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The Set of Filter Cluster Functions

Hüseyin Albayrak[a], Serpil Pehlivan[b]

[a]*Süleyman Demirel University, Faculty of Arts and Sciences, Department of Statistics, Isparta, TURKEY*
[b]*Süleyman Demirel University, Faculty of Arts and Sciences, Department of Mathematics, Isparta, TURKEY*

Abstract. In this work, we are concerned with the concepts of $\mathcal{F}$-α-convergence, $\mathcal{F}$-pointwise convergence and $\mathcal{F}$-uniform convergence for sequences of functions on metric spaces, where $\mathcal{F}$ is a filter on $\mathbb{N}$. We define the concepts of $\mathcal{F}$-limit function, $\mathcal{F}$-cluster function and limit function respectively for each of these three types of convergence, and obtain some results about the sets of $\mathcal{F}$-cluster and $\mathcal{F}$-limit functions for sequences of functions on metric spaces. We use the concept of $\mathcal{F}$-exhaustiveness to characterize the relations between these points.

1. Introduction

The concept of α-convergence which is stronger than the pointwise convergence has been studied by many mathematicians under different names since the beginning of the 20th century (see [9], [13], [20], [23], [36]). In 2008, Gregoriades and Papanastassiou ([20]) defined the concept of exhaustiveness for both families and sequences of functions, and using this notion they gave a generalization of the Ascoli theorem. This notion is close to equicontinuity and describes the relation between the pointwise convergence and the α-convergence. The notion of exhaustiveness enables us to view the convergence of a sequence of functions in terms of properties of the sequence and not of properties of functions as single members. In 2011, Boccuto et al. ([7]) defined the notions of ideal exhaustiveness and $(\mathcal{I}\alpha)$-convergence as generalizations of exhaustiveness and α-convergence for lattice group-valued functions. Later, Caserta and Kočinac ([10]) studied statistical exhaustiveness and statistical α-convergence. In [3], some results were given with respect to ideal exhaustiveness and ideal α-convergence for sequences of functions defined from metric spaces into $\mathbb{R}$. Finally, in [2], we defined the generalizations of α-convergence, pointwise convergence, uniform convergence and exhaustiveness for sequences of functions using the filters on $\mathbb{N}$, and obtained some results related to subsequences of sequences of functions.

The concepts of statistical limit points and statistical cluster points were defined for real sequences for the first time in [17], and studied by many mathematicians (see [12], [14], [25], [29]). The set of statistical cluster points was applied in Turnpike Theory [33]. The concept of the set of statistical cluster points

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Email addresses: huseyinalbayrak@sdu.edu.tr (Hüseyin Albayrak), serpilpehlivan@sdu.edu.tr (Serpil Pehlivan)

which is nonempty and compact was extended to the concept of the Γ-statistical convergence in [34]. Some applications of this convergence were studied on the asymptotic behavior of optimal paths, variational and optimal control problems in discrete time (see [28], [29], [33], [34], [38]). Then Kostyrko et al. ([24]) defined the concepts of $\mathcal{I}$-limit point and $\mathcal{I}$-cluster point which are the generalizations of statistical ones (see also [11], [26]).

This article extends the concepts of statistical cluster (or limit) points for sequences of real numbers given in [17] to the concepts of $\mathcal{F}$-cluster (or $\mathcal{F}$-limit) functions for sequences of functions. We define the concepts of limit function, $\mathcal{F}$-limit function and $\mathcal{F}$-cluster function for pointwise convergence, uniform convergence and α-convergence. From these concepts, we define nine sets which consist of the same type functions, and examine the properties of these sets and the relations between them. We show that the inclusions between the sets of limit functions, $\mathcal{F}$-limit functions and $\mathcal{F}$-cluster functions are similar to the ones considered for real sequences. Similars of relations among $\mathcal{F}$-pointwise convergence, $\mathcal{F}$-uniform convergence and $\mathcal{F}$-α-convergence is obtained for $\mathcal{F}$-cluster functions (or $\mathcal{F}$-limit functions) of these concepts too.

Let us start by giving some basic concepts. $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a *filter* if $\mathcal{F}$ is closed under taking supersets and finite intersections (see [15], [37]). A filter is said to be *free* if the intersection of all its members is empty, and *fixed* otherwise. If $\mathcal{F}$ is a filter on $\mathbb{N}$, then the set $\mathcal{I}(\mathcal{F}) = \{\mathbb{N} \setminus A : A \in \mathcal{F}\}$ is an *ideal* on $\mathbb{N}$; and conversely, if $\mathcal{I}$ is an ideal on $\mathbb{N}$, then the set $\mathcal{F}(\mathcal{I}) = \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$ is a filter on $\mathbb{N}$. Filter and ideal are dual concepts. So, the notions defined by an ideal are equivalent to the ones defined by a filter. For example, the concepts of $\mathcal{F}$-convergence and $\mathcal{I}$-convergence are equivalent.

Let $\mathcal{F}$ be a filter. A subset A of $\mathbb{N}$ is called *$\mathcal{F}$-stationary* if it has nonempty intersection with each member of the filter $\mathcal{F}$. We denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^+$. In brief, for an $A \subseteq \mathbb{N}$ we have

$$A \in \mathcal{F}^+ \Longleftrightarrow A \notin \mathcal{I}(\mathcal{F}),$$

where $\mathcal{I}(\mathcal{F})$ is the ideal corresponding to $\mathcal{F}$. Let us give some properties of $\mathcal{F}^+$ which will be used constantly in our proofs:

(1) $\mathcal{F}^+ \supseteq \mathcal{F}$ for every filter $\mathcal{F}$. $\mathcal{F}^+ = \mathcal{F}$ holds if and only if $\mathcal{F}$ is an ultrafilter.
(2) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Longrightarrow \mathcal{F}_2^+ \subseteq \mathcal{F}_1^+$.
(3) If $A \in \mathcal{F}^+$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}^+$.
(4) If $A, B \in \mathcal{F}^+$ then $A \cup B \in \mathcal{F}^+$.
(5) If $A \in \mathcal{F}^+$ and $A \subseteq B$ then $B \in \mathcal{F}^+$.
(6) Let $\mathcal{F}$ be free. If $A \in \mathcal{F}^+$ and B is a finite set, then $A \setminus B \in \mathcal{F}^+$.

In this work, the symbol $|.|$ denotes either the cardinality for sets or the absolute value for real numbers. The symbol $\lfloor . \rfloor$ denotes the greatest integer function.

A filter $\mathcal{F}$ is said to be a *P-filter*, if for every sequence $(K_n)_{n\in\mathbb{N}}$ of the sets in $\mathcal{F}$ there exists a $K \in \mathcal{F}$ such that $|K \setminus K_n| < \infty$ for each $n \in \mathbb{N}$ (see [5], [6]). P-filters are duals of P-ideals.

A filter $\mathcal{F}$ is said to be a *weak P-filter*, if for every sequence $(K_n)_{n\in\mathbb{N}}$ of the sets in $\mathcal{F}$ there exists a $K \in \mathcal{F}^+$ such that $|K \setminus K_n| < \infty$ for each $n \in \mathbb{N}$ (see [19]). It is clear that a P-filter is also a weak P-filter.

The filter convergence is a well-known concept in mathematics. The readers can see [4, 21, 22] for details.

Definition 1.1. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, ρ_X) is said to be $\mathcal{F}$-convergent to $\xi \in X$ if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : \rho_X(x_n, \xi) < \varepsilon\}$ belongs to $\mathcal{F}$. In this case, we write $\mathcal{F} - \lim x_n = \xi$ or $x_n \xrightarrow{\mathcal{F}} \xi$.*

Now we present some examples of filters.

