When $j < k$, we put $I = \{i \mid 1 \le i \le j, k_i \ge 2\}$ and $\tilde{k}_i = k_i - 1$, $i \in I$. Note that

$$k = \sum_{i=1}^{j} k_i = \sum_{i\in I} k_i + \sum_{i\notin I, 1\le i\le j} k_i = \sum_{i\in I} k_i + j - \operatorname{card} I = \sum_{i\in I} \tilde{k}_i + j, \tag{2.8}$$

and since $\Big(\dfrac{M_p^{\tau,\sigma}}{p!}\Big)^{1/p}$ is almost increasing, then (2.3) implies that

$$\frac{M_j^{\tau,\sigma}}{j!}\prod_{i\in I}\frac{M_{\tilde{k}_i}^{\tau,\sigma}}{\tilde{k}_i!}\le C^k\frac{M_k^{\tau,\sigma}}{k!}, \tag{2.9}$$

Moreover, from $\overline{(M.2)'}$ and $k_i=\tilde{k}_i+1$, $i\in I$, we obtain

$$\frac{M_{k_i}^{\tau,\sigma}}{k_i!}\le C_1^{\tilde{k}_i^{\sigma}}\frac{M_{\tilde{k}_i}^{\tau,\sigma}}{\tilde{k}_i!}, \tag{2.10}$$

for some constant $C_1>0$.

By combining (2.8), (2.9) and (2.10) we obtain

$$\begin{aligned}\frac{M_j^{\tau,\sigma}}{j!}\prod_{i=1}^{j}\frac{M_{k_i}^{\tau,\sigma}}{k_i!}&\le\Big(\frac{M_1^{\tau,\sigma}}{1!}\Big)^{j-\operatorname{card}I}\frac{M_j^{\tau,\sigma}}{j!}\prod_{i\in I}\frac{M_{k_i}^{\tau,\sigma}}{k_i!}\le\frac{M_j^{\tau,\sigma}}{j!}\prod_{i\in I}C_1^{\tilde{k}_i^{\sigma}}\frac{M_{\tilde{k}_i}^{\tau,\sigma}}{\tilde{k}_i!}\\&\le C_1^{(k-j)^{\sigma}}\frac{M_j^{\tau,\sigma}}{j!}\prod_{i\in I}\frac{M_{\tilde{k}_i}^{\tau,\sigma}}{\tilde{k}_i!}\le C_2^{k^{\sigma}}\frac{M_k^{\tau,\sigma}}{k!},\end{aligned}$$

for some constant $C_2>0$ and the Lemma is proved. □

The main result of this section reads as follows.

Theorem 2.2. *Let $\tau>0$, $\sigma>1$, and let U and V be open sets in $\mathbf{R}$ and $\mathbf{R}^d$, respectively. If $f\in\mathcal{E}_{\tau,\sigma}(U)$ and $g\in\mathcal{E}_{\tau,\sigma}(V)$ is such that $g:V\to U$, then $f\circ g\in\mathcal{E}_{\tau,\sigma}(V)$.*

Proof. For simplicity, we show that if $f\in\mathcal{E}_{\{\tau,\sigma\}}(U)$ and $g\in\mathcal{E}_{\{\tau,\sigma\}}(V)$ is such that $g:V\to U$, then $f\circ g\in\mathcal{E}_{\{\tau,\sigma\}}(V)$. We leave the (so-called Beurling) case $f\in\mathcal{E}_{(\tau,\sigma)}(U)$ and $g\in\mathcal{E}_{(\tau,\sigma)}(V)$ to the reader.

Let $K\subset\subset V$ and $h>0$ be fixed so that $g\in\mathcal{E}_{\tau,\sigma,h}(K)$. Put $I=\{g(x),x\in K\}$ and note that I is a compact set, $I\subset\subset U$. Therefore $f\in\mathcal{E}_{\tau,\sigma,h_1}(I)$ for some $h_1>0$. By the Faá di Bruno formula (2.5), for any $x\in K$ we have the following estimate

$$\begin{aligned}|\partial^{\alpha}(f\circ g)(x)|&\le|\alpha|!\sum_{(s,p,m)\in\pi}|f^{(m)}(g(x))|\prod_{k=1}^{s}\frac{1}{m_k!}\Big(\frac{1}{p_k!}|\partial^{p_k}g(x)|\Big)^{m_k}\\&\le A^{|\alpha|+1}|\alpha|!\sum\Big(h_1^{m^{\sigma}}\prod_{k=1}^{s}h^{m_k|p_k|^{\sigma}}\Big)\frac{m!}{m_1!\dots m_s!}\frac{m^{\tau m^{\sigma}}}{m!}\prod_{k=1}^{s}\Big(\frac{|p_k|^{\tau|p_k|^{\sigma}}}{|p_k|!}\Big)^{m_k}\end{aligned} \tag{2.11}$$

for some $A>0$, and the second sum being taken over all decompositions $|\alpha|=\sum_{k=1}^{s}m_k|p_k|$ where $m=\sum_{k=1}^{s}m_k$, $m_k\in\{0,1,\dots,|\alpha|\}$, $|p_k|\in\{1,\dots,|\alpha|\}$, $k=1,\dots,s$ and $s\le|\alpha|$.

By Lemma 2.3 we have

$$\frac{m^{\tau m^{\sigma}}}{m!}\prod_{k=1}^{s}\Big(\frac{|p_k|^{\tau|p_k|^{\sigma}}}{|p_k|!}\Big)^{m_k}\le C^{|\alpha|^{\sigma}}\frac{|\alpha|^{\tau|\alpha|^{\sigma}}}{|\alpha|!}. \tag{2.12}$$

Moreover,

$$m^{\sigma}+\sum_{k=1}^{s}m_k|p_k|^{\sigma}\le|\alpha|^{\sigma}+|\alpha|^{\sigma-1}\sum_{k=1}^{s}m_k|p_k|=2|\alpha|^{\sigma}$$

wherefrom

$$h_1^{m^\sigma}\prod_{k=1}^{s} h^{m_k|p_k|^\sigma} \leq C_1^{m^\sigma+\sum_{k=1}^{s} m_k|p_k|^\sigma} \leq C_1^{2|\alpha|^\sigma}, \tag{2.13}$$

where $C_1 = \max\{h, h_1\}$. From (2.12), (2.13) and (2.11) we conclude that there is a constant $C_2 > 0$ such that

$$|\partial^\alpha(f\circ g)(x)| \leq C_2^{|\alpha|^\sigma+1}|\alpha|^{\tau|\alpha|^\sigma}\sum\frac{m!}{m_1!\ldots m_s!}, \quad x\in K. \tag{2.14}$$

It remains to estimate $\sum\frac{m!}{m_1!\ldots m_s!}$. Without loss of generality we may assume that $s = |\alpha|$ (for $s < |\alpha|$ we may put $m_k = 0$, for $s < k \leq |\alpha|$). Since $|p_k| \in \{1,\ldots,|\alpha|\}$, we can write

$$|\alpha| = \sum_{k=1}^{|\alpha|} m_k|p_k| = \sum_{k=1}^{|\alpha|} km'_k,$$

where $m = \sum_{k=1}^{|\alpha|} m'_k$. Hence, we conclude that the summation in (2.14) can be taken over all $(m_1,\ldots,m_s)\in\mathbf{N}^s$, $s = |\alpha|$, such that $|\alpha| = \sum_{k=1}^{|\alpha|} km_k$ and $m = \sum_{k=1}^{|\alpha|} m_k$. Therefore,

$$\sum\frac{m!}{m_1!\ldots m_s!} = 2^{m_1+2m_2+\cdots+|\alpha|m_{|\alpha|}-1} = 2^{|\alpha|-1},$$

and the proof is completed. $\square$

As an immediate consequence of Theorem 2.2, we conclude the following:

Corollary 2.1. *Let $U \subseteq \mathbf{R}^d$ be open. Classes $\mathcal{E}_{\tau,\sigma}(U)$, $\tau > 0$, $\sigma > 1$, are inverse-closed in $C^\infty(U)$.*

We refer to [25] for a direct proof of Corollary 2.1.

