Filomat 32:9 (2018), 3155–3169
https://doi.org/10.2298/FIL1809155K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Generalized 2D Ostrowski Type Inequalities on Time Scales with k^2 Points Using a Parameter

Seth Kermausuor[a], Eze R. Nwaeze[b]

[a]*Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36104, USA*
[b]*Department of Mathematics, Tuskegee University, Tuskegee, AL 36088, USA*

Abstract. Recently, a new Ostrowski type inequality on time scales for k points was proved in [G. Xu, Z. B. Fang: A Generalization of Ostrowski type inequality on time scales with k points. Journal of Mathematical Inequalities (2017), 11(1):41–48]. In this article, we extend this result to the 2-dimensional case. Besides extension, our results also generalize the three main results of Meng and Feng in the paper [Generalized Ostrowski type inequalities for multiple points on time scales involving functions of two independent variables. Journal of Inequalities and Applications (2012), 2012:74]. In addition, we apply some of our theorems to the continuous, discrete, and quantum calculus to obtain more interesting results in this direction. We hope that results obtained in this paper would find their place in approximation and numerical analysis.

1. Introduction

To unify the theory of continuous and discrete calculus, the German mathematician Stefan Hilger [8] came up with the theory of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For the sake of brevity, we will only recall definitions and properties (in the theory of time scales) that will be needed in the sequel. We recommend the books [2, 3] for a thorough study of this subject.

Definition 1. *The forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by* $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ *for* $t \in \mathbb{T}$.

Definition 2. *The function* $f : \mathbb{T} \to \mathbb{R}$, *is called differentiable at* $t \in \mathbb{T}^k$, *with delta derivative* $f^\Delta(t) \in \mathbb{R}$, *if for any given* $\epsilon > 0$ *there exist a neighborhood* U *of* t *such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \epsilon|\sigma(t) - s|, \quad \forall s \in U.$$

We call $f^\Delta(t)$ *the delta derivative of* f *at* t. *Moreover, we say that* f *is delta differentiable (or in short: differentiable) on* $\mathbb{T}^k$ *provided* $f^\Delta(t)$ *exists for all* $t \in \mathbb{T}^k$. *The function* $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ *is then called the delta derivative of* f *on* $\mathbb{T}^k$.

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \dfrac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$.

2010 *Mathematics Subject Classification*. Primary 35A23; Secondary 26E70, 34N05
Keywords. Montgomery identity, Ostrowski type inequality, time scales, parameter
Received: 17 August 2017; Accepted: 15 October 2017
Communicated by Fuad Kittaneh
Email addresses: skermausour@alasu.edu (Seth Kermausuor), enwaeze@tuskegee.edu (Eze R. Nwaeze)

Definition 3. *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd−continuous if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left dense points $t \in \mathbb{T}$. The set of all rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. Also, the set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$.*

Definition 4. *Let $f \in C_{rd}(\mathbb{T},\mathbb{R})$. Then $g : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we have*

$$\int_a^b f(s)\Delta s = g(b) - g(a).$$

Definition 5. *The function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined as*

$$f^{\sigma}(t) := f(\sigma(t))$$

for any $t \in \mathbb{T}$.

Theorem 6. *If $a, b, c \in \mathbb{T}$ with $a < c < b$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T},\mathbb{R})$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.$

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t.$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t.$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$

(v) $\left|\int_a^b f(t)\Delta t\right| \le \int_a^b |f(t)|\Delta t.$

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t.$

Definition 7. *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}$ be functions that are recursively defined as*

$$h_0(t, s) = 1$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad for\ all\ \ s, t \in \mathbb{T}.$$

In 2008, Bohner and Matthews [1] extended the well known Ostrowski inequality [6] to time scales by proving the following theorem.

Theorem 8. *Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then for all $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f^{\sigma}(t)\Delta t \right| \le \frac{M}{b-a}\left(h_2(x, a) + h_2(x, b)\right), \tag{1}$$

where $h_2(\cdot,\cdot)$ is given in Definition 7 and $M = \sup_{a<t<b} \left|f^{\Delta}(t)\right| < \infty$. This inequality is sharp in the sense that the right-hand side of (1) cannot be replaced by a smaller one.

Following thereafter, Liu and Ngô [11] generalized Theorem 8 as follows:

Theorem 9. *Suppose that*

1. *$a, b \in \mathbb{T}$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a, b]$ for $x_0, x_1, \cdots, x_k \in \mathbb{T}$,*
2. *$\alpha_j \in \mathbb{T}$ $(j = 0, 1, \cdots k + 1)$ is $k + 2$ points so that $\alpha_0 = a$, $\alpha_j \in [x_{j-1}, x_j]$ $(j = 1, \cdots k)$ and $\alpha_{k+1} = b$,*
3. *$f : [a, b] \to \mathbb{R}$ is differentiable function.*

Then we have

$$\left| \int_a^b f^\sigma(t)\Delta t - \sum_{j=0}^{k} \left(\alpha_{j+1} - \alpha_j\right) f(x_j) \right| \le M \sum_{j=0}^{k-1} \left(h_2(x_j, \alpha_{j+1}) + h_2(x_{j+1}, \alpha_{j+1}) \right), \tag{2}$$

where $M = \sup_{a<t<b} |f^\Delta(t)|$. *This inequality is sharp in the sense that the right hand side of (2) cannot be replaced by a smaller one.*

In 2012, Feng and Meng [7] extended, among other things, Theorem 9 to the 2-dimensional case. For more on this and related results in this direction, see the papers [9, 10, 12–14, 16] and the references therein. Recently, by introducing a parameter $\lambda \in [0, 1]$, Xu and Fang [17] further generalized Theorem 9 by proving the following result.