1. **Fréchet Filter.** The family $\mathcal{F}_r = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$ is called the *Fréchet filter*. $\mathcal{F}_r$ is the minimum free filter with respect to the inclusion relation. Therefore, we can characterize free filters as follows: If $\mathcal{F} \supseteq \mathcal{F}_r$ then $\mathcal{F}$ is a free filter. $\mathcal{F}_r$-convergence coincides with the ordinary convergence. The family of $\mathcal{F}_r$-stationary sets is denoted by

$$\mathcal{F}_r^+ = \{A \subseteq \mathbb{N} : A \text{ is infinite}\}.$$

2. **Statistical Convergence Filter.** $\underline{d}(A) = \liminf_{n\to\infty} (|A \cap [1,n]|)/n$ and $\overline{d}(A) = \limsup_{n\to\infty} (|A \cap [1,n]|)/n$ are called the *lower asymptotic density* and *upper asymptotic density* of the set A, respectively. If $\underline{d}(A) = \overline{d}(A)$, that is, $\lim_{n\to\infty} (|A \cap [1,n]|)/n$ exists, then the value of this limit is called the *asymptotic density* of the set A, and it is denoted by $d(A)$ (see [8], [31]). The family $\mathcal{F}_{st} = \{A \subseteq \mathbb{N} : d(A) = 1\}$ is a free P-filter, and it is called the *statistical convergence filter*. $\mathcal{F}_{st}$-convergence is called the *statistical convergence* (see [14], [16], [30]). Note that

$$\mathcal{F}_{st}^+ = \{A \subseteq \mathbb{N} : d(A) > 0 \text{ or } d(A) \text{ does not exist}\} = \left\{A \subseteq \mathbb{N} : \overline{d}(A) > 0\right\}.$$

3. Let us consider the Euler function φ defined by

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_m}\right)$$

for $1 < n \in \mathbb{N}$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ is the prime number decomposition of n, and $\varphi(1) = 1$ ([32]). Then

$$d_\varphi(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{d|n} \varphi(d)\, \chi_A(d)$$

is called the *φ-density* of the set A, provided that this limit exists ([26]; see also [1], [27]). If $d_\varphi(A)$ exists, then $d_\varphi(A) \in \{0,1\}$. The family $\mathcal{F}_\varphi = \left\{A \subseteq \mathbb{N} : d_\varphi(A) = 1\right\}$ is a free filter. Hence we have

$$\mathcal{F}_\varphi^+ = \left\{A \subseteq \mathbb{N} : d_\varphi(A) = 1 \text{ or } d_\varphi(A) \text{ does not exist}\right\}.$$

Using the results in [35], E. Kováč ([27]) showed that if a set A has φ-density then it also has asymptotic density, and $d_\varphi(A) = d(A)$. So he argues that $\mathcal{I}_\varphi \subset \mathcal{I}_d$ for the dual ideals of $\mathcal{F}_\varphi$ and $\mathcal{F}_{st}$, respectively. This inclusion is strict since $d(\mathbb{P}) = 0$, but $d_\varphi(\mathbb{P})$ does not exist, where $\mathbb{P}$ is the set of all prime numbers (see [27], [35]). From these results we can say that $\mathcal{F}_\varphi \subset \mathcal{F}_{st}$.

The analogue of Lemma 1.2 occurs from the results given for the ideal in [5] and [24]. Lemma 1.2 was given for ultrafilters in [18]. In this lemma, the property "P-filter" is used for the necessity, the property "free" for the sufficiency. The proof of the necessity of Lemma 1.2 and the proof of Lemma 1.3 are almost same. We only give the proof of Lemma 1.3.

Lemma 1.2. *Let (X, ρ_X) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and $\xi \in X$. Let $\mathcal{F}$ be a free P-filter on $\mathbb{N}$. Then $\mathcal{F} - \lim x_n = \xi$ if and only if there is a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}$ such that $\lim_{k\to\infty} x_{n_k} = \xi$.*

Lemma 1.3. *Let (X, ρ_X) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and $\xi \in X$. Let $\mathcal{F}$ be a weak P-filter on $\mathbb{N}$. If $\mathcal{F} - \lim x_n = \xi$ then there is a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}^+$ such that $\lim_{k\to\infty} x_{n_k} = \xi$.*

Proof. Let us assume that $\mathcal{F}-\lim x_n = \xi$. Then we have

$$K_t = \left\{ n \in \mathbb{N} : \rho_X (x_n, \xi) < \frac{1}{t} \right\} \in \mathcal{F}$$

where $t \in \mathbb{N}$. Since $\mathcal{F}$ is a weak P-filter there is a set $K = \{n_1 < ... < n_k < ...\} \in \mathcal{F}^+$ such that $|K \setminus K_t| < \infty$ for each $t \in \mathbb{N}$. Let $\varepsilon > 0$. There is a $t_0 \in \mathbb{N}$ such that $\frac{1}{t_0} < \varepsilon$. Then we get

$$K_{t_0} \subseteq \{n \in \mathbb{N} : \rho_X (x_n, \xi) < \varepsilon\},$$

and so

$$K \setminus K_{t_0} \supseteq K \setminus \{n \in \mathbb{N} : \rho_X (x_n, \xi) < \varepsilon\}.$$

Since the set on the left-hand side is finite, the one on the right-hand side is also finite. Therefore, there is a $k(\varepsilon) \in \mathbb{N}$ such that for every $k \geq k(\varepsilon)$ we have $\rho_X (x_{n_k}, \xi) < \varepsilon$. So, $\lim_{k\to\infty} x_{n_k} = \xi$ holds. □

Now we introduce the concepts of $\mathcal{F}$-α-convergence, $\mathcal{F}$-pointwise convergence, $\mathcal{F}$-uniform convergence and $\mathcal{F}$-exhaustiveness which were defined in [2]. By $S(\xi, \delta)$, we denote the open ball with center ξ and radius δ.

Definition 1.4. *Given $D \subseteq (X, \rho_X)$, let $f, f_n : D \longrightarrow (Y, \rho_Y)$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $\xi \in D$. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$-exhaustive at the point ξ provided that, for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\{n \in \mathbb{N} : \rho_Y (f_n(\eta), f_n(\xi)) < \varepsilon \text{ for all } \eta \in S(\xi, \delta)\} \in \mathcal{F}.$$

Definition 1.5. *Given $D \subseteq (X, \rho_X)$, let $f, f_n : D \longrightarrow (Y, \rho_Y)$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $\xi \in D$. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$-α-convergent to f at the point ξ if for every sequence $(x_n)_{n\in\mathbb{N}}$ which is $\mathcal{F}$-convergent to ξ, the sequence $(f_n(x_n))_{n\in\mathbb{N}}$ is $\mathcal{F}$-convergent to $f(\xi)$ (i.e., $\mathcal{F}-\lim f_n(x_n) = f(\xi)$), and we write $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ (at ξ).*

Definition 1.6. *Given $D \subseteq (X, \rho_X)$, let $f, f_n : D \longrightarrow (Y, \rho_Y)$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$-pointwise convergent to f on D if $\mathcal{F}-\lim f_n(\xi) = f(\xi)$ for each $\xi \in D$, i.e.,*

$$\{n \in \mathbb{N} : \rho_Y (f_n(\xi), f(\xi)) < \varepsilon\} \in \mathcal{F}$$

for every $\varepsilon > 0$. In this case, we write $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$ (on D).

Definition 1.7. *Given $D \subseteq (X, \rho_X)$, let $f, f_n : D \longrightarrow (Y, \rho_Y)$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$-uniformly convergent to f on D provided that*

$$\{n \in \mathbb{N} : \rho_Y (f_n(\xi), f(\xi)) < \varepsilon \text{ for all } \xi \in D\} \in \mathcal{F}$$

for every $\varepsilon > 0$. In this case, we write $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ (on D).

2. $\mathcal{F}$-Limit Functions and $\mathcal{F}$-Cluster Functions

In this section, (X, ρ_X) and (Y, ρ_Y) denote two metric spaces. Let $D \subseteq X$. Then $C(D, Y)$ denotes the family of all continuous functions from D into Y. We assume that f and f_n's are functions from D to Y.

Now we will define the concepts of $\mathcal{F}$-limit function, $\mathcal{F}$-cluster function and limit function respectively for each type of convergence mentioned above.