Note that the proof of Theorem 2.2 holds even if $\sigma = 1$ and $\tau \geq 1$, so that we recover the well known superposition property of Gevrey classes (see [5, 8, 9, 19]).

3. Wave front sets related to $\mathcal{E}_{\tau,\sigma}(U)$

Let $\tau > 0$, $\sigma > 1$, $\overline{\Omega} \subseteq K \subset\subset U \subseteq \mathbf{R}^d$, where Ω and U are open in $\mathbf{R}^d$.

Let $u \in \mathcal{D}'(U)$. We studied in [17] the nature of regularity related to the condition

$$|\widehat{u}_N(\xi)| \leq A\frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}}, \quad N\in\mathbf{N},\ \xi\in\mathbf{R}^d\backslash\{0\}. \tag{3.1}$$

where $\{u_N\}_{N\in\mathbf{N}}$ is bounded sequence in $\mathcal{E}'(U)$ such that $u_N = u$ in Ω and A, h are some positive constants.

Note that (3.1) can be replaced by another condition when instead of N we use another positive, increasing sequence a_N such that $a_N \to \infty$, $N \to \infty$. This change of variables called *enumeration*, "speeds up" or "slows down" the decay estimates of single members of the corresponding sequences, without changing the asymptotic behavior of the whole sequence when $N \to \infty$. After applying the enumeration $N \to a_N$ we can write again u_N instead of u_{a_N}, since we are only interested in the asymptotic behavior.

For example, Stirling's formula and enumeration $N \to N^\sigma$ applied to (3.1) give an equivalent estimate of the form

$$|\widehat{u}_N(\xi)| \leq A_1\frac{h_1^{N^\sigma}N^{\tau N^\sigma}}{|\xi|^N}, \quad N\in\mathbf{N},\ \xi\in\mathbf{R}^d\backslash\{0\}, \tag{3.2}$$

for some constants $A_1, h_1 > 0$. We refer to [18] for more details on enumeration.

Wave-front sets $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ (see Remark 3.2 for $\mathrm{WF}_{(\tau,\sigma)}(u)$) are introduced in [18] in the study of local regularity in $\mathcal{E}_{\{\tau,\sigma\}}(U)$. Together with enumeration we used sequences of cutoff functions in a similar way as it is done in [6] in the context of analytic wave front set WF_A. We recall the definition of $\mathrm{WF}_{\{\tau,\sigma\}}(u)$.

Definition 3.1. *Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and* (3.1) *holds for all $\xi \in \Gamma$ and for some constants $A, h > 0$.*

Let $u \in \mathcal{D}'(U)$. Then, immediately follows that $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ is a closed subset of $U \times \mathbf{R}^d \backslash \{0\}$. Note that for $\tau > 0$ and $\sigma > 1$

$$\mathrm{WF}_{\{\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{1,1\}}(u) = \mathrm{WF}_A(u), \quad u \in \mathcal{D}'(U),$$

where $\mathrm{WF}_A(u)$ denoted the analytic wave front set of a distribution $u \in \mathcal{D}'(U)$, cf. [6].

Next, we prove that in the definition of $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ a bounded sequence of cut-off functions $\{u_N\}_{N\in\mathbf{N}} \subset \mathcal{E}'(U)$ can be replaced by a single function from $\mathcal{D}_{\{\tau,\sigma\}}(U)$. First, we give an example of $\phi \in \mathcal{D}_{\{\tau,\sigma\}}(U)$ such that $\phi = 1$ on particular open sets.

Example 3.1. *Let $x_0 \in \mathbf{R}^d$, $\tau > 0$, $\sigma > 1$, and let $d = \displaystyle\sum_{p=1}^{\infty} \frac{1}{(2(p+1))^{\tau p^{\sigma-1}}}$. By Lemma 2.2 and [6, Theorem 1.4.2], there exists $\psi \in \mathcal{D}^{\overline{B_{d/2}(x_0)}}_{\{\tau,\sigma\}}$ such that $\displaystyle\int \psi(x)\,dx = 1$. If χ denotes the characteristic function of*

$$\{y \in \mathbf{R}^d \,|\, |x - y| \le d/2,\ x \in \overline{B_{d/2}(x_0)}\},$$

then $\phi = \chi * \psi = 1$ on an open neighborhood Ω of $\overline{B_{d/2}(x_0)}$. In particular, if U is an open set such that

$$\inf\{|x - y| \,:\, x \in U^c,\ y \in \overline{B_{d/2}(x_0)}\} > d,$$

then $\phi \in \mathcal{D}_{\{\tau,\sigma\}}(U)$.

Remark 3.1. *In the sequel we will use the following Paley-Wiener type estimates. If $u \in \mathcal{E}'(U)$, then $|\widehat{u}(\xi)| \le C\langle\xi\rangle^M$, $\xi \in \mathbf{R}^d$, for some constant $C > 0$.*

Similarly, if $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$, where K is a compact set in $\mathbf{R}^d$, then

$$|\widehat{\phi}(\xi)| \le A h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma} \langle\xi\rangle^{-|\alpha|}, \quad \alpha \in \mathbf{N}^d, \xi \in \mathbf{R}^d, \tag{3.3}$$

for some constants $A, h > 0$.

Theorem 3.1. *Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and let $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ if and only if there exists a conic neighborhood Γ_0 of ξ_0, a compact set $K \subset\subset U$ and $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ such that $\phi = 1$ on a neighborhood of x_0, and such that*

$$|\widehat{\phi u}(\xi)| \le A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \Gamma_0, \tag{3.4}$$

for some $A, h > 0$.

Proof. The necessity is trivial, since if there is a $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$, $K \subset\subset U$, $\phi = 1$ on a neighborhood Ω of x_0 and such that (3.4) holds in a conic neighborhood Γ_0 of ξ_0, then putting $u_N = \phi u$, for every $N \in \mathbf{N}$, it follows that $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$.

Now assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$, i.e. that there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and such that

$$|\widehat{u_N}(\xi)| \le A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \Gamma. \tag{3.5}$$

Choose $\phi \in \mathcal{D}^{K_{x_0}}_{\{\tau,\sigma\}}$, $K_{x_0} \subset\subset \Omega$, $\phi = 1$ on some neighborhood of x_0, and choose a conic neighborhood Γ_0 of ξ_0 with the closure contained in Γ. Let $\varepsilon > 0$ be chosen so that $\xi - \eta \in \Gamma$ when $\xi \in \Gamma_0$ and $|\eta| < \varepsilon|\xi|$.