Theorem 10. *Suppose that* $\mathbb{T}$ *is a time scale and*

1. $a, b \in \mathbb{T}$, $\lambda \in [0, 1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a, b]$ *for* $x_0, x_1, \cdots, x_k \in \mathbb{T}$,
2. $\alpha_i \in \mathbb{T}$ $(i = 0, 1, \cdots, k + 1)$ *is* $k + 2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ *and* $\alpha_{k+1} = b$,
3. $f : [a, b] \to \mathbb{R}$ *is a differentiable function*

$$\begin{aligned}
&\left| (1-\lambda) \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) + \frac{\lambda}{2} \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i)\Big(f(\alpha_i) + f(\alpha_{i+1}) \Big) - \int_a^b f^\sigma(t)\Delta t \right| \\
&\le M \sum_{i=0}^{k-1} \Big[h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1} - \alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1} - \alpha_i}{2}\Big) \\
&+ h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big) \Big],
\end{aligned}$$

where $M = \sup_{a<t<b} \left| f^\Delta(t) \right| < \infty$.

It is our purpose in this paper to establish three new Ostrowski type result for multiple points on time scales, for functions of two independent variables, via a parameter. Our first result will extend Theorem 10 to the 2-dimensional case (see Remark 15). As a special case (for $\lambda = 0$) of our results, we will obtain the main theorems of Feng and Meng in [7] (see Remarks 15, 17 and 20); and for $\lambda \in (0, 1]$, we obtain completely new results in this direction.

This paper is arranged in the following fashion: in Section 2, our results are composed and justified. Section 3 houses application of Theorem 16 in the continuous, discrete and quantum calculus.

2. Main Results

The following lemma is given in [15, 17] but with some typos. We present here the correct version.

Lemma 11. *Suppose that* $\mathbb{T}$ *is a time scale and*

1. $a, b \in \mathbb{T}$, $\lambda \in [0, 1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a, b]$ *for* $x_0, x_1, \cdots, x_k \in \mathbb{T}$,
2. $\alpha_i \in \mathbb{T}$ $(i = 0, 1, \cdots, k + 1)$ *is* $k + 2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ *and* $\alpha_{k+1} = b$,
3. $f : [a, b] \to \mathbb{R}$ *is a differentiable fuunction.*

$$\int_a^b K(t,I_k)f^{\Delta}(t)\Delta t$$
$$= (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-\int_a^b f^{\sigma}(t)\Delta t,$$

where

$$K(t,I_k)=\begin{cases} t-\left(\alpha_1-\lambda\frac{\alpha_1-a}{2}\right), & t\in[a,\alpha_1),\\ t-\left(\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\right), & t\in[\alpha_1,x_1),\\ t-\left(\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\right), & t\in[x_1,\alpha_2),\\ \vdots & \\ t-\left(\alpha_{k-1}+\lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & t\in[\alpha_{k-1},x_{k-1}),\\ t-\left(\alpha_k-\lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & t\in[x_{k-1},\alpha_k),\\ t-\left(\alpha_k+\lambda\frac{\alpha_{k+1}-\alpha_k}{2}\right), & t\in[\alpha_k,b]. \end{cases} \tag{3}$$

In what follows, we let $\mathbb{T}_1$ and $\mathbb{T}_2$ denote two arbitrary time scales, and for an interval $[a,b]$, $[a,b]_{\mathbb{T}_i} := [a,b]\cap\mathbb{T}_i, i=1,2$. For $a<b$ and $c<d$, we define the rectangle $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$ as follows: $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2} = \{(x,y): x\in[a,b]_{\mathbb{T}_1}, y\in[c,d]_{\mathbb{T}_2}\}$. Finally, we assume through out the rest of this paper that $\mathbb{T}_1$ and $\mathbb{T}_2$ have a uniform forward jump operator σ. For more on the two-variable time scale calculus, we invite the interested reader to the papers [4, 5].

Lemma 12 (2D Generalized Montgomery Identity with a parameter). *Let $\lambda\in[0,1]; a,b\in\mathbb{T}_1;\ c,d\in\mathbb{T}_2$ with $a<b,\ c<d$. Suppose that*

1. *$I_k: a=x_0<x_1<\cdots<x_{k-1}<x_k=b$ is a partition of the interval $[a,b]_{\mathbb{T}_1}$ for $x_0,x_1,\cdots,x_k\in\mathbb{T}_1$, and $J_k: c=y_0<y_1<\cdots<y_{k-1}<y_k=d$ is a partition of the interval $[c,d]_{\mathbb{T}_2}$ for $y_0,y_1,\cdots,y_k\in\mathbb{T}_2$;*
2. *$\alpha_i\in\mathbb{T}_1,\beta_i\in\mathbb{T}_2$ $(i=0,1,\cdots,k+1)$ is $k+2$ points so that $\alpha_0=a,\ \alpha_i\in[x_{i-1},x_i]_{\mathbb{T}_1}$ $(i=1,\cdots,k)$ and $\alpha_{k+1}=b$, $\beta_0=c,\ \beta_i\in[y_{i-1},y_i]_{\mathbb{T}_2}$ $(i=1,\cdots,k)$ and $\beta_{k+1}=d$;*
3. *$f:[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}\to\mathbb{R}$ is a $\Delta_1\Delta_2$ differentiable function.*

