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Attaching a Topological Space to a Module

Dawood Hassanzadeh-Lelekaami[a]

[a]*Department of Basic Sciences, Arak University of Technology, Arak, Iran.*

Abstract. Let R be a commutative ring with identity and let M be an R-module. We investigate when the strongly prime spectrum of M has a Zariski topology analogous to that for R. We provide some examples of such modules.

1. Introduction

In this paper all rings are commutative with nonzero identity and all modules are unital. Throughout R will denote an arbitrary ring unless stated otherwise.

Recall that the spectrum $\mathrm{Spec}(R)$ of a ring R consists of all prime ideals of R. For any ideal I of R, we set $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. Then the sets $V(I)$, where I is an ideal of R, satisfy the axioms for the closed sets of a topology on $\mathrm{Spec}(R)$, called the *Zariski topology* (see, for example, [7]). In this paper, our concern is to extend this notion to modules. First we need to define what we shall mean by a (strongly) prime submodule of a module.

Let M be an R-module and N be a submodule of M. Then $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(0_M :_R M)$. If there is no ambiguity, we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$). N is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If N is prime, then ideal $\mathfrak{p} := (N : M)$ is a prime ideal of R. In this case, N is said to be $\mathfrak{p}$-*prime* (see [11, 15]). Naghipour in [18] defined a proper submodule N of an R-module M to be *strongly prime* if $((N + Rx) :_R M)y \subseteq N$ implies $x \in N$ or $y \in N$ for $x, y \in M$. The set of all strongly prime submodules of an R-module M is called the *strongly prime spectrum* of M and is denoted by X_M. Prime submodules and its variants such as strongly prime submodule [18], weakly prime submodules [6], almost prime submodules [10] and so on have been studied recently in a number of papers, for a common generalization of these concepts we refer the reader to [9]. By [18, Proposition 1.1] every strongly prime submodule is prime.

In section 2, we introduce modules whose strongly prime spectrum admits a topology. We investigate its algebraic properties and relationship with other type of modules in Theorems 2.1 and 2.2. Also, we study the Noetherian property of the topological space X_M (see Theorem 2.3). To provide more examples of modules that X_M admits a topology we introduce a new family of modules. Section 3 is devoted to a study of this new family of modules in details. In particular, we show that distributive modules satisfy the new family of modules. Also, we generalize some results of previous literature. The results are supported by examples.

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Email address: lelekaami@gmail.com and Dhmath@arakut.ac.ir (Dawood Hassanzadeh-Lelekaami)

2. Topology on X_M

For any submodule N of an R-module M we define $V(N)$ to be the set of all strongly prime submodules of M containing N. Of course, $V(M) = \emptyset$ just the empty set and $V(0)$ is X_M. Note that for any family of submodules $\{N_i\}_{i\in I}$ of M,

$$\bigcap_{i\in I} V(N_i) = V(\sum_{i\in I} N_i).$$

Thus if $Z(M)$ denotes the collection of all subsets $V(N)$ of X_M then $Z(M)$ contains the empty set and X_M, and $Z(M)$ is closed under arbitrary intersections. We shall say that M is a *module with a Zariski topology* if $Z(M)$ is closed under finite unions, i.e. for any submodules N and L of M there exists a submodule J of M such that $V(N) \cup V(L) = V(J)$, for in this case $Z(M)$ satisfies the axioms for the closed subsets of a topological space. Note that we are not excluding the trivial case where X_M is empty.

In this section, we are going to give examples of modules with a Zariski topology and we are interested in investigating the algebraic properties of this class of modules as well as topological properties of X_M.

A submodule N of M is said to be *strongly semiprime* if it is an intersection of strongly prime submodules (e.g. every proper submodule of a co-semisimple module is a strongly-semiprime submodule (see [2, p.122])). Our definition differs from that of [18]. Let $\mathcal{A}(R)$ denote the set of all R-modules M such that either $X_M = \emptyset$ or for every strongly prime submodule P of M and strongly semiprime submodules N and L of M with $N \cap L \subseteq P$ we infer that either $L \subseteq P$ or $N \subseteq P$. Let N be a submodule of an R-module M. Then we define the *strongly prime radical* of N as $S\mathrm{rad}(N) = \bigcap_{P\in V(N)} P$.