Since $\phi u = \phi u_N$,

$$\widehat{\phi u}(\xi) = \Big(\int_{|\eta|<\varepsilon|\xi|} + \int_{|\eta|\geq\varepsilon|\xi|} \Big) \widehat{\phi}(\eta)\widehat{u}_N(\xi - \eta)\, d\eta = I_1 + I_2 , \quad \xi \in \Gamma_0.$$

In order to estimate I_1, we use that $|\eta| < \varepsilon|\xi|$ implies $|\xi-\eta| \geq |\xi|-|\eta| > (1-\varepsilon)|\xi|$. By (3.5) and $|\widehat{\phi}(\eta)| \leq B\langle\eta\rangle^{-d-1}$ for some $B > 0$, we have

$$\begin{aligned} |I_1| = \Big| \int_{|\eta|<\varepsilon|\xi|} \widehat{\phi}(\eta)\widehat{u_N}(\xi - \eta)\, d\eta \Big| &\leq \int_{|\eta|<\varepsilon|\xi|} |\widehat{\phi}(\eta)| A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi - \eta|^N} d\eta \\ &\leq AB \frac{h^{N^\sigma} N^{\tau N^\sigma}}{((1-\varepsilon)|\xi|)^N} \int_{\mathbf{R}^d} \langle\eta\rangle^{-d-1} d\eta \leq A_1 \frac{h_1^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad \xi \in \Gamma_0, N \in \mathbf{N}, \end{aligned} \tag{3.6}$$

for some constants $A_1, h_1 > 0$. For the last estimate we have used $(1-\varepsilon)^{-N} < (1-\varepsilon)^{-N^\sigma}$ when $\sigma > 1$.

To estimate I_2, we use that $|\eta| \geq \varepsilon|\xi|$ implies $|\xi - \eta| \leq |\xi| + |\eta| \leq (1 + 1/\varepsilon)|\eta|$. For a given $N \in \mathbf{N}$, we put $|\alpha| = N + M + d + 1$. Then, by (3.3) there exist constants $A, h > 0$ such that

$$\begin{aligned} |I_2| &= \Big| \int_{|\eta|\geq\varepsilon|\xi|} \widehat{\phi}(\eta)\widehat{u_N}(\xi - \eta)\, d\eta \Big| \\ &\leq \frac{A h^{(N+M+d+1)^\sigma} (N + M + d + 1)^{\tau(N+M+d+1)^\sigma}}{(\varepsilon|\xi|)^N} \cdot \int_{|\eta|\geq\varepsilon|\xi|} \langle\eta\rangle^{-M-d-1} C \langle\xi - \eta\rangle^M\, d\eta \\ &\leq \frac{A_1 h_1^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N} \quad \xi \in \Gamma_0, N \in \mathbf{N}, \end{aligned} \tag{3.7}$$

where $h_1 = \max\{h, h^{2^{\sigma-1}}\}$, $A_1 = A\ \max\{1, h^{2^{\sigma-1}(M+d+1)}\}$.

In the last inequality we used

$$|\alpha|^\sigma + |\beta|^\sigma \leq |\alpha + \beta|^\sigma \leq 2^{\sigma-1}(|\alpha|^\sigma + |\beta|^\sigma), \quad \alpha, \beta \in \mathbf{N}^d,$$

and $\overline{(M.2)'}$ property of $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$.

Thus (3.4) follows and the theorem is proved. □

Remark 3.2. *In the Beurling case, for $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$ we have that $(x_0, \xi_0) \notin \mathrm{WF}_{(\tau,\sigma)}(u)$ if there exists open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and such that for every $h > 0$ there exists $A > 0$ such that*

$$|\widehat{u_N}(\xi)| \leq A \frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\ \xi \in \Gamma.$$

Note that Theorem 3.1 can be formulated for the Beurling case as well with $\phi \in \mathcal{D}^K_{(\tau,\sigma)}$ such that (3.4) *holds for every $h > 0$ and for some $A = A(h) > 0$. More precisely, for any $h > 0$ we can choose $\phi \in \mathcal{D}^K_{\tau,\sigma,C_h}$ where $C_h = \min\{h, h^{\frac{1}{2^{\sigma-1}}}\}$ and obtain $\phi \in \mathcal{D}^K_{(\tau,\sigma)}$ with the desired properties.*

Thus, the results concerning $\mathrm{WF}_{(\tau,\sigma)}(u)$ are analogous to those for $\mathrm{WF}_{\{\tau,\sigma\}}(u)$, and we will consider only the later wave-front sets in the sequel.

We end this section by an auxiliary result which will be used in the proof of Theorem 4.1.

Lemma 3.1. *Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, $\Omega \subset K \subset\subset U$, where U and Ω are open. If F is a closed cone such that $\mathrm{WF}_{\{\tau,\sigma\}}(u) \cap (K \times F) = \emptyset$ and $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$, $\phi = 1$ on Ω, then for some $A, h > 0$,*

$$|\widehat{\phi u}(\xi)| \leq A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in F. \tag{3.8}$$

Proof. Let $(x_0, \xi_0) \in K \times F$, and set $r_0 := r_{x_0,\xi_0} > 0$. Furthermore, let $\phi \in \mathcal{D}_{\{\tau,\sigma\}}(B_{r_0}(x_0))$, $\overline{B_{r_0}(x_0)} \subseteq \Omega \subseteq K$.

Since $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$, by Theorem 3.1 there exists $\psi \in \mathcal{D}_{\{\tau,\sigma\}}(U)$, $\psi = 1$ on Ω, and a conical neighborhood Γ of ξ_0, such that

$$|\widehat{\psi u}(\xi)| \leq A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in \Gamma, \tag{3.9}$$

for some $A, h > 0$.

Let Γ_0 be an open conical neighborhood of ξ_0 with the closure contained in Γ. We write

$$\widehat{\phi u}(\xi) = \Big(\int_{|\eta|<\varepsilon|\xi|} + \int_{|\eta|\geq\varepsilon|\xi|} \Big) \widehat{\phi}(\eta) \widehat{\psi u}(\xi - \eta)\, d\eta = I_1 + I_2, \quad \xi \in \Gamma_0, ,$$

and arguing in a similar way as in the proof of Theorem 3.1, we obtain (3.8) for $(x, \xi) \in B_{r_0}(x_0) \times \Gamma_0$.

In order to extend the result to $K \times F$, we use the same idea as in the proof of [6, Lemma 8.4.4]. Since the intersection of F with the unit sphere is a compact set, there exists a finite number n of balls $B_{r_{x_0,\xi_j}}(x_0)$, such that $F \subset \cup_{j=1}^n \Gamma_j$. Note that (3.8) remains true if ϕ is chosen so that $\operatorname{supp} \phi \subseteq B_{r_{x_0}} := \bigcap_{j=1}^n B_{r_{x_0,\xi_j}}(x_0)$, $\xi_j \in \Gamma_j$.

Moreover, since K is compact set, it can be covered by a finite number m of balls $B_{r_{x_k}}$, $k \leq m$. Since $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ satisfies $(M.1)$ and $(M.3)'$, then there exist non-negative functions $\phi_k \in \mathcal{D}_{\{\tau,\sigma\}}(B_{r_{x_k}})$, $k \leq n$, such that $\sum_{k=1}^n \phi_k = 1$ on a neighborhood of K (cf. [10, Lemma 5.1.]).

To conclude the proof, we note that if $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ then $\phi\phi_k \in \mathcal{D}_{\{\tau,\sigma\}}(B_{r_{x_k}})$ and therefore (3.8) holds if ϕ is replaced by $\phi\phi_k$. Since $\sum_{k=1}^n \phi\phi_k = \phi$, the proof is finished. □

4. The propagation of singularities

To set the stage we recall the notion of the characteristic set of an operator and the main property of its principal symbol, cf. [20].