Then we have the identity

$$\int_a^b\int_c^d K(s,t,I_k,J_k)f^{\Delta_1\Delta_2}(s,t)\Delta_2t\Delta_1s$$
$$=(1-\lambda)^2\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j)+\frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j)$$
$$+f(\alpha_{i+1},y_j)+f(x_i,\beta_j)+f(x_i,\beta_{j+1})\Big)-(1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1s$$
$$+\frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big)$$
$$-\frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j)+f(\sigma(s),\beta_{j+1})\Big)\Delta_1s-(1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2t$$
$$-\frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)\Delta_2t+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s, \tag{4}$$

where $K(s,t,I_k,J_k)=K_1(s,I_k)K_2(t,J_k)$ *and*

$$K_1(s,I_k)=\begin{cases} s-\left(\alpha_1-\lambda\frac{\alpha_1-a}{2}\right), & s\in[a,\alpha_1)_{\mathbb{T}_1},\\ s-\left(\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\right), & s\in[\alpha_1,x_1)_{\mathbb{T}_1},\\ s-\left(\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\right), & s\in[x_1,\alpha_2)_{\mathbb{T}_1},\\ \vdots & \\ s-\left(\alpha_{k-1}+\lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & t\in[\alpha_{k-1},x_{k-1})_{\mathbb{T}_1},\\ s-\left(\alpha_k-\lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & s\in[x_{k-1},\alpha_k)_{\mathbb{T}_1},\\ s-\left(\alpha_k+\lambda\frac{\alpha_{k+1}-\alpha_k}{2}\right), & s\in[\alpha_k,b]_{\mathbb{T}_1},\end{cases}$$

$$K_2(t,J_k)=\begin{cases} t-\left(\beta_1-\lambda\frac{\beta_1-c}{2}\right), & t\in[c,\beta_1)_{\mathbb{T}_2},\\ t-\left(\beta_1+\lambda\frac{\beta_2-\beta_1}{2}\right), & t\in[\beta_1,y_1)_{\mathbb{T}_2},\\ t-\left(\beta_2-\lambda\frac{\beta_2-\beta_1}{2}\right), & t\in[y_1,\beta_2)_{\mathbb{T}_2},\\ \vdots & \\ t-\left(\beta_{k-1}+\lambda\frac{\beta_k-\beta_{k-1}}{2}\right), & t\in[\beta_{k-1},y_{k-1})_{\mathbb{T}_2},\\ t-\left(\beta_k-\lambda\frac{\beta_k-\beta_{k-1}}{2}\right), & t\in[y_{k-1},\beta_k)_{\mathbb{T}_2},\\ t-\left(\beta_k+\lambda\frac{\beta_{k+1}-\beta_k}{2}\right), & t\in[\beta_k,d]_{\mathbb{T}_2}.\end{cases}$$

Proof. By using Lemma 11, we deduce that for each $s\in\mathbb{T}_1$

$$\begin{aligned}&\int_c^d K_2(t,J_k)f^{\Delta_1\Delta_2}(s,t)\Delta_2 t\\ &=(1-\lambda)\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)f^{\Delta_1}(s,y_j)+\frac{\lambda}{2}\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\Big(f^{\Delta_1}(s,\beta_j)+f^{\Delta_1}(s,\beta_{j+1})\Big)-\int_c^d f^{\Delta_1}(s,\sigma(t))\Delta_2 t.\end{aligned}\tag{5}$$

Similarly, for each $t\in\mathbb{T}_2$, we have

$$\begin{aligned}&\int_a^b K_1(s,I_k)f^{\Delta_1}(s,t)\Delta_1 s\\ &=(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i,t)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,t)+f(\alpha_{i+1},t)\Big)-\int_a^b f(\sigma(s),t)\Delta_1 s.\end{aligned}\tag{6}$$

Using (5) and (6), we obtain

$$\begin{aligned}&\int_a^b\int_c^d K(s,t,I_k,J_k)f^{\Delta_1\Delta_2}(s,t)\Delta_2 t\Delta_1 s=\int_a^b\int_c^d K_1(s,I_k)K_2(t,J_k)f^{\Delta_1\Delta_2}(s,t)\Delta_2 t\Delta_1 s\\ &=\int_a^b K_1(s,I_k)\Big[\int_c^d K_2(t,J_k)f^{\Delta_1\Delta_2}(s,t)\Delta_2 t\Big]\Delta_1 s\\ &=\int_a^b K_1(s,I_k)\Big[(1-\lambda)\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)f^{\Delta_1}(s,y_j)+\frac{\lambda}{2}\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\Big(f^{\Delta_1}(s,\beta_j)+f^{\Delta_1}(s,\beta_{j+1})\Big)\\ &-\int_c^d f^{\Delta_1}(s,\sigma(t))\Delta_2 t\Big]\Delta_1 s\end{aligned}$$

$$= (1-\lambda)\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\int_a^b K_1(s,I_k)f^{\Delta_1}(s,y_j)\Delta_1 s$$
$$+\frac{\lambda}{2}\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\int_a^b K_1(s,I_k)\Big(f^{\Delta_1}(s,\beta_j)+f^{\Delta_1}(s,\beta_{j+1})\Big)\Delta_1 s$$
$$-\int_c^d\int_a^b K_1(s,I_k)f^{\Delta_1}(s,\sigma(t))\Delta_1 s\Delta_2 t$$