Definition 2.1. *A function f is said to be an $\mathcal{F}$-uniform limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ if there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^+$ such that the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent to f on D. We denote the set of all $\mathcal{F}$-uniform limit functions of $(f_n)_{n\in\mathbb{N}}$ by $\Lambda^u_{f_n}(\mathcal{F})$.*

Definition 2.2. *A function f is said to be an $\mathcal{F}$-uniform cluster function of the sequence $(f_n)_{n\in\mathbb{N}}$ if*

$$\{n \in \mathbb{N} : \rho_Y (f_n(\xi), f(\xi)) < \varepsilon \text{ for all } \xi \in D\} \in \mathcal{F}^+$$

for every $\varepsilon > 0$. We denote the set of all $\mathcal{F}$-uniform cluster functions of $(f_n)_{n\in\mathbb{N}}$ by $\Gamma^u_{f_n}(\mathcal{F})$.

For the filter $\mathcal{F}_r$ both definitions are equivalent to each other.

Definition 2.3. *A function f is said to be a uniform limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ if there is an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\}$ such that the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent to f on D. We denote the set of all uniform limit functions of $(f_n)_{n\in\mathbb{N}}$ by $L^u_{f_n}$.*

We will show in the following lemma that the concepts of $\mathcal{F}_r$-uniform limit function, $\mathcal{F}_r$-uniform cluster function and uniform limit function are all equivalent.

Lemma 2.4. *For the Fréchet filter $\mathcal{F}_r$, we have $\Lambda^u_{f_n}(\mathcal{F}_r) = \Gamma^u_{f_n}(\mathcal{F}_r) = L^u_{f_n}$.*

Proof. Since $\mathcal{F}_r^+$ consists of infinite sets, it is easy to see that the concepts of $\mathcal{F}_r$-uniform limit function and uniform limit function are equivalent. Now we only need to show that $\Lambda^u_{f_n}(\mathcal{F}_r) = \Gamma^u_{f_n}(\mathcal{F}_r)$. Assume that $f \in \Lambda^u_{f_n}(\mathcal{F}_r)$. Hence we can find a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}_r^+$ such that the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent to f on D. Take $\varepsilon > 0$. Then there is a $k(\varepsilon) \in \mathbb{N}$ such that for every $k \geq k(\varepsilon)$ and every $\xi \in D$ we have $\rho_Y (f_{n_k}(\xi), f(\xi)) < \varepsilon$. Therefore, we get

$$K \setminus \{1, 2, ..., k(\varepsilon)\} \subseteq \{n \in \mathbb{N} : \rho_Y (f_n(\xi), f(\xi)) < \varepsilon \text{ for all } \xi \in D\} \in \mathcal{F}_r^+,$$

and so $f \in \Gamma^u_{f_n}(\mathcal{F}_r)$.
Now, let $f \in \Gamma^u_{f_n}(\mathcal{F}_r)$. Thus for every $\varepsilon > 0$ the sets

$$K_\varepsilon := \{n \in \mathbb{N} : \rho_Y (f_n(\xi), f(\xi)) < \varepsilon \text{ for all } \xi \in D\}$$

belong to $\mathcal{F}_r^+$, i.e., they are infinite. Then we can choose an infinite set K such that $K \setminus K_\varepsilon$ is finite for each $\varepsilon > 0$. This set K is the desired set in the definitions of $\mathcal{F}$-limit function or limit function. Consequently, we get $f \in \Lambda^u_{f_n}(\mathcal{F}_r)$. $\square$

The above result also holds for pointwise limit functions and α-limit functions which we define as follows.

Definition 2.5. *A function f is said to be an $\mathcal{F}$-pointwise limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ if for each finite set $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^+$ such that $\lim_{k\to\infty} f_{n_k}(\xi_i) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$. We denote the set of all $\mathcal{F}$-pointwise limit functions of $(f_n)_{n\in\mathbb{N}}$ by $\Lambda^{pw}_{f_n}(\mathcal{F})$.*

Definition 2.6. *A function f is said to be an $\mathcal{F}$-pointwise cluster function of the sequence $(f_n)_{n\in\mathbb{N}}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each $\varepsilon > 0$ we have*

$$\{n \in \mathbb{N} : \rho_Y (f_n(\xi_i), f(\xi_i)) < \varepsilon \text{ for all } i \in \{1, 2, ..., m\}\} \in \mathcal{F}^+.$$

We denote the set of all $\mathcal{F}$-pointwise cluster functions of $(f_n)_{n\in\mathbb{N}}$ by $\Gamma^{pw}_{f_n}(\mathcal{F})$.

Definition 2.7. *A function f is said to be a pointwise limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ there is an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\}$ such that $\lim_{k\to\infty} f_{n_k}(\xi_i) = f(\xi_i)$ for every $i \in \{1, ..., m\}$. We denote the set of all pointwise limit functions of $(f_n)_{n\in\mathbb{N}}$ by $L^{pw}_{f_n}$.*

Definition 2.8. *A function f is said to be an $\mathcal{F}$-α-limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each sequence $(x_{i,n})_{n\in\mathbb{N}}$ in D such that $\mathcal{F}-\lim x_{i,n} = \xi_i$ ($i \in \{1, ..., m\}$) there is a set $K = \{n_1 < ... < n_k < ...\} \in \mathcal{F}^+$ such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$. We denote the set of all $\mathcal{F}$-α-limit functions of $(f_n)_{n\in\mathbb{N}}$ by $\Lambda^{\alpha}_{f_n}(\mathcal{F})$.*

Definition 2.9. *A function f is said to be an $\mathcal{F}$-α-cluster function of the sequence $(f_n)_{n\in\mathbb{N}}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each sequence $(x_{i,n})_{n\in\mathbb{N}}$ in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ ($i \in \{1, 2, ..., m\}$) and each $\varepsilon > 0$ we have*

$$\{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon \text{ for all } i \in \{1, 2, ..., m\}\} \in \mathcal{F}^+.$$

We denote the set of all $\mathcal{F}$-α-cluster functions of $(f_n)_{n\in\mathbb{N}}$ by $\Gamma^{\alpha}_{f_n}(\mathcal{F})$.

Definition 2.10. *A function f is said to be an α-limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each sequence $(x_{i,n})_{n\in\mathbb{N}}$ in D such that $\lim x_{i,n} = \xi_i$ ($i \in \{1, 2, ..., m\}$) there is an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\}$ such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$. We denote the set of all α-limit functions of $(f_n)_{n\in\mathbb{N}}$ by $L^{\alpha}_{f_n}$.*

Now we give two examples related to $\mathcal{F}$-limit functions. The second example shows the difference between $\mathcal{F}$-pointwise limit function, $\mathcal{F}$-uniform limit function and $\mathcal{F}$-α-limit function.

Example 2.11. *Let the sequence $(f_n)_{n\in\mathbb{N}}$ be defined by*

$$f_n(\xi) = \begin{cases} \cos\left(\left\lfloor \frac{n+1}{2} \right\rfloor \pi\right) . |\xi|^{\frac{n-1}{n}} & \text{if } \xi \geq 0 \\ \sin\left(\left\lfloor \frac{n}{2} \right\rfloor \pi + \frac{\pi}{2}\right) . |\xi|^{\frac{n-1}{n}} & \text{if } \xi < 0 \end{cases}$$

where $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ for all $n \in \mathbb{N}$. We will find the statistical limit functions of $(f_n)_{n\in\mathbb{N}}$.
For the sets $K_1 := \{n \in \mathbb{N} : n \equiv 1 \pmod 4\}$, $K_2 := \{n \in \mathbb{N} : n \equiv 2 \pmod 4\}$, $K_3 := \{n \in \mathbb{N} : n \equiv 3 \pmod 4\}$ and $K_4 := \{n \in \mathbb{N} : n \equiv 0 \pmod 4\}$ we have

$$\begin{array}{ll} \lim\limits_{\substack{n\to\infty \\ n\in K_1}} f_n(\xi) = -\xi := f(\xi), & \lim\limits_{\substack{n\to\infty \\ n\in K_2}} f_n(\xi) = -|\xi| := g(\xi), \\ \lim\limits_{\substack{n\to\infty \\ n\in K_3}} f_n(\xi) = \xi := h(\xi), & \lim\limits_{\substack{n\to\infty \\ n\in K_4}} f_n(\xi) = |\xi| := l(\xi). \end{array}$$