If $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ is a differential operator of order m on U and $a_\alpha \in C^\infty(U)$, $|\alpha| \leq m$, then its characteristic variety at $\overline{x} \in U$ is given by

$$\mathrm{Char}_{\overline{x}}(P) = \{(\overline{x}, \xi) \in U \times \mathbf{R}^d \backslash \{0\} \,|\, P_m(\overline{x}, \xi) = 0\},$$

and its characteristic set on U is given by

$$\mathrm{Char}(P) = \bigcup_{\overline{x}\in U} \mathrm{Char}_{\overline{x}}(P).$$

Here $P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in C^\infty(U \times \mathbf{R}^d \backslash \{0\})$ is the principal symbol of $P(x, D)$.

By the homogeneity of the principal symbol, it follows that $\mathrm{Char}(P)$ is a closed conical subset of $U \times \mathbf{R}^d \backslash \{0\}$.

If $(x_0, \xi_0) \notin \mathrm{Char}(P)$, then there exists an open neighborhood Ω of x_0 and a conical neighborhood Γ of ξ_0 such that $P_m(x, \xi) \neq 0$, $x \in \Omega$ and $\xi \in \Gamma$. Moreover, since the principal symbol is homogeneous we have

$$\Big| P_m(x, \frac{\xi}{|\xi|}) \Big| = \frac{1}{|\xi|^m} |P_m(x, \xi)| \geq C, \quad x \in \Omega, \xi \in \Gamma,$$

so that for any compact set $K \subset\subset \Omega$ there are constants $0 < C_1 < C_2$ such that

$$C_1|\xi|^m \leq |P_m(x,\xi)| \leq C_2|\xi|^m, \quad x \in K, \xi \in \Gamma.$$

Next we extend [18, Theorem 1.1] to operators with variable coefficients. We recall that in [6, Theorem 8.6.1] operators with real analytic coefficients are observed, while in Theorem 4.1 we allow extended Gevrey regular coefficients. In particular, by the inspection of the proof, we conclude that Theorem 4.1 remains true even if $\sigma = 1$ and $\tau > 1$, that is, if the coefficients are Gevrey regular. We refer to [21, Corollary 3.4.14.] for a related result in the context of pseudo-differential operators.

Theorem 4.1. *Let $\tau > 0$, $\sigma > 1$, $u \in \mathcal{D}'(U)$ and let $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be partial differential operator of order m such that $a_\alpha(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, $|\alpha| \leq m$. Then*

$$\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(f) \subseteq \mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(f) \cup \mathrm{Char}(P(x,D)), \tag{4.1}$$

where $P(x,D)u = f$ in $\mathcal{D}'(U)$. In particular,

$$\mathrm{WF}_{0,\infty}(f) \subseteq \mathrm{WF}_{0,\infty}(u) \subseteq \mathrm{WF}_{0,\infty}(f) \cup \mathrm{Char}(P(x,D)), \tag{4.2}$$

where $\mathrm{WF}_{0,\infty}(u) = \bigcup_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\{\tau,\sigma\}}(u)$.

Proof. The pseudolocal property $\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(f) \subseteq \mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u)$ is already proved in [24], see also [18], so it remains to prove the second inclusion in (4.1).

Assume that $(x_0,\xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(f) \cup \mathrm{Char}(P(x,D))$. Then, there exists a compact set K containing x_0 and a closed cone Γ containing ξ_0 such that $P_m(x,\xi) \neq 0$ when $(x,\xi) \in K \times \Gamma$ and such that

$$(K \times \Gamma) \cap \Big(\mathrm{WF}_{\{\tau,\sigma\}}(f) \cup \mathrm{Char}(P(x,D))\Big) = \emptyset.$$

Let $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ such that $\phi = 1$ on some neighborhood of x_0. By Theorem 3.1 it is enough to prove that

$$|\widehat{\phi u}(\xi)| \leq A \frac{h^{N^\sigma} N^{2^{\sigma-1}\tau N^\sigma}}{|\xi|^N}, \quad \xi \in \Gamma, N \in \mathbf{N}.$$

We divide the proof in several steps.

Step 1. Note that the Paley-Wiener type estimate (see Remark 3.1) implies

$$|\xi|^N |\widehat{\phi u}(\xi)| \leq A (N^{2^{\sigma-1}\tau N^{\sigma-1}})^N (N^{2^{\sigma-1}\tau N^{\sigma-1}})^M \leq A h^{N^\sigma} N^{2^{\sigma-1}\tau N^\sigma}, \quad N \in \mathbf{N},$$

where $A, h > 0$ do not depend on N, and the last inequality follows from $M 2^{\sigma-1}\tau N^{\sigma-1} \ln N \leq M 2^{\sigma-1}\tau N^\sigma$ after taking the exponentials. This gives the desired estimate when $|\xi| \leq N^{2^{\sigma-1}\tau N^{\sigma-1}}$, $\xi \in \Gamma$.

Step 2. It remains to estimate $|\widehat{\phi u}(\xi)|$ when $\xi \in \Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$ and for $N \in \mathbf{N}$ large enough. We refer to Subsection 4.1 for the calculations which lead to

$$\phi(x) = e^{ix\cdot\xi} P^T(x,D) \left(\frac{e^{-ix\cdot\xi}}{P_m(x,\xi)} w_N(x,\xi) \right) + e_N(x,\xi), \quad x \in K, \xi \in \Gamma, \tag{4.3}$$

where

$$w_N(x,\xi) = \sum_{k \in \mathcal{K}_1} \sum_{\mathfrak{S}_k = 0}^{N-m} (R_{j_1} R_{j_2} \ldots R_{j_k} \phi)(x,\xi), \tag{4.4}$$

$$e_N(x,\xi) = \sum_{k \in \mathcal{K}_2} \sum_{\mathfrak{S}_k = N-m+1}^{N} (R_{j_1} R_{j_2} \ldots R_{j_k} \phi)(x,\xi), \tag{4.5}$$

$\mathfrak{S}_k = j_1 + j_2 + \cdots + j_k$, $j_i \in \{1, \ldots, m\}$, $1 \le i \le k$, and we put

$$\mathcal{K}_1 = \{k \in \mathbf{N} \,|\, 0 \le mk \le N - m\}, \tag{4.6}$$

and

$$\mathcal{K}_2 = \{k \in \mathbf{N} \,|\, N - m < mk \le N\}. \tag{4.7}$$

Functions R_j in (4.4) and (4.5) can be written as

$$R_j(x,\xi) = \sum_{|\alpha|\le j} c_{\alpha,j}(x,\xi) D^\alpha, \tag{4.8}$$

for suitable functions $c_{\alpha,j}(x,\xi)$ which are homogeneous of order $-j$ (with respect to ξ) and such that

$$|D^\beta c_{\alpha,j}(x,\xi)| \le |\xi|^{-j} A h^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma}, \quad \beta \in \mathbf{N}^d, x \in K, \xi \in \Gamma$$

for some $A, h > 0$ and for all $|\alpha| \le j$, see Subsections 4.1 and 4.2.

From (4.3) it follows that

$$\begin{aligned}\widehat{\phi u}(\xi) &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int u(x) P^T(x,D) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(x,\xi)} \right) dx \\ &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int P(x,D) u(x) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(x,\xi)} \right) dx, \end{aligned} \tag{4.9}$$

$x \in K$, $\xi \in \Gamma$, and in the next steps we estimate terms on the right hand side of (4.9).