$$= (1-\lambda)\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\Big[(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i,y_j)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j)+f(\alpha_{i+1},y_j)\Big)$$
$$-\int_a^b f(\sigma(s),y_j)\Delta_1 s\Big]$$
$$+\frac{\lambda}{2}\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\Big[(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(x_i,\beta_j)+f(x_i,\beta_{j+1})\Big)$$
$$+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big)$$
$$-\int_a^b\Big(f(\sigma(s),\beta_j)+f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s\Big]$$
$$-\int_c^d\Big[(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)$$
$$-\int_a^b f(\sigma(s),\sigma(t))\Delta_1 s\Big]\Delta_2 t$$
$$= (1-\lambda)^2\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j)+\frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)(f(\alpha_i,y_j)+f(\alpha_{i+1},y_j))$$
$$-(1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s$$
$$+\frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(x_i,\beta_j)+f(x_i,\beta_{j+1})\Big)$$
$$+\frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big)$$
$$-\frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j)+f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s$$
$$-(1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2 t-\frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t$$
$$+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$

$$= (1-\lambda)^2 \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i) f(x_i, y_j) + \frac{(1-\lambda)\lambda}{2} \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i, y_j)$$

$$+ f(\alpha_{i+1}, y_j) + f(x_i, \beta_j) + f(x_i, \beta_{j+1})\Big) - (1-\lambda) \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j) f(\sigma(s), y_j)\Delta_1 s$$

$$+ \frac{\lambda^2}{4} \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i, \beta_j) + f(\alpha_i, \beta_{j+1}) + f(\alpha_{i+1}, \beta_j) + f(\alpha_{i+1}, \beta_{j+1})\Big)$$

$$- \frac{\lambda}{2} \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j)\Big(f(\sigma(s), \beta_j) + f(\sigma(s), \beta_{j+1})\Big)\Delta_1 s - (1-\lambda) \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i) f(x_i, \sigma(t))\Delta_2 t$$

$$- \frac{\lambda}{2} \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i, \sigma(t)) + f(\alpha_{i+1}, \sigma(t))\Big)\Delta_2 t + \int_a^b \int_c^d f(\sigma(s), \sigma(t))\Delta_2 t \Delta_1 s.$$

This gives the desired result. □

Remark 13. *By taking $k = 2$ in Lemma 12 and letting*

$$\begin{cases} x_1 = x,\ x_0 = \alpha_0 = \alpha_1 = a,\ \alpha_2 = \alpha_3 = x_2 = b \\ y_1 = y,\ y_0 = \beta_0 = \beta_1 = c,\ \beta_2 = \beta_3 = y_2 = d, \end{cases}$$

we recapture [19, Lemma 2.1].

We are now in position to formulate and prove our first result.

Theorem 14. *Suppose the function f satisfies the conditions of Lemma 12; and that for each $i \in \{0, 1, 2, \ldots, k-1\}$, $\alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}$ and $\alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}$ belong to $\mathbb{T}_1$, and $\beta_{i+1} - \lambda\frac{\beta_{i+1}-\beta_i}{2}$ and $\beta_{i+1} + \lambda\frac{\beta_{i+2}-\beta_{i+1}}{2}$ belong to $\mathbb{T}_2$. Then we have the inequality*

$$\Bigg|(1-\lambda)^2 \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i) f(x_i, y_j) + \frac{(1-\lambda)\lambda}{2} \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i, y_j)$$

$$+ f(\alpha_{i+1}, y_j) + f(x_i, \beta_j) + f(x_i, \beta_{j+1})\Big) - (1-\lambda) \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j) f(\sigma(s), y_j)\Delta_1 s$$

$$+ \frac{\lambda^2}{4} \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i, \beta_j) + f(\alpha_i, \beta_{j+1}) + f(\alpha_{i+1}, \beta_j) + f(\alpha_{i+1}, \beta_{j+1})\Big)$$

$$- \frac{\lambda}{2} \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j)\Big(f(\sigma(s), \beta_j) + f(\sigma(s), \beta_{j+1})\Big)\Delta_1 s - (1-\lambda) \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i) f(x_i, \sigma(t))\Delta_2 t$$

$$- \frac{\lambda}{2} \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i, \sigma(t)) + f(\alpha_{i+1}, \sigma(t))\Big)\Delta_2 t + \int_a^b \int_c^d f(\sigma(s), \sigma(t))\Delta_2 t \Delta_1 s\Bigg|$$

$$\leq M\Bigg\{ \sum_{i=0}^{k-1} \Big[h_2\Big(x_i, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$

$$+ h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big]$$

$$\times \sum_{j=0}^{k-1} \Big[h_2\Big(y_j, \beta_{j+1} - \lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) + h_2\Big(\beta_{j+1}, \beta_{j+1} - \lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)$$

$$+ h_2\Big(\beta_{j+1}, \beta_{j+1} + \lambda\frac{\beta_{j+2} - \beta_{j+1}}{2}\Big) + h_2\Big(y_{j+1}, \beta_{j+1} + \lambda\frac{\beta_{j+2} - \beta_{j+1}}{2}\Big)\Big]\Big\}, \tag{7}$$

where $M = \sup_{a<s<b, c<t<d} \left|f^{\Delta_1\Delta_2}(s,t)\right| < \infty.$

Proof. By applying Lemma 12, and item (v) of Theorem 6 we get

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j)\\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s\\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j) + f(\alpha_i,\beta_{j+1}) + f(\alpha_{i+1},\beta_j) + f(\alpha_{i+1},\beta_{j+1})\Big)\\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j) + f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s - (1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2 t\\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t)) + f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t + \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg|\\
&\le M\int_a^b\int_c^d |K(s,t,I_k,J_k)|\Delta_2 t\Delta_1 s = M\int_a^b |K_1(s,I_k)|\Delta_1 s\int_c^d |K_2(t,J_k)|\Delta_2 t.
\end{aligned} \tag{8}$$