Since $d(K_1) = d(K_2) = d(K_3) = d(K_4) = \frac{1}{4}$ (i.e. $K_1, K_2, K_3, K_4 \in \mathcal{F}^+_{st}$) we get $\Lambda^{pw}_{f_n}(\mathcal{F}_{st}) = \{f, g, h, l\}$. Since the uniform convergence does not hold here, we have $\Lambda^{u}_{f_n}(\mathcal{F}_{st}) = \emptyset$.
Let $\xi_1, \xi_2, ..., \xi_m \in \mathbb{R}$ and $(x_{i,n})_{n\in\mathbb{N}}$ be sequences such that $\mathcal{F}_{st} - \lim x_{i,n} = \xi_i$ for every $i \in \{1, ..., m\}$. Then there exist some sets $L_i \in \mathcal{F}$ such that $\lim_{n\to\infty, n\in L_i} x_{i,n} = \xi_i$ for every $i \in \{1, 2, ..., m\}$. Let $L := \bigcup_{i=1}^{m} L_i \in \mathcal{F}$. Then $K_1 \cap L \in \mathcal{F}^+$, and we obtain

$$\lim_{\substack{n\to\infty \\ n\in K_1\cap L}} f_n(x_{i,n}) = -\xi_i = f(\xi_i)$$

for each $i \in \{1, 2, ..., m\}$. Since the set $\{\xi_1, \xi_2, ..., \xi_m\}$ and the sequences $(x_{i,n})_{n\in\mathbb{N}}$ are arbitrary, $f \in \Lambda^{\alpha}_{f_n}(\mathcal{F}_{st})$ holds. Similarly, we get $g, h, l \in \Lambda^{\alpha}_{f_n}(\mathcal{F}_{st})$. □

Example 2.12. *Let us consider the functions* $f_n : [0,1] \longrightarrow \mathbb{R}$*,* $n \in \mathbb{N}$ *defined by*

$$f_n(\xi) = \begin{cases} \dfrac{\xi}{n} & \text{if } n \in \{1,5,...,4k-3,...\} \\ \xi^n & \text{if } n \in \{3,7,...,4k-1,...\} \\ 0 & \text{if } n \in \{2,4,...,2k,...\} \text{ and } \xi < \frac{1}{2} \\ \dfrac{n+1}{2n} & \text{if } n \in \{2,4,...,2k,...\} \text{ and } \xi \geq \frac{1}{2} \end{cases}.$$

Let $K_1 := \{1,5,...,4k-3,...\}$, $K_2 := \{3,7,...,4k-1,...\}$ *and* $K_3 := \{2,4,...,2k,...\}$*. Since* $d(K_1) = \frac{1}{4}, d(K_2) = \frac{1}{4}$ *and* $d(K_3) = \frac{1}{2}$ *we have* $K_1, K_2, K_3 \in \mathcal{F}_{st}^+$*. Then we get* $\Lambda_{f_n}^{pw}(\mathcal{F}_{st}) = \{f,g,h\}$, $\Lambda_{f_n}^{u}(\mathcal{F}_{st}) = \{f,h\}$ *and* $\Lambda_{f_n}^{\alpha}(\mathcal{F}_{st}) = \{f\}$*, where* $f,g,h : [0,1] \longrightarrow \mathbb{R}$ *and*

$$f(\xi) = 0, \; g(\xi) = \begin{cases} 0 & \text{if } \xi < 1 \\ 1 & \text{if } \xi = 1 \end{cases}, \; h(\xi) = \begin{cases} 0 & \text{if } \xi < 1/2 \\ \dfrac{1}{2} & \text{if } \xi \geq 1/2 \end{cases}.$$

We will only show that $g, h \notin \Lambda_{f_n}^{\alpha}(\mathcal{F}_{st})$*. Let* $\xi_1 = 1$ *and* $(x_{1,n})_{n\in\mathbb{N}} = (1-1/2n)_{n\in\mathbb{N}}$*. Then* $\mathcal{F}_{st} - \lim x_{1,n} = \xi_1$*, and we obtain*

$$f_n(x_{1,n}) = \begin{cases} \dfrac{2n-1}{2n^2} & \text{if } n \in K_1 \\ \left(1 - \dfrac{1}{2n}\right)^n & \text{if } n \in K_2 \\ \dfrac{n+1}{2n} & \text{if } n \in K_3 \end{cases} \text{ and } \lim_{n\to\infty} f_n(x_{1,n}) = \begin{cases} 0 & \text{if } n \in K_1 \\ e^{-\frac{1}{2}} & \text{if } n \in K_2 \\ \dfrac{1}{2} & \text{if } n \in K_3 \end{cases}.$$

Therefore there is not any $K \in \mathcal{F}_{st}^+$ *such that* $\lim_{n\to\infty, n\in K} f_n(x_{1,n}) = 1 = g(\xi_1)$*. Similarly, for the point* $\xi_2 = \frac{1}{2}$ *and the sequence* $(x_{2,n}) = \left(\frac{1}{2} - \frac{1}{2n}\right)_{n\in\mathbb{N}}$*, we get* $\mathcal{F}_{st} - \lim x_{2,n} = \xi_2$ *and*

$$f_n(x_{2,n}) = \begin{cases} \dfrac{n-1}{2n^2} & \text{if } n \in K_1 \\ \left(\dfrac{1}{2} - \dfrac{1}{2n}\right)^n & \text{if } n \in K_2 \\ 0 & \text{if } n \in K_3 \end{cases} \text{ and } \lim_{n\to\infty} f_n(x_{2,n}) = 0.$$

Hence, we can not find any $K \in \mathcal{F}_{st}^+$ *such that* $\lim_{n\to\infty, n\in K} f_n(x_{2,n}) = \frac{1}{2} = h(\xi_2)$*.* □

For a sequence of real numbers, $\mathcal{F}$-convergence implies that the $\mathcal{F}$-limit point and the $\mathcal{F}$-cluster point of the sequence are unique, respectively. We generalize this result to sequences of functions in the next theorem. This theorem and the latter show that the notions which we have given above are well-defined.
In the proof of the following theorem, it is enough that $\mathcal{F}$ is only free for $f \in \Gamma_{f_n}(\mathcal{F})$.

Theorem 2.13. *Let* (X, ρ_X) *and* (Y, ρ_Y) *be two metric spaces, and* $D \subseteq X$*. Let* $f, f_n : D \longrightarrow Y$ *(*$n \in \mathbb{N}$*), and* $\mathcal{F}$ *be a free weak P-filter on* $\mathbb{N}$*. Then the following holds:*

(i) If $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ on D then $\Lambda_{f_n}^{\alpha}(\mathcal{F}) = \Gamma_{f_n}^{\alpha}(\mathcal{F}) = \{f\}$.

(ii) If $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ on D then $\Lambda_{f_n}^{u}(\mathcal{F}) = \Gamma_{f_n}^{u}(\mathcal{F}) = \{f\}$.

(iii) If $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$ on D then $\Lambda_{f_n}^{pw}(\mathcal{F}) = \Gamma_{f_n}^{pw}(\mathcal{F}) = \{f\}$.

Proof. We will prove only (i) and (ii). The proof of (iii) is similar.

(i) Let us assume that $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ on D. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and $(x_{i,n})_{n\in\mathbb{N}}$ be sequences in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ for each $i \in \{1, 2, ..., m\}$. Then, for each $i \in \{1, 2, ..., m\}$ we have $\mathcal{F} - \lim f_n(x_{i,n}) = f(\xi_i)$, and so

$$K_{i,\varepsilon} := \{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon\} \in \mathcal{F}$$

for every $\varepsilon > 0$. Since $\bigcap_{i=1}^m K_{i,\varepsilon} \in \mathcal{F} \subseteq \mathcal{F}^+$, we get

$$\{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon \text{ for all } i \in \{1, 2, ..., m\}\} \in \mathcal{F}^+$$

for every $\varepsilon > 0$. Hence $f \in \Gamma^{\alpha}_{f_n}(\mathcal{F})$ holds.