Remark 4.1. *Since the number of summands in $w_N(x,\xi)$ and $e_N(x,\xi)$ is the same as in the case when R_j have constant coefficients we refer to [18, Subsection 4.1] where it is shown that the upper bound for the number of summands is of the form $A \cdot C^N$ for suitable constants $A, C > 0$. In fact, from [18, Subsection 4.1] it follows that the number of summands in*

$$e_N(x,\xi) = \sum_{k\in\mathcal{K}_2} \sum_{\mathfrak{S}_k = \lfloor (N/\tilde\tau)^{1/\sigma} \rfloor - m + 1}^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor} (R_{j_1} R_{j_2} \ldots R_{j_k} \phi)(x,\xi)$$

is bounded by $A \cdot C^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor}$ and the calculations remain the same after replacing $\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor$ by N.

Step 3. Note that the operators R_j, $1 \le j \le m$, given in (4.8) do not commute. For that reason we must use different arguments than those given in [18] where the operators with constant coefficients were studied. The estimates of $D^\beta(R_{j_1} ... R_{j_k} \phi)$ from Subsection 4.3 (cf. (4.29)) imply

$$\begin{aligned}|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| &\le A \sum_{|\alpha|\le M} |D_x^\alpha (e_N(x,\xi) e^{-ix\xi})| \\ &= A \sum_{|\alpha|\le M} |D_x^\alpha (\sum_{k\in\mathcal{K}_2} \sum_{\mathfrak{S}_k = N-m+1}^{N} (R_{j_1} R_{j_2} \ldots R_{j_k} \phi)(x,\xi) e^{-ix\xi})| \\ &\le A_1 |\xi|^M |\xi|^{m-N} h^{N^\sigma} (N+M)^{\tau(N+M)^\sigma} = A_1 \frac{h^{N^\sigma} (N+M)^{\tau(N+M)^\sigma}}{|\xi|^{N-m-M}}, \end{aligned} \tag{4.10}$$

$x \in K$, $\xi \in \Gamma$, for suitable constants $A_1, h > 0$, and $N \in \mathbf{N}$ large enough. After enumeration $N \to N + m + M$ we conclude that (4.10) is equivalent to

$$\begin{aligned}|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| &\le A_1 \frac{h^{(N+m+N)^\sigma} (N+m+2M)^{\tau(N+m+2M)^\sigma}}{|\xi|^N} \\ &\le A_2 \frac{h_1^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad x \in K, \xi \in \Gamma, \end{aligned}$$

for some $A_2 > 0$ where for the last inequality we used $\overline{(M.2)'}$ of the sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$. This is the estimate for the first term on the righthand side of (4.9).

Step 4. It remains to estimate the second term on the righthand side of (4.9) for $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$. Since $(x_0,\xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(f)$, by Lemma 3.1, there exists a compact set $\tilde K \subset\subset U$ such that $\psi \in \mathcal{D}_{\{\tau,\sigma\}}(U)$, $\psi = 1$ on a neighborhood of $\tilde K$, and a conical neighborhood V of ξ_0 such that $\Gamma \subset V$ and

$$|\mathcal{F}(\psi f)(\eta)| \leq A\frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad \eta \in V, N \in \mathbf{N}, \tag{4.11}$$

for some $A, h > 0$. In the sequel we write $v = \psi f$. Since $w_N f = w_N v$ in $\mathcal{D}'(U)$, we have

$$\begin{aligned}\langle f(\cdot)e^{-i\xi\cdot}, w_N(\cdot,\xi)/P_m(\cdot,\xi)\rangle &= \mathcal{F}_{x\to\eta}(v(x)\frac{w_N(x,\xi)}{P_m(x,\xi)})(\xi)\\ &= \int_{\mathbf{R}^d} \mathcal{F}(v)(\xi-\eta)\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)\,d\eta = I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_{|\eta|<\varepsilon|\xi|} \mathcal{F}(v)(\xi-\eta)\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)\,d\eta,$$

$$I_2 = \int_{|\eta|\geq\varepsilon|\xi|} \mathcal{F}(v)(\xi-\eta)\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)\,d\eta,$$

and $0 < \varepsilon < 1$ is chosen so that $\xi - \eta \in V$ when $\xi \in \Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$, and $|\eta| < \varepsilon|\xi|$.

To finish the proof we estimate I_1 and I_2 in Step 5 and Step 6, respectively.

Step 5. Let $j_1,\dots,j_k \in \{1,\dots,m\}$ be fixed. Since the coefficients of $P_m(\cdot,\xi)$ are in $C^\infty(U)$, and $P_m(x,\xi) \neq 0$ when $x \in K$ and $\xi \in \Gamma$, it follows that $\dfrac{R_{j_1}R_{j_2}\dots R_{j_k}\phi(\cdot,\xi)}{P_m(\cdot,\xi)}$ belongs to $C^\infty(K)$ when $\xi \in \Gamma$, and moreover it is homogeneous of order $-m-\mathfrak{S}_k$. Hence, by Paley-Wiener type estimates it follows that there exist a constant $C > 0$, such that

$$\begin{aligned}|\mathcal{F}_{x\to\eta}\Big(\frac{R_{j_1}R_{j_2}\dots R_{j_k}\phi(x,\xi)}{P_m(x,\xi)}\Big)(\eta,\xi)| &\leq C|\xi|^{-m-\mathfrak{S}_k}\langle\eta\rangle^{-d-1}\\ &\leq C\langle\eta\rangle^{-d-1}, \quad \eta \in \mathbf{R}^d,\end{aligned}$$

when $\xi \in \Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$.

This estimate, and the estimate for number of terms in (4.4) (see remark 4.1) imply that there exist constants $A, C > 0$ such that

$$\begin{aligned}\Big|\mathcal{F}_{x\to\eta}\Big(\frac{w_N(x,\xi)}{P_m(x,\xi)}\Big)\Big| &\leq \sum_{k\in\mathcal{K}_1}\sum_{\mathfrak{S}_k=0}^{N-m}\Big|\mathcal{F}_{x\to\eta}\Big(\frac{R_{j_1}R_{j_2}\dots R_{j_k}\phi(x,\xi)}{P_m(x,\xi)}\Big)(\eta,\xi)\Big|\\ &\leq AC^N\langle\eta\rangle^{-d-1}.\end{aligned} \tag{4.12}$$

Since $|\eta| < \varepsilon|\xi| \Rightarrow |\xi - \eta| \geq (1-\varepsilon)|\xi|$, by using (4.11) and (4.12), we obtain the desired estimate for I_1:

$$\begin{aligned} |I_1| &\leq \int_{|\eta|<\varepsilon|\xi|} |\mathcal{F}(v)(\xi-\eta)||\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)}(\eta,\xi)|\,d\eta \\ &\leq \int_{|\eta|<\varepsilon|\xi|} A\frac{h^{N^\sigma}N^{\tau N^\sigma}}{|\xi-\eta|^N}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)}(\eta,\xi)|\,d\eta \\ &\leq A\frac{h^{N^\sigma}N^{\tau N^\sigma}}{((1-\varepsilon)|\xi|)^N}\int_{\mathbf{R}^d} C^N\langle\eta\rangle^{-d-1}\,d\eta \\ &\leq A_1\frac{h_1^{N^\sigma}N^{\tau N^\sigma}}{|\xi|^N}, \qquad \xi\in\Gamma, |\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}, \end{aligned}$$

for some constants $A_1, h_1 > 0$.

