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Statistical σ–Convergence of Double Sequences with Application

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Abstract. The concepts of σ-statistical convergence, statistical σ-convergence and strong σ_q-convergence of single (ordinary) sequences have been introduced and studied in [M. Mursaleen, O.H.H. Edely, On the invariant mean and statistical convergence, App. Math. Lett. 22, (2011), 1700–1704] which were obtained by unifying the notions of density and invariant mean. In this paper, we extend these ideas from single to double sequences. We also use the concept of statistical σ-convergence of double sequences to prove a Korovkin-type approximation theorem for functions of two variables and give an example to show that our Korovkin-type approximation theorem is stronger than those proved earlier by other authors.

1. Introduction

The idea of statistical convergence was first introduced by Fast [11] and also independently by Buck [6] and Schoenberg [33] for real and complex sequences.

A double sequence $x = \left(x_{jk}\right)$ of real numbers, $i, j \in \mathbb{N}$, the set of all positive integers, is said to be convergent in the Pringsheim's sense (or $P-$convergent) if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{jk} - L\right| < \epsilon$ whenever $j, k \geq N$. We shall write this as

$$\lim_{j,k} x_{jk} = L$$

where j and k tending to infinity independent of each other [30].

A double sequence x is bounded if there exists a positive number M such that $\left|x_{jk}\right| < M$ for all j and k, i.e., if

$$\|x\|_{(\infty,2)} = \sup_{j,k} \left|x_{jk}\right| < \infty.$$

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Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. Let us denote by ℓ_∞^2 the space of all bounded double sequences.

The concepts of double natural density and statistical convergence of double sequences were introduced by Mursaleen and Edely [22]: Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of (j,k) in K such that $j \leq m$ and $k \leq n$. In case the sequence $(K(m,n)/mn)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

A real double sequence $x = \left(x_{jk}\right)$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set

$$\left\{(j,k), j \leq m \text{ and } k \leq n : \left|x_{jk} - L\right| \geq \epsilon\right\}$$

has double natural density zero. In this case we write $st_2 - \lim x = L$ and we denote the set of all statistically convergent double sequences by st_2.

Let σ be a one-to-one mapping from $\mathbb{N}$ into itself. A continuous linear functional φ on the space ℓ_∞ of all bounded sequences is called an invariant mean or a σ-mean if and only if (i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, (ii) $\varphi(e) = 1$, where $e = (1, 1, \cdots)$, and (iii) $\varphi(x) = \varphi\left(\left(x_{\sigma(k)}\right)\right)$ for all $x \in \ell_\infty$.

The idea of σ-convergence for single sequences was defined in [32] and for double sequences in [7]. A double sequence $x = \left(x_{jk}\right)$ of real numbers is said to be σ-convergent to a number L if and only if $x \in V_2^\sigma$, where

$$V_2^\sigma = \left\{x \in \ell_\infty^2 : \lim_{p,q} \tau_{pqst}(x) = L, \text{ uniformly in } s, t; L = \sigma\text{-}lim x\right\},$$

$$\tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s), \sigma^k(t)}$$

and $\tau_{-1,q,s,t}(x) = \tau_{p,-1,s,t}(x) = \tau_{-1,-1,s,t}(x) = 0$.

For $\sigma(k) = k + 1$, the set V_2^σ is reduced to the set f_2 of almost convergent double sequences [21]. Note that $c_2^\infty \subset V_2^\sigma \subset \ell_\infty^2$.

2. Main Results

The concepts of σ-density of a subset of $\mathbb{N}$, and σ-statistical convergence, statistical σ-convergence and strong σ_v-convergence $(0 < v < \infty)$ of a single sequence have been recently introduced and examined by Mursaleen and Edely in [23]. In this section, we introduce double analogues of these concepts and we also prove some relations between our newly defined methods. There are many situations when we need to study convergence problems for matrix sequences. These new concepts provide the tools to deal with these types of convergence methods for double sequences which can be considered as matrix sequences. We also apply the concept of statistical σ-convergence of double sequences to prove a Korovkin-type approximation theorem for functions of two variables.

Definition 2.1. *A bounded double sequence $x = \left(x_{jk}\right)$ is said to be strongly σ-convergent to L if*

$$\lim_{p,q} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} \left| x_{\sigma^j(s),\sigma^k(t)} - L \right| = 0, \quad \text{uniformly in } s,t.$$

and we write this as $x_{j,k} \to L\left[V_2^\sigma\right]$ (cf. [24]).