Let $K_t := \{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < 1/t \text{ for all } i \in \{1, 2, ..., m\}\} \in \mathcal{F}$, $t \in \mathbb{N}$. Since $\mathcal{F}$ is a weak P-filter, we can find a set $K = \{n_1 < n_2 < ...n_k < ...\} \in \mathcal{F}^+$ such that $|K \setminus K_t| < \infty$ for every $t \in \mathbb{N}$. Fix $i \in \{1, 2, ..., m\}$ and $\varepsilon > 0$. Then there exist $t_0 \in \mathbb{N}$ with $1/t_0 < \varepsilon$ and $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $n_k \in K_{t_0} \subseteq K_{i,\varepsilon}$ for every $k \geq k_0$. Hence for every $k \geq k_0$ we have $\rho_Y(f_{n_k}(x_{i,n_k}), f(\xi_i)) < \varepsilon$. Then we obtain $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$. Consequently, $f \in \Lambda^{\alpha}_{f_n}(\mathcal{F})$ holds.

Now we prove that if $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ on D then $\Gamma^{\alpha}_{f_n}(\mathcal{F})$ is equal to the singleton set $\{f\}$. Let us assume that $g \in \Gamma^{\alpha}_{f_n}(\mathcal{F})$ where $g : D \longrightarrow Y$ and $g \neq f$. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$, $(x_{i,n})_{n\in\mathbb{N}}$ be sequences in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ for each $i \in \{1, 2, ..., m\}$, and $\varepsilon > 0$. Then we have

$$M_\varepsilon = \{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), g(\xi_i)) < \varepsilon/2 \text{ for all } i \in \{1, 2, ..., m\}\} \in \mathcal{F}^+.$$

Since $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ on D, we get

$$K_\varepsilon = \{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon/2 \text{ for all } i \in \{1, 2, ..., m\}\} \in \mathcal{F}.$$

Then $M_\varepsilon \cap K_\varepsilon \in \mathcal{F}^+$, we obtain

$$\rho_Y(f(\xi_i), g(\xi_i)) \leq \rho_Y(f(\xi_i), f_{n_0}(x_{i,n_0})) + \rho_Y(f_{n_0}(x_{i,n_0}), g(\xi_i)) < \varepsilon$$

for an $n_0 \in M_\varepsilon \cap K_\varepsilon$ and all $i \in \{1, 2, ..., m\}$. Hence we get $f(\xi_i) = g(\xi_i)$ for $i \in \{1, 2, ..., m\}$. Since the points $\xi_1, \xi_2, ..., \xi_m$ are arbitrary in D, we get $f = g$ on D.

(ii) Let $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ on D. Then we have

$$K_\varepsilon := \{n \in \mathbb{N} : \rho_Y(f_n(\xi), f(\xi)) < \varepsilon \text{ for all } \xi \in D\} \in \mathcal{F} \subseteq \mathcal{F}^+$$

for every $\varepsilon > 0$. Hence $f \in \Gamma^{u}_{f_n}(\mathcal{F})$ holds.

Let $K_t := \{n \in \mathbb{N} : \rho_Y(f_n(\xi), f(\xi)) < 1/t \text{ for all } \xi \in D\} \in \mathcal{F}$, $t \in \mathbb{N}$. Since $\mathcal{F}$ is a weak P-filter, we can find a set $K = \{n_1 < n_2 < ...n_k < ...\} \in \mathcal{F}^+$ such that $|K \setminus K_t| < \infty$ for every $t \in \mathbb{N}$. Take $\varepsilon > 0$. Then there exist $t_0 \in \mathbb{N}$ with $1/t_0 < \varepsilon$ and $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $n_k \in K_{t_0} \subseteq K_\varepsilon$ for every $k \geq k_0$. Therefore, for every $k \geq k_0$ and every $\xi \in D$ we have $\rho_Y(f_{n_k}(\xi), f(\xi)) < \varepsilon$. Consequently, the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent to f on D and so $f \in \Lambda^{u}_{f_n}(\mathcal{F})$.

Now we will show that the $\mathcal{F}$-uniform limit function is unique. Let us assume that $g \in \Lambda^{u}_{f_n}(\mathcal{F})$ where $g : D \longrightarrow Y$ and $g \neq f$. Then we can find a set $M \in \mathcal{F}^+$ such that the subsequence $(f_n)_{n\in M}$ is uniformly convergent to g on D. Let $\varepsilon > 0$. Then there is an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \in M_\varepsilon$ and every $\xi \in D$ we have

$$\rho_Y(f_n(\xi), g(\xi)) < \varepsilon/2,$$

where $M_\varepsilon = M \setminus \{1, 2, ..., n(\varepsilon)\} \in \mathcal{F}^+$. Since $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ on D, for every $n \in K_\varepsilon \in \mathcal{F}$ and every $\xi \in D$ we get

$$\rho_Y(f_n(\xi), f(\xi)) < \varepsilon/2.$$

Then $M_\varepsilon \cap K_\varepsilon \in \mathcal{F}^+$ since $K_\varepsilon \in \mathcal{F}$ and $M_\varepsilon \in \mathcal{F}^+$. Let $n_0 \in M_\varepsilon \cap K_\varepsilon$. So we obtain

$$\rho_Y(f(\xi), g(\xi)) \leq \rho_Y(f(\xi), f_{n_0}(\xi)) + \rho_Y(f_{n_0}(\xi), g(\xi)) < \varepsilon$$

for every $\xi \in D$. Consequently, we get $f = g$ on D.
It can be similarly shown that if $f_n \xrightarrow{\mathcal{F}-u} f$ on D then the $\mathcal{F}$-uniform cluster function f is unique. $\square$

Theorem 2.14. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, $D \subseteq X$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filters on $\mathbb{N}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Then the following holds:*

- **(i)** $\Lambda^\alpha_{f_n}(\mathcal{F}_2) \subseteq \Lambda^\alpha_{f_n}(\mathcal{F}_1)$ and $\Gamma^\alpha_{f_n}(\mathcal{F}_2) \subseteq \Gamma^\alpha_{f_n}(\mathcal{F}_1)$.
- **(ii)** $\Lambda^u_{f_n}(\mathcal{F}_2) \subseteq \Lambda^u_{f_n}(\mathcal{F}_1)$ and $\Gamma^u_{f_n}(\mathcal{F}_2) \subseteq \Gamma^u_{f_n}(\mathcal{F}_1)$.
- **(iii)** $\Lambda^{pw}_{f_n}(\mathcal{F}_2) \subseteq \Lambda^{pw}_{f_n}(\mathcal{F}_1)$ and $\Gamma^{pw}_{f_n}(\mathcal{F}_2) \subseteq \Gamma^{pw}_{f_n}(\mathcal{F}_1)$.

Proof. The items (ii) and (iii) can be simply proved due to the inclusion $\mathcal{F}_2^+ \subseteq \mathcal{F}_1^+$. We only prove the first part of (i). The second part can be proved similarly. Assume that $f \in \Lambda^\alpha_{f_n}(\mathcal{F}_2)$. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and $(x_{i,n})_{n\in\mathbb{N}}$ be sequences in D such that $\mathcal{F}_1 - \lim x_{i,n} = \xi_i$ $(i \in \{1, 2, ..., m\})$. For each $i \in \{1, 2, ..., m\}$ we get $\mathcal{F}_2 - \lim x_{i,n} = \xi_i$ from $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since $f \in \Lambda^\alpha_{f_n}(\mathcal{F}_2)$, there is a set

$$K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}_2^+ \subseteq \mathcal{F}_1^+$$

such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$. Therefore, $f \in \Lambda^\alpha_{f_n}(\mathcal{F}_1)$ is obtained. $\square$

Using the filters $\mathcal{F}_{st}$ and $\mathcal{F}_\varphi$ we give an example for the above theorem.