Now, we observe that

$$\begin{aligned}
\int_a^b |K_1(s,I_k)|\Delta_1 s = \sum_{i=0}^{k-1}\Big[&h_2\Big(x_i, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\\
&+ h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big]
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\int_c^d |K_2(t,J_k)|\Delta_2 t = \sum_{j=0}^{k-1}\Big[&h_2\Big(y_j, \beta_{j+1} - \lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) + h_2\Big(\beta_{j+1}, \beta_{j+1} - \lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)\\
&+ h_2\Big(\beta_{j+1}, \beta_{j+1} + \lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) + h_2\Big(y_{j+1}, \beta_{j+1} + \lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big].
\end{aligned} \tag{10}$$

The inequality in (7) is obtained by substituting (9) and (10) into (8). This completes the proof. □

Remark 15. *It is important to note that Theorem 14 is an extension of Theorem 10 to the 2D case. Furthermore, by taking* $\lambda = 0$ *in Theorem 14, we get* [7, Theorem 2.1].

Theorem 16. *Suppose the function* f *satisfies the conditions of Theorem 14 and there exist* $M_1, M_2 \in \mathbb{R}$ *such that* $M_1 \le f^{\Delta_1\Delta_2}(s,t) \le M_2$ *for all* $s \in [a,b]_{\mathbb{T}_1}, t \in [c,d]_{\mathbb{T}_2}$. *Then we have the inequality*

$$\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j)$$

$$+ f(\alpha_{i+1}, y_j) + f(x_i, \beta_j) + f(x_i, \beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b (\beta_{j+1}-\beta_j) f(\sigma(s), y_j)\Delta_1 s$$

$$+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j) + f(\alpha_i,\beta_{j+1}) + f(\alpha_{i+1},\beta_j) + f(\alpha_{i+1},\beta_{j+1})\Big)$$

$$- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b (\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j) + f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s - (1-\lambda)\sum_{i=0}^{k}\int_c^d (\alpha_{i+1}-\alpha_i) f(x_i,\sigma(t))\Delta_2 t$$

$$- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d (\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t)) + f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t + \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$

$$- \frac{M_1+M_2}{2}\sum_{i=0}^{k-1}\Big[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$

$$+ h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big]$$

$$\times\sum_{j=0}^{k-1}\Big[h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) - h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)$$

$$+ h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Bigg|$$

$$\le \frac{M_2-M_1}{2}\Bigg\{\sum_{i=0}^{k-1}\Big[h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$

$$+ h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big]$$

$$\times\sum_{j=0}^{k-1}\Big[h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) + h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)$$

$$+ h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) + h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Bigg\}. \tag{11}$$

Proof. By a simple computation, one gets

$$\int_a^b K_1(s,I_k)\Delta_1 s = \sum_{i=0}^{k-1}\Big[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+ h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \tag{12}$$

and

$$\int_c^d K_2(t,J_k)\Delta_2 t = \sum_{j=0}^{k-1}\Big[h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) - h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)$$
$$+ h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]. \tag{13}$$

Using (12) and (13), we have that

$$\int_a^b\int_c^d K(s,t,I_k,J_k)\Delta_2 t\Delta_1 s = \int_a^b K_1(s,I_k)\Delta_1 s\int_c^d K_2(t,J_k)\Delta_2 t$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} \Big[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1} - \alpha_i}{2}\Big) - h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1} - \alpha_i}{2}\Big) \\
&+ h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big)\Big] \\
&\times \sum_{j=0}^{k-1} \Big[h_2\Big(\beta_{j+1}, \beta_{j+1} - \lambda \frac{\beta_{j+1} - \beta_j}{2}\Big) - h_2\Big(y_j, \beta_{j+1} - \lambda \frac{\beta_{j+1} - \beta_j}{2}\Big) \\
&+ h_2\Big(y_{j+1}, \beta_{j+1} + \lambda \frac{\beta_{j+2} - \beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1}, \beta_{j+1} + \lambda \frac{\beta_{j+2} - \beta_{j+1}}{2}\Big)\Big]. \qquad (14)
\end{aligned}
$$

From the assumption that $M_1 \leq f^{\Delta_1\Delta_2}(s,t) \leq M_2$ for all $s \in [a,b]_{\mathbb{T}_1}, t \in [c,d]_{\mathbb{T}_2}$, it follows that

$$
\sup_{a<s<b, c<t<d} \left| f^{\Delta_1\Delta_2}(s,t) - \frac{M_1 + M_2}{2} \right| \leq \frac{M_2 - M_1}{2}. \qquad (15)
$$

On the other hand,

$$
\begin{aligned}
&\int_a^b \int_c^d K(s,t,I_k,J_k) f^{\Delta_1\Delta_2}(s,t)\Delta_2 t\Delta_1 s - \frac{M_1+M_2}{2} \int_a^b \int_c^d K(s,t,I_k,J_k)\Delta_2 t\Delta_1 s \\
&\qquad = \int_a^b \int_c^d K(s,t,I_k,J_k)\Big(f^{\Delta_1\Delta_2}(s,t) - \frac{M_1+M_2}{2}\Big)\Delta_2 t\Delta_1 s. \qquad (16)
\end{aligned}
$$