Step 6. It remains to estimate I_2. We note that $|\xi-\eta| \leq (1+1/\varepsilon)|\eta|$ when $|\eta| \geq \varepsilon|\xi|$. The Paley-Wiener type estimate for $v = \psi f \in \mathcal{E}'(U)$ implies that $|\mathcal{F}(v)(\eta)| \leq C\langle\eta\rangle^M$, for some constant $M, C > 0$. Therefore

$$\begin{aligned} |I_2| &\leq \int_{|\eta|\geq\varepsilon|\xi|} |\mathcal{F}(v)(\xi-\eta)||\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|\,d\eta \\ &\leq \int_{|\eta|\geq\varepsilon|\xi|} \langle\xi-\eta\rangle^M\langle\eta\rangle^{N+d+1}\frac{|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|}{\langle\eta\rangle^{N+d+1}}\,d\eta \\ &\leq C^{N+1}\frac{\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{N+M+d+1}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|}{\langle\xi\rangle^N}, \end{aligned}$$

when $\xi\in\Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$.

To finish the proof, it remains to show that $\xi\in\Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$, implies that there exist constants $A, h > 0$ such that

$$\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{N+M+d+1}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)| \leq Ah^{N^\sigma}N^{2^{\sigma-1}\tau N^\sigma}, \tag{4.13}$$

for a sufficiently large $N\in\mathbf{N}$, and then we use this estimate to bound $|I_2|$.

Arguing in the similar way as in the proof of [18, Theorem 1.1], it is sufficient to prove

$$\sup_{x\in K}\Big|D^\beta\Big(\frac{w_N(x,\xi)}{P_m(x,\xi)}\Big)\Big| \leq Ah^{N^\sigma}N^{2^{\sigma-1}\tau N^\sigma}, \quad \beta\in\mathbf{N}^d, |\beta| = N+M+d+1, \tag{4.14}$$

for some constants $A, h > 0$, when $\xi\in\Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$. Recall (see Subsection 4.2),

$$\sup_{x\in K}\Big|D^\gamma\frac{1}{P_m(x,\xi)}\Big| \leq |\xi|^{-m}C^{|\gamma|^\sigma+1}|\gamma|^{\tau|\gamma|^\sigma}, \quad \gamma\in\mathbf{N}^d, \xi\in\Gamma,$$

for some constant $C > 0$. Moreover, from (4.29) (see Subsection 4.3), it follows that

$$\sup_{x\in K}|D^\gamma w_N(x,\xi)| \leq Ah^{N^\sigma}\sum_{k\in\mathcal{K}_1}\sum_{\mathfrak{S}_k=0}^{N-m}|\xi|^{-\mathfrak{S}_k}(\mathfrak{S}_k+|\gamma|)^{\tau(\mathfrak{S}_k+|\gamma|)^\sigma},$$

for some constants $A, h > 0$, when $\xi\in\Gamma$.

Hence, for $x \in K$ and $\xi \in \Gamma$, $|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}}$, we obtain

$$\begin{aligned}
\left| D^{\beta} \frac{w_N(x,\xi)}{P_m(x,\xi)} \right| &\leq \sum_{k \in \mathcal{K}_1} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |D^{\beta-\gamma} \frac{1}{P_m(x,\xi)}| |D^{\gamma} w_N(x,\xi)| \\
&\leq Ah^{N^{\sigma}} \sum_{\gamma \leq \beta} \sum_{k \in \mathcal{K}_1} \sum_{\mathfrak{S}_k = 0}^{N-m} |\xi|^{-\mathfrak{S}_k - m} \binom{\beta}{\gamma} C^{|\beta-\gamma|^{\sigma}+1} |\beta-\gamma|^{\tau|\beta-\gamma|^{\sigma}} (\mathfrak{S}_k + |\gamma|)^{\tau(\mathfrak{S}_k + |\gamma|)^{\sigma}} \\
&\leq A_1 h_1^{N^{\sigma}} \sum_{\gamma \leq \beta} \sum_{k \in \mathcal{K}_1} \sum_{\mathfrak{S}_k = 0}^{N-m} \binom{\beta}{\gamma} |\xi|^{-\mathfrak{S}_k} (\mathfrak{S}_k + |\beta|)^{\tau(\mathfrak{S}_k + |\beta|)^{\sigma}}, \qquad (4.15)
\end{aligned}$$

for some $A_1, h_1 > 0$, $\beta \in \mathbf{N}^d$, $|\beta| = N + M + d + 1$, where we used $(M.1)$ property of the sequence $M_p^{\tau,\sigma} = p^{\tau p^{\sigma}}$.

Since $\mathfrak{S}_k \leq N - m$, it follows that $N > \mathfrak{S}_k$ and therefore

$$|\xi| > N^{2^{\sigma-1}\tau N^{\sigma-1}} > \mathfrak{S}_k^{\,2^{\sigma-1}\tau \mathfrak{S}_k^{\,\sigma-1}}.$$

Now, $\overline{(M.2)}$ property of $M_p^{\tau,\sigma} = p^{\tau p^{\sigma}}$ implies

$$\begin{aligned}
|\xi|^{-\mathfrak{S}_k} (\mathfrak{S}_k + |\beta|)^{\tau(\mathfrak{S}_k + |\beta|)^{\sigma}} &\leq \frac{(\mathfrak{S}_k + |\beta|)^{\tau(\mathfrak{S}_k + |\beta|)^{\sigma}}}{\mathfrak{S}_k^{\,2^{\sigma-1}\tau \mathfrak{S}_k^{\,\sigma}}} \\
&\leq C^{\mathfrak{S}_k^{\,\sigma} + |\beta|^{\sigma}} \frac{\mathfrak{S}_k^{\,2^{\sigma-1}\tau \mathfrak{S}_k^{\,\sigma}} |\beta|^{2^{\sigma-1}\tau|\beta|^{\sigma}}}{\mathfrak{S}_k^{\,2^{\sigma-1}\tau \mathfrak{S}_k^{\,\sigma}}} \\
&= C^{\mathfrak{S}_k^{\,\sigma} + |\beta|^{\sigma}} (N + M + d + 1)^{2^{\sigma-1}\tau(N+M+d+1)^{\sigma}} \leq C_1^{N^{\sigma}} N^{2^{\sigma-1}\tau N^{\sigma}}, \qquad (4.16)
\end{aligned}$$

for some constant $C_1 > 0$ where the last inequality follows from $\overline{(M.2)'}$ property of $M_p^{\tau,\sigma}$. Using the estimate for number of terms in w_N, by (4.15) and (4.16), the estimate (4.14) follows.

By the similar arguments as in the proof of [18, Theorem 1.1], (4.13) follows from (4.14) since

$$\pi_1(\operatorname{supp} \frac{w_N(x,\xi)}{P_m(x,\xi)}) \subseteq K.$$

Therefore,

$$|I_2| \leq A \frac{h^{N^{\sigma}} N^{2^{\sigma-1}\tau N^{\sigma}}}{|\xi|^N} \qquad (4.17)$$

for suitable constants $A, h > 0$ and N sufficiently large, and the theorem is proved. □

4.1. *Representing $\widehat{\phi u}(\xi)$ by an approximate solution*

In this subsection we derive (4.3), (4.4) and (4.5).