Definition 2.2. *Let $s,t \geq 0$, $p,q \geq 1$ be integers, $K \subset N \times N$ and $K_\sigma(s+1, s+p; t+1, t+q)$ denote the cardinality of the set*

$$\{\sigma(s) \leq j \leq \sigma^p(s), \sigma(t) \leq k \leq \sigma^q(t) : (j,k) \in K\}.$$

If we write

$$N_{p,q} = \min_{s,t} K_\sigma(s+1, s+p; t+1, t+q)$$

and

$$N^{p,q} = \max_{s,t} K_\sigma(s+1, s+p; t+1, t+q),$$

then $\underline{\delta}_{2,\sigma}(K) = \lim_{p,q} \frac{N_{p,q}}{pq}$ and $\overline{\delta}_{2,\sigma}(K) = \lim_{p,q} \frac{N^{p,q}}{pq}$ exist. These are called respectively lower and upper double σ-density of the set K. If $\underline{\delta}_{2,\sigma}(K) = \overline{\delta}_{2,\sigma}(K)$, then the common value $\delta_{2,\sigma}(K)$ is called the double σ-density of the set K. For $\sigma(i) = i+1$, double σ-density is reduced to double uniform density (cf. [37]).

Definition 2.3. *A double sequence $x = (x_{jk})$ is said to be σ-statistically convergent to L if for every $\epsilon > 0$ the set $K_\epsilon = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \left|x_{jk} - L\right| \geq \epsilon\right\}$ has double σ-density zero, i.e. $\delta_{2,\sigma}(K_\epsilon) = 0$. In this case we write $\sigma(\delta_2)$-$\lim x = L$. That is*

$$\lim_{p,q} \frac{1}{pq} \left| \left\{ \sigma(s) \leq j \leq \sigma^p(s), \sigma(t) \leq k \leq \sigma^q(t) : \left|x_{jk} - L\right| \geq \epsilon \right\} \right| = 0, \text{ uniformly in } s,t.$$

Definition 2.4. *A double sequence $x = (x_{jk})$ is said to be statistically σ-convergent to L if for every $\epsilon > 0$*

$$\lim_{m,n} \frac{1}{mn} \left| \left\{ p \leq m, q \leq n : \left|\tau_{pqst}(x) - L\right| \geq \epsilon \right\} \right| = 0, \text{ uniformly in } s,t.$$

In this case we write $\delta_2(\sigma)$-$\lim x = L$.

The extended form of the definition of $\left[V_2^\sigma\right]$ is as follows.

Definition 2.5. *A double sequence $x = (x_{jk})$ is said to be strongly σ_v-convergent $(0 < v < \infty)$ to L if*

$$\lim_{p,q} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} \left| x_{\sigma^j(s),\sigma^k(t)} - L \right|^v = 0, \text{ uniformly in } s,t$$

and we write this as $x_{j,k} \to L\left[V_2^\sigma\right]_v$. Note that for $v = 1$, $\left[V_2^\sigma\right]_v \cap \ell_2^\infty = \left[V_2^\sigma\right]$.

From definitions, we can note the following remarks for double sequences.

Remark 2.6. *(i) σ-statistical convergence implies statistical convergence (and hence statistical σ-convergence) and $\sigma(\delta_2)$-$\lim x = st_2$-$\lim x$.*
(ii) σ-convergence implies statistical σ-convergence but not σ-statistical convergence.

Example 2.7. *The double sequence $z = \left(z_{jk}\right)$ defined by*

$$z_{jk} = \begin{cases} 1, & \text{if } j = k \text{ odd}, \\ -1, & \text{if } j = k \text{ even}, \\ 0, & j \neq k; \end{cases} \tag{1}$$

is σ-convergent to zero (for $\sigma(n) = n + 1$) and hence statistically σ-convergent to zero but it is neither statistically convergent nor σ-statistically convergent.

Now we prove the following relation between the concepts of σ-statistical convergence and statistical σ-convergence for a double sequence.