By applying item (v) of Theorem 6 to (16) and using (15), we have

$$
\begin{aligned}
&\Big|\int_a^b \int_c^d K(s,t,I_k,J_k) f^{\Delta_1\Delta_2}(s,t)\Delta_2 t\Delta_1 s - \frac{M_1+M_2}{2} \int_a^b \int_c^d K(s,t,I_k,J_k)\Delta_2 t\Delta_1 s\Big| \\
&\qquad \leq \frac{M_2 - M_1}{2} \int_a^b \int_c^d |K(s,t,I_k,J_k)|\Delta_2 t\Delta_1 s. \qquad (17)
\end{aligned}
$$

The desired inequality follows from (17) by using Lemma 12 and (14). □

Remark 17. *If we take $\lambda = 0$ in Theorem 16, we recapture* [7, Theorem 2.6].

Next, we establish a generalized Ostrowski–Grüss type inequality on time scales for double integrals for k^2 points via a parameter. For this, we will need the following 2-dimensional Grüss inequality on time scales.

Lemma 18 (see [7]). *Let $f, g \in C_{rd}\big([a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}, \mathbb{R}\big)$ such that $\phi \leq f(x,y) \leq \Phi$ and $\gamma \leq g(x,y) \leq \Gamma$ for all $x \in [a,b]_{\mathbb{T}_1}, y \in [c,d]_{\mathbb{T}_2}$, where ϕ, Φ, γ, Γ are constants. Then we have*

$$
\begin{aligned}
&\Big|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)g(s,t)\Delta_2 t\Delta_1 s - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\Delta_2 t\Delta_1 s \frac{1}{(b-a)(d-c)} \\
&\times \int_a^b \int_c^d g(s,t)\Delta_2 t\Delta_1 s\Big| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \qquad (18)
\end{aligned}
$$

We now state and prove our last main theorem of this paper.

Theorem 19. *Suppose the conditions of Theorem 16 hold. Then we have*

$$
\Big|(1-\lambda)^2 \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1} - \beta_j)(\alpha_{i+1} - \alpha_i) f(x_i, y_j) + \frac{(1-\lambda)\lambda}{2} \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1} - \beta_j)(\alpha_{i+1} - \alpha_i)\Big(f(\alpha_i, y_j)
$$

$$
\begin{aligned}
&+ f(\alpha_{i+1}, y_j) + f(x_i, \beta_j) + f(x_i, \beta_{j+1})\Big) - (1-\lambda) \sum_{j=0}^{k} \int_a^b (\beta_{j+1} - \beta_j) f(\sigma(s), y_j) \Delta_1 s \\
&+ \frac{\lambda^2}{4} \sum_{j=0}^{k} \sum_{i=0}^{k} (\beta_{j+1} - \beta_j)(\alpha_{i+1} - \alpha_i)\Big(f(\alpha_i, \beta_j) + f(\alpha_i, \beta_{j+1}) + f(\alpha_{i+1}, \beta_j) + f(\alpha_{i+1}, \beta_{j+1})\Big) \\
&- \frac{\lambda}{2} \sum_{j=0}^{k} \int_a^b (\beta_{j+1} - \beta_j)\Big(f(\sigma(s), \beta_j) + f(\sigma(s), \beta_{j+1})\Big)\Delta_1 s - (1-\lambda) \sum_{i=0}^{k} \int_c^d (\alpha_{i+1} - \alpha_i) f(x_i, \sigma(t)) \Delta_2 t \\
&- \frac{\lambda}{2} \sum_{i=0}^{k} \int_c^d (\alpha_{i+1} - \alpha_i)\Big(f(\alpha_i, \sigma(t)) + f(\alpha_{i+1}, \sigma(t))\Big)\Delta_2 t + \int_a^b \int_c^d f(\sigma(s), \sigma(t)) \Delta_2 t \Delta_1 s \\
&- \frac{f(b,d) - f(a,d) - f(b,c) + f(a,c)}{(b-a)(d-c)} \\
&\times \sum_{i=0}^{k-1} \Big[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1} - \alpha_i}{2}\Big) - h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1} - \alpha_i}{2}\Big) \\
&+ h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big)\Big] \\
&\times \sum_{j=0}^{k-1} \Big[h_2\Big(\beta_{j+1}, \beta_{j+1} - \lambda \frac{\beta_{j+1} - \beta_j}{2}\Big) - h_2\Big(y_j, \beta_{j+1} - \lambda \frac{\beta_{j+1} - \beta_j}{2}\Big) \\
&+ h_2\Big(y_{j+1}, \beta_{j+1} + \lambda \frac{\beta_{j+2} - \beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1}, \beta_{j+1} + \lambda \frac{\beta_{j+2} - \beta_{j+1}}{2}\Big)\Big]\Bigg| \\
&\le \frac{[(b-a)(d-c)]^2}{4}(M_2 - M_1). \qquad (19)
\end{aligned}
$$