Example 2.15. *Let $f_n : [0, 1) \longrightarrow \mathbb{R}, n \in \mathbb{N}$ be defined by*

$$f_n(\xi) = \begin{cases} 1 & \text{if } n \in \mathbb{P} \\ \xi^n & \text{if } n \in \mathbb{N} \setminus \mathbb{P} \end{cases},$$

where $\mathbb{P}$ is the set of all prime numbers. Since $d(\mathbb{P}) = 0$, we have $d(\mathbb{N} \setminus \mathbb{P}) = 1$ and so $\mathbb{N} \setminus \mathbb{P} \in \mathcal{F}_{st}$. Also, neither $\mathbb{P}$ nor $\mathbb{N} \setminus \mathbb{P}$ does not have φ-density, so $\mathbb{P}$, $\mathbb{N} \setminus \mathbb{P} \in \mathcal{F}_\varphi^+$. Now consider the functions $f, g : [0, 1) \longrightarrow \mathbb{R}$, $f(\xi) = 0$ and $g(\xi) = 1$. Let $\xi \in [0, 1)$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\mathcal{F}_{st} - \lim x_n = \xi$. Then we can find a set $L \in \mathcal{F}_{st}$ such that $\lim_{n\to\infty} x_n = \xi$ (since $\mathcal{F}_{st}$ is a P-filter). So we get

$$\lim_{\substack{n\to\infty \\ n\in M}} f_n(x_n) = \lim_{n\to\infty} (x_n)^n = 0 = f(\xi),$$

where $M := L \cap (\mathbb{N} \setminus \mathbb{P}) \in \mathcal{F}_{st}$. Then we obtain $\mathcal{F}_{st} - \lim f_n(x_n) = f(\xi)$, and so $f_n \xrightarrow{\mathcal{F}_{st}-\alpha} f$ on $[0, 1)$. From Theorem 2.13, $\Gamma^\alpha_{f_n}(\mathcal{F}_{st}) = \{f\}$ holds.
Now let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq [0, 1)$, $(x_{i,n})_{n\in\mathbb{N}}$ be sequences in $[0, 1)$ such that $\mathcal{F}_\varphi - \lim x_{i,n} = \xi_i$ for each $i \in \{1, 2, ..., m\}$ and $\varepsilon > 0$. For each $i \in \{1, 2, ..., m\}$ and every $n \in \mathbb{P}$ we have

$$f_n(x_{i,n}) - g(\xi_i) = 1 - 1 = 0.$$

Therefore we get

$$\mathbb{P} \subseteq \left\{n \in \mathbb{N} : \left|f_n(x_{i,n}) - g(\xi_i)\right| < \varepsilon \text{ for all } i \in \{1, 2, ..., m\}\right\} \in \mathcal{F}_\varphi^+,$$

and so $g \in \Gamma^{\alpha}_{f_n}\left(\mathcal{F}_{\varphi}\right)$.
From $\mathcal{F}_{\varphi} - \lim x_{i,n} = \xi_i$ *we have*

$$K_i := \left\{ n \in \mathbb{N} : \left| x_{i,n} - \xi_i \right| < \frac{\varepsilon}{2} \right\} \in \mathcal{F}_{\varphi}$$

for each $i \in \{1,2,...,m\}$. *Let* $K := \bigcap_{i=1}^{m} K_i \in \mathcal{F}_{\varphi}$. *Let* $n_i(\varepsilon) = \left\lfloor \frac{\ln(\varepsilon/2)}{\ln \xi_i} \right\rfloor$ *for each* $i \in \{1,2,...,m\}$ *and* $n_0(\varepsilon) = \max\{n_i(\varepsilon) : i \in \{1,2,...,m\}\}$. *For each* $i \in \{1,2,...,m\}$ *and every* $n \in K \cap (\mathbb{N} \setminus (\mathbb{P} \cup \{1,...,n_0(\varepsilon)\})) \in \mathcal{F}^{+}_{\varphi}$ *we get*

$$\begin{aligned} \left| f_n(x_{i,n}) - f(\xi_i) \right| &= \left| x^n_{i,n} - 0 \right| = \left| x^n_{i,n} - \xi^n_i + \xi^n_i \right| \\ &\leq \left| x^n_{i,n} - \xi^n_i \right| + \left| \xi^n_i \right| \\ &\leq \left| x_{i,n} - \xi_i \right| + \left| \xi^n_i \right| < \varepsilon. \end{aligned}$$

Then we obtain

$$\left\{ n \in \mathbb{N} : \left| f_n(x_{i,n}) - f(\xi_i) \right| < \varepsilon \text{ for all } i \in \{1,2,...,m\} \right\} \in \mathcal{F}^{+}_{\varphi},$$

that is, $f \in \Gamma^{\alpha}_{f_n}\left(\mathcal{F}_{\varphi}\right)$. *Hence* $\Gamma^{\alpha}_{f_n}\left(\mathcal{F}_{\varphi}\right) = \{f, g\}$ *holds.*
Consequently, we have $\Gamma^{\alpha}_{f_n}(\mathcal{F}_{st}) \subseteq \Gamma^{\alpha}_{f_n}\left(\mathcal{F}_{\varphi}\right)$. □

Theorem 2.16. *Let* (X, ρ_X) *and* (Y, ρ_Y) *be two metric spaces,* $D \subseteq X$, $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions defined from* D *to* Y, *and* $\mathcal{F}$ *be a free filter on* $\mathbb{N}$. *Then the following holds:*

(i) $\Lambda^{\alpha}_{f_n}(\mathcal{F}) \subseteq \Gamma^{\alpha}_{f_n}(\mathcal{F}) \subseteq L^{\alpha}_{f_n}$.
(ii) $\Lambda^{u}_{f_n}(\mathcal{F}) \subseteq \Gamma^{u}_{f_n}(\mathcal{F}) \subseteq L^{u}_{f_n}$.
(iii) $\Lambda^{pw}_{f_n}(\mathcal{F}) \subseteq \Gamma^{pw}_{f_n}(\mathcal{F}) \subseteq L^{pw}_{f_n}$.

Proof. We will prove only (i). The items (ii) and (iii) can be proved by the similar technique.
(i) Let us assume that $f \in \Lambda^{\alpha}_{f_n}(\mathcal{F})$. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$, $(x_{i,n})$ be sequences in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ $(i \in 1,2,...,m)$, and $\varepsilon > 0$. Then we can find a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^{+}$ such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. For each $i \in \{1,2,...,m\}$ there is a $k_i \in \mathbb{N}$ such that $\rho_Y(f_{n_k}(x_{i,n_k}), f(\xi_i)) < \varepsilon$ for every $n_k \geq n_{k_i}$. Take $k_0 = \max\{k_1, k_2, ..., k_m\}$. Then we get $K \setminus \{n_1, n_2, ..., n_{k_0}\} \in \mathcal{F}^{+}$ and so

$$K \setminus \{n_1, ..., n_{k_0}\} \subseteq \{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon \text{ for all } i \in \{1,...,m\}\} \in \mathcal{F}^{+}.$$

Therefore, $f \in \Gamma^{\alpha}_{f_n}(\mathcal{F})$ holds.
Now let us assume that $f \in \Gamma^{\alpha}_{f_n}(\mathcal{F})$. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and $(x_{i,n})$ be sequences in D such that $\lim_{n\to\infty} x_{i,n} = \xi_i$ $(i \in 1,2,...,m)$. Since $\mathcal{F}$ is free, we have $\mathcal{F} - \lim x_{i,n} = \xi_i$ $(i \in 1,2,...,m)$. Then for each $\varepsilon > 0$ we get

$$K_{\varepsilon} := \{n \in \mathbb{N} : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon \text{ for all } i \in \{1,2,...,m\}\} \in \mathcal{F}^{+}.$$

Since $\mathcal{F}$ is free, all K_{ε} are infinite sets for every $\varepsilon > 0$. So we can choose an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\}$ such that $K \setminus K_{\varepsilon}$ is finite for every $\varepsilon > 0$. Then we get $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. Therefore, $f \in L^{\alpha}_{f_n}$ is obtained. □

To show the difference between the set of $\mathcal{F}$-limit functions and the set of $\mathcal{F}$-cluster functions we are going to give an example which shows that this inclusion is true except $\mathcal{F}_r$. It is similar to Example 3 in [17].