Theorem 2.8. *If a sequence $x = \left(x_{jk}\right)$ is bounded and σ -statistically convergent to L then it is statistically σ-convergent to L but not conversely.*

Proof. Let x be bounded and σ-statistically convergent to L, and
$K(\epsilon) := \left\{\sigma(s) \leq j \leq \sigma^p(s), \sigma(t) \leq k \leq \sigma^q(t) : \ \left|x_{jk} - L\right| \geq \epsilon\right\}$. Then

$$\begin{aligned}
\left|\tau_{pqst}(x) - L\right| &= \left|\frac{1}{pq}\sum_{j=1}^{p}\sum_{k=1}^{q} x_{\sigma^j(s),\sigma^k(t)} - L\right| = \left|\frac{1}{pq}\sum_{j=\sigma(s)}^{\sigma^p(s)}\sum_{k=\sigma(t)}^{\sigma^q(t)} \left(x_{jk} - L\right)\right| \\
&\leq \left|\frac{1}{pq}\sum_{(j,k)\in K(\epsilon)} \left(x_{jk} - L\right)\right| \\
&\leq \frac{1}{pq}\left(\sup_{j,k}\left|x_{jk} - L\right|\right)\max_{s,t} K_\sigma\left(s+1, s+p; t+1, t+q\right) \\
&= \frac{N^{p,q}}{pq}\sup_{j,k}\left|x_{jk} - L\right| \to 0 \ \text{ as } p, q \to \infty,
\end{aligned}$$

which implies that $\lim_{p,q} \tau_{pqst}(x) = L$, uniformly in s, t. Thus x is σ-convergent to L and hence statistically σ-convergent to L
For the converse, consider the double sequence $z = \left(z_{jk}\right)$ defined by (1). It is statistically σ-convergent but not σ-statistically convergent. This completes the proof of the theorem.

In the following result we establish the relation between σ-statistical convergence and strong σ_v-convergence by the same technique used in Theorem 4.1 of [22].

Theorem 2.9. ***(i)*** *If $0 < v < \infty$ and a double sequence $x = \left(x_{jk}\right)$ is strongly σ_v-convergent to L, then it is σ -statistically convergent to L.*
(ii) *If x is bounded and σ-statistically convergent to L then x is strongly σ_v-convergent to the L.*

Proof. **(i)** Let $0 < v < \infty$, $x_{j,k} \to L\left[V_2^\sigma\right]_v$ and $K^\epsilon := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left|x_{jk} - L\right| \geq \epsilon\right\}$. Then we have

$$\begin{aligned}
\frac{1}{pq}\sum_{j=1}^{p}\sum_{k=1}^{q}\left|x_{\sigma^j(s),\sigma^k(t)} - L\right|^v &\geq \frac{1}{pq}\sum_{\substack{j=1\\ \left|x_{\sigma^j(s),\sigma^k(t)}-L\right|\geq\epsilon}}^{p}\sum_{k=1}^{q}\left|x_{\sigma^j(s),\sigma^k(t)} - L\right|^v \\
&= \frac{1}{pq}\sum_{\substack{j=\sigma(s)\\ \left|x_{jk}-L\right|\geq\epsilon}}^{\sigma^p(s)}\sum_{k=\sigma(t)}^{\sigma^q(t)}\left|x_{jk} - L\right|^v \\
&\geq \frac{\epsilon^v}{pq}K_\sigma^\epsilon\left(s+1, s+p; t+1, t+q\right).
\end{aligned}$$

Taking limit as $p, q \to \infty$ (in any manner) on both sides of the last inequality, we obtain $\delta_{2,\sigma}(K^\epsilon) = 0$, that is, $\sigma\left(\delta_2\right)$-$\lim x = L$.

(ii) Let x be bounded, say $M := \|x\| + L$, and $\sigma\left(\delta_2\right)$-$\lim x = L$. Then for $\epsilon > 0$, we have $\delta_{2,\sigma}(K^\epsilon) = 0$. So for all s, t, we have

$$\begin{aligned}
\frac{1}{pq}\sum_{j=1}^{p}\sum_{k=1}^{q}\left|x_{\sigma^j(s),\sigma^k(t)} - L\right|^v &= \frac{1}{pq}\sum_{\substack{j=\sigma(s)\\ (j,k)\notin K^\epsilon}}^{\sigma^p(s)}\sum_{k=\sigma(t)}^{\sigma^q(t)}\left|x_{jk} - L\right|^v + \frac{1}{pq}\sum_{\substack{j=\sigma(s)\\ (j,k)\in K^\epsilon}}^{\sigma^p(s)}\sum_{k=\sigma(t)}^{\sigma^q(t)}\left|x_{jk} - L\right|^v \\
&\leq \epsilon^v + \frac{1}{pq}\left(\sup_{j,k}\left|x_{jk} - L\right|\right)\max_{s,t}K_\sigma^\epsilon\left(s+1, s+p; t+1, t+q\right) \\
&\leq \epsilon^v + \frac{M}{pq}N^{p,q}.
\end{aligned}$$

Letting $p, q \to \infty$ on both sides of the last inequality, and using the fact that $\delta_{2,\sigma}(K^\epsilon) = 0$, we conclude that $x_{j,k} \to L\left[V_2^\sigma\right]_v$.