Proof. By Lemma 18, we have

$$
\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(s,t,I_k,J_k) f^{\Delta_1\Delta_2}(s,t) \Delta_2 t \Delta_1 s \\
&- \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(s,t,I_k,J_k) \Delta_2 t \Delta_1 s \\
&\times \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f^{\Delta_1\Delta_2}(s,t) \Delta_2 t \Delta_1 s\Bigg| \\
&\le \frac{1}{4}\Big(\sup K(s,t,I_k,J_k) - \inf K(s,t,I_k,J_k)\Big)(M_2 - M_1). \qquad (20)
\end{aligned}
$$

We also have that,

$$
\int_a^b \int_c^d f^{\Delta_1\Delta_2}(s,t) \Delta_2 t \Delta_1 s = f(b,d) - f(a,d) - f(b,c) + f(a,c). \qquad (21)
$$

On the other hand, we observe that

$$
\sup K(s,t,I_k,J_k) - \inf K(s,t,I_k,J_k) \le (b-a)(d-c). \qquad (22)
$$

The desired inequality is obtained by combining (4), (14), (20), (21) and (22). □

Remark 20. *Theorem 19 amounts to* [7, Theorem 2.9] *for the case* $\lambda = 0$.

3. Applications

In this section, we apply Theorem 16 to the continuous, discrete, and quantum calculus. Similar results can be obtained from Theorems 14 and 19.

Corollary 21 (Continuous case). *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a function such that $M_1 \le \frac{\partial^2 f(s,t)}{\partial s \partial t} \le M_2$ for all $(s,t) \in [a,b]\times[c,d]$ and some $M_1, M_2 \in \mathbb{R}$. Then we have the inequality*

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i) f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b (\beta_{j+1}-\beta_j) f(s,y_j)ds \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j) + f(\alpha_i,\beta_{j+1}) + f(\alpha_{i+1},\beta_j) + f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b (\beta_{j+1}-\beta_j)\Big(f(s,\beta_j)+f(s,\beta_{j+1})\Big)ds - (1-\lambda)\sum_{i=0}^{k}\int_c^d (\alpha_{i+1}-\alpha_i) f(x_i,t)dt \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d (\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,t)+f(\alpha_{i+1},t)\Big)dt + \int_a^b\int_c^d f(s,t)dtds \\
&- \frac{M_1+M_2}{128}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2 - \Big(2x_i - \lambda\alpha_i + (\lambda-2)\alpha_{i+1}\Big)^2 \\
&+ \Big(2x_{i+1} - \lambda\alpha_{i+2} + (\lambda-2)\alpha_{i+1}\Big)^2 - \lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big] \\
&\times \sum_{j=0}^{k-1}\Big[\lambda^2\Big(\beta_{i+1}-\beta_i\Big)^2 - \Big(2y_i - \lambda\beta_i + (\lambda-2)\beta_{i+1}\Big)^2 \\
&+ \Big(2y_{i+1} - \lambda\beta_{i+2} + (\lambda-2)\beta_{i+1}\Big)^2 - \lambda^2\Big(\beta_{i+2}-\beta_{i+1}\Big)^2\Big]\Bigg| \\
&\le \frac{M_2-M_1}{128}\Bigg\{\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2 + \Big(2x_i - \lambda\alpha_i + (\lambda-2)\alpha_{i+1}\Big)^2 \\
&+ \Big(2x_{i+1} - \lambda\alpha_{i+2} + (\lambda-2)\alpha_{i+1}\Big)^2 + \lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big] \\
&\times \sum_{j=0}^{k-1}\Big[\lambda^2\Big(\beta_{i+1}-\beta_i\Big)^2 + \Big(2y_i - \lambda\beta_i + (\lambda-2)\beta_{i+1}\Big)^2 \\
&+ \Big(2y_{i+1} - \lambda\beta_{i+2} + (\lambda-2)\beta_{i+1}\Big)^2 + \lambda^2\Big(\beta_{i+2}-\beta_{i+1}\Big)^2\Big]\Bigg\}.
\end{aligned} \tag{23}$$

Proof. The proof follows by setting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 16 and using the fact that

$$h_2(s,t) = \frac{(s-t)^2}{2}.$$

□

Remark 22. *By taking $k=2$ in Corollary 21 and letting*

$$\begin{cases} x_1 = x,\ x_0 = \alpha_0 = \alpha_1 = a,\ \alpha_2 = \alpha_3 = x_2 = b \\ y_1 = y,\ y_0 = \beta_0 = \beta_1 = c,\ \beta_2 = \beta_3 = y_2 = d, \end{cases}$$

we get [18, Theorem 4].

Corollary 23 (Discrete case). *Let $f : \{a, a+1, \cdots, b-1, b\} \times \{c, c+1, \cdots, d-1, d\} \to \mathbb{R}$ be a function such that*

$$M_1 \le f(s+1,t+1) - f(s+1,t) - f(s,t+1) + f(s,t) \le M_2$$

for all $(s,t) \in \{a, a+1, \cdots, b-1, b\} \times \{c, c+1, \cdots, d-1, d\}$ and for some $M_1, M_2 \in \mathbb{R}$. Then the following inequality holds.