Example 2.17. *For each $p \in \mathbb{N}$ let us define the functions g_p from $\mathbb{R}$ to $\mathbb{R}$ as*

$$g_p(\xi) = \begin{cases} \frac{1}{p} & \text{if } \xi > 0 \\ 0 & \text{if } \xi \le 0 \end{cases}.$$

Let $J_p = \left\{2^{p-1}(2q-1) : q \in \mathbb{N}\right\}$, and let the sequence of functions $(f_n)_{n\in\mathbb{N}}$ be defined by $f_n(\xi) = g_p(\xi)$ if $n \in J_p$. For each $p \in \mathbb{N}$ we have $d\left(J_p\right) = \frac{1}{2^p}$ and so $J_p \in \mathcal{F}_{st}^+$. It is clear that for each $p \in \mathbb{N}$ the subsequences $(f_n)_{n\in J_p}$ are uniformly convergent to g_p on $\mathbb{R}$. Therefore we have $g_p \in \Lambda_{f_n}^u(\mathcal{F}_{st}) \subseteq \Gamma_{f_n}^u(\mathcal{F}_{st})$ for each $p \in \mathbb{N}$.
Let the function f be defined by $f(\xi) = 0$ for every $\xi \in \mathbb{R}$. Then we get $f \in \Gamma_{f_n}^u(\mathcal{F}_{st})$, but $f \notin \Lambda_{f_n}^u(\mathcal{F}_{st})$. Indeed, for every $\varepsilon > 0$ we obtain

$$\bigcup_{p>\frac{1}{\varepsilon}} J_p \subseteq \left\{n \in \mathbb{N} : \left|f_n(\xi) - f(\xi)\right| < \varepsilon \text{ for all } \xi \in \mathbb{R}\right\} \in \mathcal{F}_{st}^+$$

since the set on the left-hand side has positive asymptotic density. Then $f \in \Gamma_{f_n}^u(\mathcal{F}_{st})$ holds. Now let us suppose that there is a set $K \in \mathcal{F}_{st}^+$ such that the subsequence $(f_n)_{n\in K}$ is uniformly convergent to f. Then $K \cap J_p$ is finite for each $p \in \mathbb{N}$. Therefore, for each $p \in \mathbb{N}$ we get

$$\begin{aligned} K &\subseteq \bigcup_{i=p+1}^{\infty} J_i \cup M_p \\ d(K) &\le d\left(\bigcup_{i=p+1}^{\infty} J_i\right) = 1 - \sum_{i=1}^{p} d(J_i) = \frac{1}{2^p} \end{aligned}$$

where M_p is a finite set. So, we have $d(K) = 0$ as $p \to \infty$. This contradicts to $K \in \mathcal{F}_{st}^+$. Hence $f \notin \Lambda_{f_n}^u(\mathcal{F}_{st})$. □

Theorem 2.18. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, $D \subseteq X$, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}-$exhaustive then we have*

$$\Lambda_{f_n}^{pw}(\mathcal{F}) \subseteq C(D, Y) \text{ and } \Gamma_{f_n}^{pw}(\mathcal{F}) \subseteq C(D, Y).$$

Proof. Let us assume that $f \in \Lambda_{f_n}^{pw}(\mathcal{F})$. We will prove that f is continuous on D. Let $\xi \in D$ and $\varepsilon > 0$. Since the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}-$exhaustive at the point ξ, there is a $\delta > 0$ such that

$$L := \{n \in \mathbb{N} : \rho_Y(f_n(\xi), f_n(\eta)) < \varepsilon/3\} \in \mathcal{F}$$

for all $\eta \in S(\xi, \delta) \cap D$. Take $\zeta \in S(\xi, \delta) \cap D$. Since $f \in \Lambda_{f_n}^{pw}(\mathcal{F})$, for the set $\{\xi, \zeta\}$ we can find a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^+$ such that

$$\lim_{k\to\infty} f_{n_k}(\xi) = f(\xi) \text{ and } \lim_{k\to\infty} f_{n_k}(\zeta) = f(\zeta).$$

Then there is a $k_0 \in \mathbb{N}$ such that for each $n \in K_0$ we have

$$\rho_Y(f_n(\xi), f(\xi)) < \varepsilon/3 \text{ and } \rho_Y(f_n(\zeta), f(\zeta)) < \varepsilon/3$$

where $K_0 := K \setminus \{n_1, n_2, ..., n_{k_0}\} \in \mathcal{F}^+$. Let $M = K_0 \cap L \in \mathcal{F}^+$. We get

$$\rho_Y(f(\xi), f(\zeta)) \le \rho_Y(f(\xi), f_n(\xi)) + \rho_Y(f_n(\xi), f_n(\zeta)) + \rho_Y(f_n(\zeta), f(\zeta)) < \varepsilon$$

for each $n \in M$. Thus f is continuous at ξ. This completes the proof of the theorem. The proof for $\Gamma_{f_n}^{pw}(\mathcal{F})$ is similar. □

Theorem 2.19. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, $D \subseteq X$, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y, and $\mathcal{F}$ be a filter on $\mathbb{N}$. Then the following holds:*

- **(i)** $\Lambda^{\alpha}_{f_n}(\mathcal{F}) \subseteq \Lambda^{pw}_{f_n}(\mathcal{F})$, $\Gamma^{\alpha}_{f_n}(\mathcal{F}) \subseteq \Gamma^{pw}_{f_n}(\mathcal{F})$ and $L^{\alpha}_{f_n} \subseteq L^{pw}_{f_n}$.
- **(ii)** $\Lambda^{u}_{f_n}(\mathcal{F}) \subseteq \Lambda^{pw}_{f_n}(\mathcal{F})$, $\Gamma^{u}_{f_n}(\mathcal{F}) \subseteq \Gamma^{pw}_{f_n}(\mathcal{F})$ and $L^{u}_{f_n} \subseteq L^{pw}_{f_n}$.

Proof. **(i)** Let us assume that $f \in \Lambda^{\alpha}_{f_n}(\mathcal{F})$. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$. Let us take the constant sequences $(x_{i,n})$ defined by $x_{i,n} = \xi_i$ $(n \in \mathbb{N})$ for each $i \in \{1, 2, ..., m\}$. Then we have $\mathcal{F} - \lim x_{i,n} = \xi_i$ for each $i \in \{1, 2, ..., m\}$, and so there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^+$ such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$ since $f \in \Lambda^{\alpha}_{f_n}(\mathcal{F})$. Consequently, we obtain $\lim_{k\to\infty} f_{n_k}(\xi_i) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$, and so $f \in \Lambda^{pw}_{f_n}(\mathcal{F})$ holds. The other inclusions can be proved similarly.
(ii) Straightforward from the definitions. □

From Theorem 2.18 and 2.19 we have the following corollary:

Corollary 2.20. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, $D \subseteq X$, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive then the following holds:*

- **(i)** $\Lambda^{\alpha}_{f_n}(\mathcal{F}) \subseteq C(D, Y)$ and $\Gamma^{\alpha}_{f_n}(\mathcal{F}) \subseteq C(D, Y)$.
- **(ii)** $\Lambda^{u}_{f_n}(\mathcal{F}) \subseteq C(D, Y)$ and $\Gamma^{u}_{f_n}(\mathcal{F}) \subseteq C(D, Y)$.