Corollary 2.10. $\ell_\infty^2 \cap \sigma\left(\delta_2\right) \subset \left[V_2^\sigma\right] \subset V_2^\sigma$.

Theorem 2.11. *A sequence $x = \left(x_{jk}\right)$ is statistically σ-convergent to L if and only if there exists a set $K = \{(j,k)\} \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_2(K) = 1$ and σ-$\lim_{(j,k)\in K} x_{jk} = L$.*

Proof. Let x be statistically σ-convergent to L and define

$$K_r(\sigma) := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left|t_{jkst}(x) - L\right| \geq \frac{1}{r}\right\} \quad (r = 1, 2, \cdots),$$

and

$$M_r(\sigma) := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left|t_{jkst}(x) - L\right| < \frac{1}{r}\right\}, (r = 1, 2, \cdots).$$

Since x is statistically σ-convergent to L, it is obvious that $\delta_2\left(K_r(\sigma)\right) = 0$. Furthermore

$$M_1(\sigma) \supset M_2(\sigma) \supset \cdots \supset M_i(\sigma) \supset M_{i+1}(\sigma) \supset \cdots \tag{2}$$

and

$$\delta_2\left(M_r(\sigma)\right) = 1 \quad (r = 1, 2, \cdots). \tag{3}$$

Assume now that for $(j,k) \in M_r(\sigma)$, $\left(x_{jk}\right)$ is not σ-convergent to L, i.e. $\left(t_{jkst}(x)\right)$ is not convergent to L uniformly in s,t. Therefore there is $\epsilon > 0$ such that $\left|t_{jkst} - L\right| \geq \epsilon$ for infinitely many terms. Let

$$M_\epsilon(\sigma) := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \ \left|t_{jkst}(x) - L\right| < \epsilon\right\} \text{ and } \epsilon > \frac{1}{r}, (r = 1,2,\cdots).$$

Then

$$\delta_2(M_\epsilon(\sigma)) = 0$$

and by (2) we obtain that $M_\epsilon(\sigma) \supset M_r(\sigma)$. Hence $\delta_2(M_r(\sigma)) = 0$ which contradicts (3). Therefore for $(j,k) \in M_r(\sigma)$, $\left(x_{jk}\right)$ is σ-convergent to L.
Conversely suppose that there exists a subset $K = \{(j,k)\} \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_2(K) = 1$ and $\sigma\text{-}\lim_{(j,k)\in K} x_{jk} = L$, i.e., for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$\left|t_{jkst} - L\right| < \epsilon$$

for all $j,k \geq N$ and for all s,t. From the inclusion

$$K_\epsilon(\sigma) = \left\{(j,k) : \ \left|t_{jkst}(x) - L\right| \geq \epsilon\right\} \subset \mathbb{N}\times\mathbb{N} - \{(j_{N+1},k_{N+1}),(j_{N+2},k_{N+2}),\cdots\}$$

we conclude that $\delta_2(K_\epsilon(\sigma)) \leq 1 - 1 = 0$. Hence x is statistically σ-convergent to L.

Note that statistical case (for single sequences) of the above theorem has been proved by Šalát [31].

3. Application to Korovokin-type approximation type theorem

Throughout this section, we shall let $C(D)$ denote the space of all continuous real valued functions on any compact subset of the real two dimensional space. Then $C(D)$ is a Banach space with the norm $\|.\|_\infty$ defined by

$$\left\|f\right\|_\infty := \sup_{(x,y)\in D} \left|f(x,y)\right|, \ (f \in C(D)).$$

Let L be a linear operator from $C(D)$ into $C(D)$. Then as usual, we say that L is positive linear operator provided that $f \geq 0$ implies $Lf \geq 0$. Also, we denote the value of Lf at a point (x,y) by $L(f(u,v);x,y)$ or only $L(f;x,y)$.