Let $P^T(x, D) = \sum_{|\alpha| \leq m} b_{\alpha}(x) D^{\alpha}$, $b_{\alpha}(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ be the transpose of $P(x, D)$. If $v(x, \xi)$ is the solution of the equation

$$e^{ix\xi} P^T(x, D) v(x, \xi) = \phi(x), \quad x \in K, \xi \in \Gamma, \qquad (4.18)$$

then

$$\widehat{\phi u}(\xi) = \int u(x)\phi(x) e^{-ix\xi} dx = \int u(x) P^T(x, D) v(x, \xi) dx, \quad \xi \in \Gamma,$$

Similarly as in [6] and [17], we may assume that $v(x,\xi) = \dfrac{e^{-ix\xi}w(x,\xi)}{P_m(x,\xi)}$, for some $w(\cdot,\xi) \in C^\infty(K)$, so that the left hand side of (4.18) becomes

$$\begin{aligned} e^{ix\xi}P^T(x,D)\Big(\frac{w(x,\xi)e^{-ix\xi}}{P_m(x,\xi)}\Big) &= e^{ix\xi}\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\binom{\alpha}{\beta} b_\alpha(x) D^{\alpha-\beta}(e^{-ix\xi})D^\beta\Big(\frac{w(x,\xi)}{P_m(x,\xi)}\Big) \\ &= \sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma} b_\alpha(x)(-\xi)^{\alpha-\beta}D^\gamma\Big(\frac{1}{P_m(x,\xi)}\Big)D^{\beta-\gamma}w(x,\xi) \\ &= (I - R(x,\xi))w(x,\xi), \quad x\in K, \xi\in\Gamma, \end{aligned} \tag{4.19}$$

where

$$R(x,\xi) = \sum_{j=1}^m R_j(x,\xi), \quad R_j(x,\xi) = \sum_{|\alpha|\le j} c_{\alpha,j}(x,\xi)D^\alpha,$$

for suitable functions $c_{\alpha,j}(x,\xi)$ which are homogeneous of order $-j$ and

$$|D^\beta c_{\alpha,j}(x,\xi)| \le |\xi|^{-j}Ah^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}, \quad \beta\in\mathbf{N}^d, x\in K, \xi\in\Gamma \tag{4.20}$$

for some $A, h > 0$ and for all $|\alpha| \le j$. We refer to Subsection 4.2 for the calculus which shows how (4.19) implies (4.20).

Therefore (4.18) can be rewritten in the following convenient form:

$$(I - R(x,\xi))w(x,\xi) = \phi(x) \quad x\in K, \xi\in\Gamma. \tag{4.21}$$

which gives rise to approximate solutions as follows.

Note that the order of operator R^k, $k\in\mathbf{N}$, is mk. We compute

$$\begin{aligned} \sum_{k\in\mathcal{K}_1} R^k - R\sum_{k\in\mathcal{K}_1} R^k &= \sum_{k\in\mathcal{K}_1} R^k - \sum_{k\in\mathcal{K}_1} R^{k+1} \\ &= \sum_{k\in\mathcal{K}_1} R^k - \sum_{\{k\in\mathbf{N}\,|\,m\le mk\le N\}} R^k = I - \sum_{k\in\mathcal{K}_2} R^k \end{aligned} \tag{4.22}$$

where $\mathcal{K}_1$ is given by (4.6) and in the last equality we used

$$\mathcal{K}_1 \cap \{k\in\mathbf{N}\,|\,m\le mk\le N\} = \{k\in\mathbf{N}\,|\,m\le mk\le N-m\}.$$

Moreover, since the operators R_j, $1\le j\le m$, do not commute we can write

$$\sum_{k\in\mathcal{K}_1} R^k = \sum_{k\in\mathcal{K}_1}\sum_{\mathfrak{S}_k=0}^{N-m} R_{j_1}R_{j_2}\dots R_{j_k},$$

and

$$\sum_{k\in\mathcal{K}_2} R^k = \sum_{k\in\mathcal{K}_2}\sum_{\mathfrak{S}_k=N-m+1}^{N} R_{j_1}R_{j_2}\dots R_{j_k}$$

where $\mathfrak{S}_k = j_1 + j_2 + \dots + j_k$, $j_i\in\{1,\dots,m\}$, $1\le i\le k$.

Now,

$$(I-R(x,\xi))\Big(\sum_{k\in\mathcal{K}_1} R^k\phi(x)\Big) = \Big(\sum_{k\in\mathcal{K}_1} R^k - R\sum_{k\in\mathcal{K}_1} R^k\Big)\phi(x) = \Big(I - \sum_{k\in\mathcal{K}_2} R^k\Big)\phi(x) = \phi(x) - \sum_{k\in\mathcal{K}_2} R^k\phi(x),$$

and if we put $w_N = \sum_{k\in\mathcal{K}_1} R^k\phi$ and $e_N = \sum_{k\in\mathcal{K}_2}\phi$ we conclude that

$$(I-R)w_N(x,\xi) = \phi(x) - e_N(x,\xi), \quad N\in\mathbf{N}, x\in K, \xi\in\Gamma,$$

with w_N and e_N given by (4.4) and (4.5) respectively. Now (4.3) follows from (4.19).

4.2. Estimates for $c_{\alpha,j}(x,\xi)$

We show in this subsection that (4.19) implies (4.20). An essential argument in this part of the proof is the inverse-closedness property presented in Theorem 2.1.

Recall,

$$D^{\alpha}\Big(\frac{1}{P_m(x,\xi)}\Big) = \alpha! \sum_{(s,p,j)\in\pi} \frac{(-1)^j j!}{(P_m(x,\xi))^{j+1}} \prod_{k=1}^{s} \frac{1}{j_k!}\Big(\frac{1}{p_k!} D^{p_k} P_m(x,\xi)\Big)^{j_k}, \tag{4.23}$$

for $\alpha \in \mathbf{N}^d$, where sum is taken over all decompositions (s,p,j) of the form

$$\alpha = j_1 p_1 + j_2 p_2 + \cdots + j_s p_s,$$

with $j = \sum_{i=1}^{s} j_i \in \{0,1,\ldots,|\alpha|\}$, $p_i \in \mathbf{N}^d$, $|p_i| \in \{1,\ldots,|\alpha|\}$ for $i \in \{1,\ldots,s\}$, $s \le |\alpha|$. (see Subsection 2.1)

Since the coefficients of $P_m(x,\xi)$ belong to $\mathcal{E}_{\{\tau,\sigma\}}(U)$, it follows that

$$\sup_{x\in K} |D^{p_k} P_m(x,\xi)| \le A h^{|p_k|^{\sigma}} |p_k|^{\tau |p_k|^{\sigma}} |\xi|^m, \tag{4.24}$$

for some $A, h > 0$. Moreover, $(K \times \Gamma) \cap \mathrm{Char}(P) = \emptyset$ implies that

$$\sup_{x\in K} |P_m(x,\xi)| \ge C' |\xi|^m. \tag{4.25}$$

Hence, by (4.23), (4.24) and (4.25), we obtain

$$\begin{aligned}
|D^{\alpha}\Big(\frac{1}{P_m(x,\xi)}\Big)| &\le |\alpha|! \sum_{(s,p,j)\in\pi} \frac{j!}{j_1!\ldots j_s! |P_m(x,\xi)|^{j+1}} \cdot \prod_{k=1}^{s} \Big(\frac{1}{p_k!} |D^{p_k} P_m(x,\xi)|\Big)^{j_k} \\
&\le |\alpha|! \sum_{(s,p,j)\in\pi} \frac{|\xi|^{mj} j!}{|\xi|^{m(j+1)} j_1!\ldots j_s! |P_m(x,\xi)|^{j+1}} \cdot \prod_{k=1}^{s} \Big(\frac{1}{p_k!} A h^{|p_k|^{\sigma}} |p_k|^{\tau|p_k|^{\sigma}}\Big)^{j_k} \\
&\le |\xi|^{-m} A_1 h_1^{|\alpha|^{\sigma}+1} |\alpha|^{\tau|\alpha|^{\sigma}},
\end{aligned}$$

for some $A_1, h_1 > 0$, where the last inequality follows by calculation from the proof of Theorem 2.1.