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i) f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\sum_{s=a}^{b-1}(\beta_{j+1}-\beta_j) f(s+1,y_j) \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j) + f(\alpha_i,\beta_{j+1}) + f(\alpha_{i+1},\beta_j) + f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\sum_{s=a}^{b-1}(\beta_{j+1}-\beta_j)\Big(f(s+1,\beta_j) + f(s+1,\beta_{j+1})\Big) - (1-\lambda)\sum_{i=0}^{k}\sum_{t=c}^{d-1}(\alpha_{i+1}-\alpha_i) f(x_i,t+1) \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\sum_{t=c}^{d-1}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,t+1) + f(\alpha_{i+1},t+1)\Big) + \sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1) \\
&- \frac{M_1+M_2}{2}\sum_{i=0}^{k-1}\Big[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \\
&\times \sum_{j=0}^{k-1}\Big[h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) - h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Bigg| \\
&\le \frac{M_2-M_1}{2}\Bigg\{\sum_{i=0}^{k-1}\Big[h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \\
&\times \sum_{j=0}^{k-1}\Big[h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) + h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) + h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Bigg\}, \qquad (24)
\end{aligned}$$

where $h_2(s,t) = \frac{(s-t)(s-t-1)}{2}$ for all $s,t \in \mathbb{Z}$.

Proof. We get the intended result by letting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorem 16. □

Corollary 24 (Quantum case). *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}, \mathbb{T}_2 = q_2^{\mathbb{N}_0}, q_1, q_2 > 1$ *in Theorem 16*

$$\begin{aligned}
&\Big|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(q_1 s,y_j)d_q s \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(q_1 s,\beta_j)+f(q_1 s,\beta_{j+1})\Big)d_q s - (1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,q_2 t)d_q t \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,q_2 t)+f(\alpha_{i+1},q_2 t)\Big)d_q t + \int_a^b\int_c^d f(q_1 s,q_2 t)d_q t d_q s \\
&- \frac{M_1+M_2}{2}\sum_{i=0}^{k-1}\Big[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \\
&\times\sum_{j=0}^{k-1}\Big[h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) - h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Big| \\
&\le \frac{M_2-M_1}{2}\Big\{\sum_{i=0}^{k-1}\Big[h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \\
&\times\sum_{j=0}^{k-1}\Big[h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) + h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) + h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Big\},
\end{aligned} \tag{25}$$

where $h_2(s,t) = \frac{(s-t)(s-qt)}{q+1}$, *for all* $t, s \in q^{\mathbb{N}_0}$.

Conclusion

In this paper we have proved generalizations of the results of Feng and Meng [7] via a parameter $\lambda \in [0,1]$. Our first theorem extends the main result in the paper [17] to the 2-dimensional case. As special cases of our results, we recapture results of Xue et al. [18], and Zheng [19]. More results can be obtained by applying Theorems 14 and 19 to the continuous, discrete and quantum calculus.

Acknowledgement

Many thanks to the anonymous referee(s).

References

[1] M. Bohner, T. Matthews, Ostrowski Inequalities on time scales, J. Inequal. Pure and Appl. Math. 9 (1) (2008) Art. 6.
[2] M. Bohner, A. Peterson, Dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2001.
[3] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Series, Birkhäuser Boston, Boston, MA, 2003.
[4] M. Bohner, G. S. Guseinov, Partial differentiation on time scales, Dyn. Syst. Appl. 13 (3–4) (2004) 351–379.
[5] M. Bohner, G. S. Guseinov, Multiple integration on time scales, Dyn. Syst. Appl. 14 (3–4) (2005) 579–606.
[6] S. S. Dragomir, Grüss inequality in inner product spaces, The Aust. Math. Soc. Gazette. 26 (2) (1999) 66–70.
[7] Q. Feng, F. Meng, Generalized Ostrowski type inequalities for multiple points on time scales involving functions of two independent variables, J. Inequal. Appl. (2012) 2012:74.
[8] S. Hilger, Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, Würzburg, Germany 1988.
[9] B. Karpuz, U. M. Özkan, Ostrowski Inequality on time scales, J. Inequal. Pure and Appl. Math. 9 (4) (2008) Art. 112.
[10] S. Kermausuor, E. R. Nwaeze, D. F. M. Torres, Generalized weighted Ostrowski and Ostrowski–Grüss type inequalities on time scale via a parameter function, J. Math. Inequal. 11 (4) (2017) 1185–1199.
[11] W. J. Liu, A. Ngô, A generalization of Ostrowski inequality on time scales for k points, Appl. Math. Comput. 203 (2) (2008) 754–760.
[12] E. R. Nwaeze, A new weighted Ostrowski type inequality on arbitrary time scale, J. King Saud Uni. Sci. 29 (2) (2017) 230–234.
[13] E. R. Nwaeze, Generalized weighted trapezoid and Grüss type inequalities on time scales, Aust. J. Math. Anal. Appl. 11 (1) (2017) Art. 4.
[14] E. R. Nwaeze, A. M. Tameru, On weighted Montgomery identity for k points and its associates on time scales, Abstr. App. Anal. (2017) Art. ID 5234181.
[15] E. R. Nwaeze, S. Kermausuor, A. M. Tameru, New time scale generalizations of the Ostrowski–Grüss type inequality for k points, J. Inequal. Appl. (2017) 2017:245.
[16] A. Tuna, D. Daghan, Generalization of Ostrowski and Ostrowski–Grüss type inequalities on time scales, Comput. Math. Appl. 60 (2010) 803–811.
[17] G. Xu, Z. B Fang, A Generalization of Ostrowski type inequality on time scales with k points, J. Math. Inequal. 11 (1) (2017) 41–48.
[18] Q. Xue, J. Zhu, W. J. Liu, A new generalization of Ostrowski-type inequality involving functions of two independent variables, Comput. Math. Appl. 60 (2010) 2219–2224.
[19] B. Zheng, Some new generalized 2D Ostrowski-Grüss type inequalities on time scales, Arab J. Math. Sci. 19 (2) (2013) 159172.