The classical Korovkin approximation theorem (see [15]) was extended from single sequences to double sequences as follows [38].

Theorem 3.1. *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then for all $f \in C(D)$,*

$$\lim_{m,n} \left\|L_{mn}(f) - f\right\|_\infty = 0$$

if and only if

$$\lim_{m,n} \left\|L_{mn}(f_i) - f_i\right\|_\infty = 0, \quad i = 0,1,2,3$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$, $f_3(x,y) = x^2 + y^2$.

After the paper of Gadjiv and Orhan [12], there are several papers published on this theme by using different type of summability methods (c.f. [1]–[5], [13], [14], [17]–[20], [25]–[29], [34]–[36]). One of the approximation theorems is the following due to Dirik and Demirci [9] for functions of two variables by using statistical convergence of double sequences.

Theorem 3.2. *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$st_2 - \lim \left\| L_{mn}(f) - f \right\|_\infty = 0$$

if and only if

$$st_2 - \lim \left\| L_{mn}(f_i) - f_i \right\|_\infty = 0, \quad i = 0, 1, 2, 3$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$.

In this section, we use the notion of statistical σ-convergence of double sequences to prove approximation theorems for functions of two variables. The following is the $\delta_2(\sigma)$-version of Theorem 3.1 and Theorem 3.2, which is followed by an example to show its importance. The case of statistical σ-convergence for single sequences is considered in [8].

Theorem 3.3. *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself and let $\tau_{pqmn}(L_{mn}(f; x, y)) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} L_{\sigma^j(m), \sigma^k(n)}(f; x, y)$. Then for all $f \in C(D)$,*

$$\delta_2(\sigma)\text{-}\lim \left\| L_{mn}(f; x, y) - f(x, y) \right\|_\infty = 0, \quad \textit{i.e.} \tag{4}$$

$$st_2\text{-}\lim \left\| \tau_{pqmn}(L_{mn}(f; x, y)) - f(x, y) \right\|_\infty = 0, \quad \textit{uniformly in } m, n$$

if and only if

$$\delta_2(\sigma)\text{-}\lim \left\| L_{mn}(f_i; x, y) - f_i(x, y) \right\|_\infty = 0, \quad i = 0, 1, 2, 3 \tag{5}$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$.

Proof. Condition (5) follows immediately from condition (4) since each $f_i \in C(D)$, $(i = 0, 1, 2, 3)$. Let us prove the converse. By the continuity of f on compact set D, we can write $\left| f(x, y) \right| \le M$, where $M = \left\| f \right\|_\infty$. Also since $f \in C(D)$, for every $\epsilon > 0$, there is a number $\delta > 0$ such that $\left| f(u, v) - f(x, y) \right| < \epsilon$ for all $(u, v) \in D$ satisfying $|u - x| < \delta$ and $\left| v - y \right| < \delta$. Hence we get

$$\left| f(u, v) - f(x, y) \right| < \epsilon + \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\}. \tag{6}$$

Now, using the linearity and monotonicity of L_{mn}, from (6) we obtain for any $m, n \in$ that

$$\begin{aligned}\left|L_{mn}(f;x,y)-f(x,y)\right| &= L_{mn}(f(u,v)-f(x,y);x,y)-f(x,y)(L_{mn}(f_0;x,y)-f_0(x,y))\\ &\le L_{mn}\left(\left|f(u,v)-f(x,y)\right|;x,y\right)+M\left|L_{mn}(f_0;x,y)-f_0(x,y)\right|\\ &\le L_{mn}\left(\epsilon+\frac{2M}{\delta^2}\left[(u-x)^2+(v-y)^2\right];x,y\right)\\ &+M\left|L_{mn}(f_0;x,y)-f_0(x,y)\right|\\ &\le \epsilon+\left(\epsilon+M+\frac{2M}{\delta^2}\left(E^2+F^2\right)\right)\left|L_{mn}(f_0;x,y)-f_0(x,y)\right|\\ &+\frac{4M}{\delta^2}E\left|L_{mn}(f_1;x,y)-f_1(x,y)\right|+\frac{4M}{\delta^2}F\left|L_{mn}(f_2;x,y)-f_2(x,y)\right|\\ &+\frac{2M}{\delta^2}\left|L_{mn}(f_3;x,y)-f_3(x,y)\right|\end{aligned}$$

where $E := \max|x|$, $F := \max\left|y\right|$. Taking supremum over $(x, y) \in D$ we get

$$\left\|L_{mn}(f;x,y)-f(x,y)\right\|_\infty \le \epsilon + B\sum_{i=0}^{3}\left\|L_{mn}(f_i;x,y)-f_i(x,y)\right\|_\infty$$

where

$$B := \max\left\{\epsilon+M+\frac{2M}{\delta^2}\left(E^2+F^2\right), \frac{2M}{\delta^2}, \frac{4M}{\delta^2}E, \frac{4M}{\delta^2}F\right\}.$$