In particular, we have proved that $\dfrac{1}{P_m(\cdot,\xi)} \in \mathcal{E}_{\{\tau,\sigma,h\}}(K)$ for some $h > 0$ and for every $\xi \in \Gamma$. From the algebra property of extended Gevrey classes it follows that $b_\alpha(\cdot)\partial^{\gamma} \dfrac{1}{P_m(\cdot,\xi)} \in \mathcal{E}_{\{\tau,\sigma,h'\}}(K)$ for some $h' > 0$, where $|\gamma| \le |\alpha| \le m$ and $b_\alpha(x)$ are the coefficients of $P^T(x,D)$.

These estimates, together with (4.19) give (4.20).

4.3. Estimates for $D^{\beta}(R_{j_1}...R_{j_k}\phi)$

In this subsection we follow the idea presented in [6, Lemmas 8.6.2 and 8.6.3]. As in Subsection 4.1 we put

$$\mathfrak{S}_k = j_1 + \cdots + j_k, \quad N - m \le \mathfrak{S}_k \le N,$$

for $k \in \mathbf{N}$ such that $mk \le N$, and let $|\beta| \le M$.

Recall that $R_j(x,\xi) = \sum_{|\alpha|\le j} c_{\alpha,j}(x,\xi) D^{\alpha}$, and note that by the successive applications of the Leibniz rule, $D^{\beta}(R_{j_1}...R_{j_k}\phi)$ can be written as a sum of terms of the form

$$(D^{\gamma_0} c_{\alpha_{j_1},j_1}(x,\xi))(D^{\gamma_1} c_{\alpha_{j_2},j_2}(x,\xi)) \ldots (D^{\gamma_{k-1}} c_{\alpha_{j_k},j_k}(x,\xi))(D^{\gamma_k}\phi(x)).$$

Put $a_i = |\gamma_i|$ so that

$$a_0 + \cdots + a_k = \mathfrak{S}_k + |\beta|, \tag{4.26}$$

$$a_0 \leq |\beta|, \tag{4.27}$$

and

$$a_i \leq \sum_{t=1}^{i} j_t + |\beta|, \quad 1 \leq i \leq k. \tag{4.28}$$

From (4.20) it follows that

$$|D^{\gamma_{i-1}} c_{\alpha_{j_i}, j_i}(x,\xi)| \leq |\xi|^{-j_i} A h^{a_{i-1}^\sigma} a_{i-1}^{\tau a_{i-1}^\sigma}, \quad \gamma_{i-1} \in \mathbf{N}^d, x \in K, \xi \in \Gamma,$$

for some constants $A, h > 0$ and for all $|\alpha_{j_i}| \leq j_i$, $i = 1, \ldots, k$.

Observe that the number of multiindices $\gamma_0, \ldots, \gamma_k$ with the property (4.26) is $\binom{\mathfrak{S}_k + |\beta|}{a_0, \ldots, a_k}$. In the sequel we write $\sum$ when the sum is taken over all multiindices $\gamma_0, \ldots, \gamma_k$ which satisfies (4.26)-(4.28).

Since $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$, for $x \in K$ and $\xi \in \Gamma$, we estimate

$$\begin{aligned}
|(D^\beta R_{j_1} ... R_{j_k} \phi)(x,\xi)| &\leq \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \ldots, a_k} \Big(\prod_{i=1}^{k} |D^{\gamma_{i-1}} c_{\alpha_{j_i}, j_i}(x,\xi)| \Big) \cdot |D^{\gamma_k} \phi(x)| \\
&\leq |\xi|^{-\mathfrak{S}_k} \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \ldots, a_k} \Big(\prod_{i=1}^{k} A h^{a_{i-1}^\sigma} a_{i-1}^{\tau a_{i-1}^\sigma} \Big) \cdot \Big(A h^{a_k{}^\sigma} a_k{}^{\tau a_k{}^\sigma} \Big) \\
&\leq |\xi|^{m-N} A^{\frac{N}{m}+1} h^{N^\sigma} \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \ldots, a_k} \Big(\prod_{i=1}^{k+1} a_{i-1}^{\tau a_{i-1}^\sigma} \Big) \\
&\leq |\xi|^{m-N} A_1 h_1^{N^\sigma} \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \ldots, a_k} \Big(\prod_{i=1}^{k+1} a_{i-1}^{\tau a_{i-1}^\sigma} \Big),
\end{aligned}$$

for some $A_1, h_1 > 0$. By the almost increasing property of $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ it follows that

$$\begin{aligned}
\prod_{i=1}^{k+1} a_{i-1}^{\tau a_{i-1}^\sigma} &\leq C^{a_0 + \cdots + a_k} \frac{a_0! \cdots a_k!}{(a_0 + \cdots + a_k)!} (a_0 + \cdots + a_k)^{\tau (a_0 + \cdots + a_k)^\sigma} \\
&= C^{\mathfrak{S}_k + |\beta|} \frac{a_0! \cdots a_k!}{(\mathfrak{S}_k + |\beta|)!} (\mathfrak{S}_k + |\beta|)^{\tau (\mathfrak{S}_k + |\beta|)^\sigma},
\end{aligned}$$

for some $C > 0$, wherefrom

$$\begin{aligned}
\sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \ldots, a_k} \Big(\prod_{i=1}^{k+1} a_{i-1}^{\tau a_{i-1}^\sigma} \Big) &\leq \sum \frac{a_0! \cdots a_k!}{(\mathfrak{S}_k + |\beta|)!} \cdot C^{\mathfrak{S}_k + |\beta|} \frac{(\mathfrak{S}_k + |\beta|)!}{a_0! \ldots a_k!} (\mathfrak{S}_k + |\beta|)^{\tau (\mathfrak{S}_k + |\beta|)^\sigma} \\
&= C^{\mathfrak{S}_k + |\beta|} (\mathfrak{S}_k + |\beta|)^{\tau (\mathfrak{S}_k + |\beta|)^\sigma} \sum_{a_0 + \cdots + a_k = \mathfrak{S}_k + |\beta|} 1 \\
&\leq C^N (N+M)^{\tau (N+M)^\sigma} \binom{\mathfrak{S}_k + |\beta| - 1}{k} \leq C_1^N (N+M)^{\tau (N+M)^\sigma}
\end{aligned}$$

for suitable $C_1 > 0$.

Hence, we conclude that there exist constants $A, h > 0$ such that

$$|(D^\beta R_{j_1} ... R_{j_k} \phi)(x,\xi)| \leq A |\xi|^{m-N} h^{N^\sigma} (N+M)^{\tau (N+M)^\sigma}, \tag{4.29}$$

$x \in K$, $\xi \in \Gamma$, which gives the desired estimate.

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