Theorem 2.21. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, $D \subseteq X$, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive then we have*

$$\Lambda^{\alpha}_{f_n}(\mathcal{F}) = \Lambda^{pw}_{f_n}(\mathcal{F}) \text{ and } \Gamma^{\alpha}_{f_n}(\mathcal{F}) = \Gamma^{pw}_{f_n}(\mathcal{F}).$$

Proof. One inclusion was shown in Theorem 2.19(i). Let us show the other inclusion. Let us assume that $f \in \Lambda^{pw}_{f_n}(\mathcal{F})$. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and $(x_{i,n})_{n\in\mathbb{N}}$ be sequences in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ $(i \in \{1, 2, ..., m\})$. Let $\varepsilon > 0$. Since the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at the points ξ_i, for each $i \in \{1, 2, ..., m\}$ there exist $\delta_i > 0$ such that

$$J_i := \{n \in \mathbb{N} : \rho_Y(f_n(\eta), f_n(\xi_i)) < \varepsilon/2 \text{ for all } \eta \in S(\xi_i, \delta_i) \cap D\} \in \mathcal{F}.$$

For each $i \in \{1, 2, ..., m\}$, since $\mathcal{F} - \lim x_{i,n} = \xi_i$ there is an $K_i \in \mathcal{F}$ such that $\rho_X(x_{i,n}, \xi_i) < \delta_i$ for every $n \in K_i$. For every $n \in L_i$ we get

$$\rho_Y(f_n(x_{i,n}), f_n(\xi_i)) < \varepsilon/2$$

where $L_i := J_i \cap K_i \in \mathcal{F}$ for every $i \in \{1, 2, ..., m\}$. Let $L := \bigcap_{i=1}^{m} L_i \in \mathcal{F}$. From $f \in \Lambda^{pw}_{f_n}(\mathcal{F})$ there is an $M \in \mathcal{F}^+$ such that $\lim_{n\to\infty, n\in M} f_n(\xi_i) = f(\xi_i)$ for every $i \in \{1, 2, ..., m\}$. In this case, there is an $n_0 \in \mathbb{N}$ such that for each $i \in \{1, 2, ..., m\}$ and each $n \in M_0 := M \setminus \{1, 2, ..., n_0\} \in \mathcal{F}^+$ we have

$$\rho_Y(f_n(\xi_i), f(\xi_i)) < \varepsilon/2.$$

Then $L \cap M_0 \in \mathcal{F}^+$ and so for every $i \in \{1, 2, ..., m\}$ we obtain

$$\rho_Y(f_n(x_{i,n}), f(\xi_i)) \leq \rho_Y(f_n(x_{i,n}), f_n(\xi_i)) + \rho_Y(f_n(\xi_i), f(\xi_i)) < \varepsilon$$

for each $n \in L \cap M_0$. Consequently, $f \in \Lambda^{\alpha}_{f_n}(\mathcal{F})$ holds. □

Theorem 2.22. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, $D \subseteq X$, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y, and $\mathcal{F}$ be a free P-filter on $\mathbb{N}$. Then the following holds:*

$$\begin{aligned} \Lambda^u_{f_n}(\mathcal{F}) \cap C(D,Y) &\subseteq \Lambda^\alpha_{f_n}(\mathcal{F}) \cap C(D,Y), \\ \Gamma^u_{f_n}(\mathcal{F}) \cap C(D,Y) &\subseteq \Gamma^\alpha_{f_n}(\mathcal{F}) \cap C(D,Y). \end{aligned}$$

Proof. Let us assume that $f \in \Lambda^u_{f_n}(\mathcal{F}) \cap C(D,Y)$. Then we can find a set $M \in \mathcal{F}^+$ such that the subsequence $(f_n)_{n\in M}$ uniformly converges to f on D. Let $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and $(x_{i,n})_{n\in\mathbb{N}}$ be sequences in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ $(i \in \{1,2,...,m\})$. Since $\mathcal{F}$ is a P-filter, for each $i \in \{1,2,...,m\}$ there is an $L_i \in \mathcal{F}$ such that $\lim_{n\to\infty, n\in L_i} x_{i,n} = \xi_i$. Let $L := \bigcap_{i=1}^m L_i \in \mathcal{F}$ and $K := L \cap M \in \mathcal{F}^+$. Let us denote the elements of K as $\{n_1 < n_2 < ... < n_k < ...\}$. We will show that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. Fix $i \in \{1,2,...,m\}$ and $\varepsilon > 0$. From the uniform convergence of $(f_{n_k})_{k\in\mathbb{N}}$, there is a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have

$$\rho_Y(f_{n_k}(\xi), f(\xi)) < \varepsilon/2 \text{ for all } \xi \in D.$$

Hence for each $k \geq k_0$ we get

$$\rho_Y(f_{n_k}(x_{i,n_k}), f(x_{i,n_k})) < \varepsilon/2. \tag{1}$$

Since the function f is continuous at the point ξ_i, there is a $\delta_i > 0$ such that $\rho_Y(f(\eta), f(\xi_i)) < \varepsilon/2$ for every $\eta \in S(\xi_i, \delta_i) \cap D$. Then there is a $k_i \in \mathbb{N}$ such that $x_{i,n_k} \in S(\xi_i, \delta_i) \cap D$ for every $k \geq k_i$. Thus we have

$$\rho_Y(f(x_{i,n_k}), f(\xi_i)) < \varepsilon/2 \tag{2}$$

for every $k \geq k_i$. Let $k'_i = \max\{k_0, k_i\}$. From the inequality (1) and (2), we get

$$\rho_Y(f_{n_k}(x_{i,n_k}), f(\xi_i)) \leq \rho_Y(f_{n_k}(x_{i,n_k}), f(x_{i,n_k})) + \rho_Y(f(x_{i,n_k}), f(\xi_i)) < \varepsilon$$

for every $k \geq k'_i$. Then we obtain $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$. Consequently, $f \in \Lambda^\alpha_{f_n}(\mathcal{F}) \cap C(D,Y)$ holds. The second inclusion can be proved similarly. □

Theorem 2.23. *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, D be a compact subset of X, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from D to Y, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive then the following holds:*

$$\Lambda^\alpha_{f_n}(\mathcal{F}) \subseteq \Lambda^u_{f_n}(\mathcal{F}) \text{ and } \Gamma^\alpha_{f_n}(\mathcal{F}) \subseteq \Gamma^u_{f_n}(\mathcal{F}).$$

Proof. We will prove the second inclusion. Let us assume that $f \in \Gamma^\alpha_{f_n}(\mathcal{F})$. Let $\varepsilon > 0$. From Corollary 2.20 f is continuous on D. Then for every $\xi \in D$ there is a $\delta_\xi > 0$ such that $\rho_X(\xi, \eta) < \delta_\xi$ implies $\rho_Y(f(\xi), f(\eta)) < \varepsilon/3$. Since $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive on D, for every $\xi \in D$ there exist a positive real number $\lambda_\xi < \delta_\xi$ and a $K(\xi) \in \mathcal{F}$ such that $\rho_Y(f_n(\eta), f_n(\xi)) < \varepsilon/3$ for all $\eta \in S(\xi, \lambda_\xi)$ and all $n \in K(\xi)$. Then $\bigcup_{\xi\in D} S(\xi, \lambda_\xi) \supseteq D$, and since D is compact there are finitely many $\xi_1, \xi_2, ..., \xi_m \in D$ such that $D \subseteq \bigcup_{i=1}^m S(\xi_i, \lambda_{\xi_i})$. Let $K := \bigcap_{i=1}^m K(\xi_i) \in \mathcal{F}$. Therefore we get

$$\rho_Y(f(\eta), f(\xi_i)) < \varepsilon/3$$

and

$$\rho_Y(f_n(\eta), f_n(\xi_i)) < \varepsilon/3$$

for all $n \in K$ and all $\eta \in S(\xi_i, \lambda_{\xi_i})$ $(i = \{1,2,...,m\})$.
We have $f \in \Gamma^{pw}_{f_n}(\mathcal{F})$ from Theorem 2.19(i). We have already observed that

$$L := \{n \in \mathbb{N} : \rho_Y(f_n(\xi_i), f(\xi_i)) < \varepsilon/3 \text{ for all } i \in \{1,2,...,m\}\} \in \mathcal{F}^+.$$

Let us take $M := K \cap L \in \mathcal{F}^+$.
Let $\zeta \in D$. Then $\zeta \in S(\xi_i, \lambda_{\xi_i})$ for some $i \in \{1, ..., m\}$ and so we get

$$\rho_Y(f_n(\zeta), f(\zeta)) \le \rho_Y(f_n(\zeta), f_n(\xi_i)) + \rho_Y(f_n(\xi_i), f(\xi_i)) + \rho_Y(f(\xi_i), f(\zeta)) < \varepsilon$$

for every $n \in M$. Therefore we obtain

$$M \subseteq \{n \in \mathbb{N} : \rho_Y(f_n(\zeta), f(\zeta)) < \varepsilon \text{ for all } \zeta \in D\} \in \mathcal{F}^+,$$

and so $f \in \Gamma^u_{f_n}(\mathcal{F})$ holds. □

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