Similarly, we have

$$\left\|\tau_{pqmn}(L_{mn}(f;x,y))-f(x,y)\right\|_\infty \le \epsilon + B\sum_{i=0}^{3}\left\|\tau_{pqmn}(L_{mn}(f_i;x,y))-f_i(x,y)\right\|_\infty .$$

Now for a given $r > 0$, choose $\epsilon > 0$ such that $\epsilon < r$ and define

$$\begin{aligned}D &:= \left\{(p,q)\in\mathbb{N}^2 : \left\|\tau_{pqmn}(L_{mn}(f;x,y))-f(x,y)\right\|_\infty \ge r\right\}\\ D_i &:= \left\{(p,q)\in\mathbb{N}^2 : \left\|\tau_{pqmn}(L_{mn}(f_i;x,y))-f_i(x,y)\right\|_\infty \ge \frac{r-\epsilon}{4B}\right\}, i = 0,1,2,3.\end{aligned}$$

Then $D \subset \bigcup_{i=0}^{3} D_i$ and so

$$\delta_2(D) \le \sum_{i=0}^{3}\delta_2(D_i).$$

Letting $p, q \to \infty$ in the last inequality and using (5), we obtain (4). This completes the proof of theorem.

In the following example, we construct a double sequence of positive linear operators which satisfies the conditions of Theorem 3.3 but does not satisfy the conditions of Theorem 3.1 and Theorem 3.2.

Example 3.4. *Consider the sequence of Bernstein operators of two variables given by*

$$B_{mn}(f;x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} f\left(\frac{j}{m},\frac{k}{n}\right)\binom{m}{j}\binom{n}{k}x^j(1-x)^{m-j}y^k(1-y)^{n-k},$$

where $0 \le x, y \le 1$ *and* $f \in C\left(I^2\right); I^2 = [0,1] \times [0,1]$. *Let* $P_{mn} : C\left(I^2\right) \to C\left(I^2\right)$ *be defined by*

$$P_{mn}(f;x,y) = (1 + z_{mn})\, B_{mn}(f;x,y) \tag{7}$$

where (z_{mn}) *is defined as in (1). Then, observe that*

$$\begin{aligned}
P_{mn}(f_0;x,y) &= (1+z_{mn})\, f_0(x,y)\\
P_{mn}(f_1;x,y) &= (1+z_{mn})\, f_1(x,y)\\
P_{mn}(f_2;x,y) &= (1+z_{mn})\, f_2(x,y)\\
P_{mn}(f_3;x,y) &= (1+z_{mn})\left(f_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n}\right)
\end{aligned}$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$, $f_3(x,y) = x^2 + y^2$. *Since* $\delta_2(\sigma)\text{-}\lim z = 0$ *(for* $\sigma(i) = i+1$*), we obtain*

$$\delta_2(\sigma)\text{-}\lim \left\|P_{mn}(f_i;x,y) - f_i(x,y)\right\|_\infty = 0,\ i = 0,1,2,3.$$

Hence by Theorem **??**, *we conclude that*

$$\delta_2(\sigma)\text{-}\lim \left\|P_{mn}(f;x,y) - f(x,y)\right\|_\infty = 0$$

for any $f \in C\left(I^2\right)$. *On the other hand, we get* $P_{mn}(f;0,0) = (1+z_{mn})\, f(0,0)$ *and hence*

$$\left\|P_{mn}(f;x,y) - f(x,y)\right\|_\infty \ge \left|P_{mn}(f;0,0) - f(0,0)\right| = z_{mn}\left|f(0,0)\right|.$$

So, we see that Theorem 3.1 and Theorem 3.2 do not work for our operators defined by (7), since (z_{mn}) *is neither P-convergent nor statistically convergent.*

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