Lemma 2.1. *Let M be an R-module. Then M is a module with a Zariski topology if and only if $M \in \mathcal{A}(R)$.*

Proof. ($\Rightarrow$) Let P be any strongly prime submodule of M and let N and L be strongly semiprime submodules of M such that $N \cap L \subseteq P$. By assumption, there exists a submodule H of M such that $V(N) \cup V(L) = V(H)$. There is a collection of strongly prime submodules $\{Q_i\}_{i\in I}$ such that $N = \bigcap_{i\in I} Q_i$. Thus, for each $i \in I$, $Q_i \in V(N) \subseteq V(H)$. Hence, $H \subseteq N$. Similarly $H \subseteq L$. Therefore, $H \subseteq N \cap L$. Now $V(N) \cup V(L) \subseteq V(N \cap L) \subseteq V(H) = V(N) \cup V(L)$. It follows that $V(N) \cup V(L) = V(N \cap L)$. But $P \in V(N \cap L)$ now gives $P \in V(N)$ or $P \in V(L)$, i.e. $N \subseteq P$ or $L \subseteq P$.

($\Leftarrow$) Let N and L be any submodules of M such that $V(N)$ and $V(L)$ are both non-empty. Then $V(N)\cup V(L) = V(S\mathrm{rad}(N)) \cup V(S\mathrm{rad}(L)) = V(S\mathrm{rad}(N) \cap S\mathrm{rad}(L))$, since $M \in \mathcal{A}(R)$. □

Example 2.2. *We give an example of a module M such that $Z(M)$ is not a topological space. Consider $M = \mathbb{R}^2$ as a vector space over the field of real numbers $\mathbb{R}$. Suppose that L_1, L_2 and L_3 are three distinct lines (maximal subspaces) of M. By [18, Proposition 1.3], L_1, L_2 and L_3 are strongly prime submodules of M. Note that $L_1 \cap L_2 \subseteq L_3$ but $L_1 \nsubseteq L_3$ and $L_2 \nsubseteq L_3$. Thus, $M \notin \mathcal{A}(R)$. It follows from Lemma 2.1 that M is not a module with a Zariski topology.*

In the next lemma we show that if M is a module with a Zariski topology over a filed, then it is a one dimensional vector space.

Lemma 2.3. *Let R be a field and M be an R-module with a strongly prime submodule P of M such that for every strongly semiprime submodules N and L of M with $N \cap L \subseteq P$ we deduce that either $L \subseteq P$ or $N \subseteq P$. Then M is a one-dimensional vector space over R.*

Proof. Suppose that M is not one-dimensional. Then $P \neq 0$, by [18, Proposition 1.3]. We take a non-zero element $p \in P$. Since $P \neq M$, there exists an element $m \in M \setminus P$. Since in a vector space over a field, every proper subspace is the intersection of all maximal (proper) subspaces which contain it (see [7, p.297, Proposition 8]), by [18, Proposition 1.3], Rm and $R(m+p)$ are strongly semiprime with $Rm \cap R(m+p) = 0 \subseteq P$. But $Rm \nsubseteq P$ and $R(m + p) \nsubseteq P$. This is a contradiction. □

Lemma 2.4. *Let M be an R-module and N be a submodule of M. Then*

$$X_{M/N} = \{P/N \,|\, P \in X_M,\ N \subseteq P\}.$$

Proof. This follows from [9, Lemma 2.9]. □

Proposition 2.5. *Let $M \in \mathcal{A}(R)$.*

1. *Any homomorphic image of M belongs to $\mathcal{A}(R)$.*
2. *$M/\mathfrak{m}M$ is a cyclic module for any maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) We have to show that for every submodule N of M, $M/N \in \mathcal{A}(R)$. By Lemma 2.4, every strongly prime submodule of M/N is in the form P/N where P is a strongly prime submodule of M and $N \subseteq P$. Thus any strongly semiprime submodule of M/N has the form Q/N where Q is a strongly semiprime submodule containing N. Now, the result immediately follows.

(2) We may assume that $M \neq \mathfrak{m}M$. Then $M/\mathfrak{m}M$ is a nonzero vector space over $R/\mathfrak{m}$, and so has a maximal subspace, namely Q. Note that Q is also an R-submodule of $M/\mathfrak{m}M$. Thus there exists a maximal submodule P of M such that $\mathfrak{m}M \subseteq P$ and $P/mM = Q$. Note that Q is a strongly prime submodule of M/mM [18, Proposition 1.3]; so P is strongly prime by Lemma 2.3. Hence $X_M \neq \emptyset$ and $M/\mathfrak{m}M \in \mathcal{A}(R/\mathfrak{m})$. Therefore by Lemma 2.3, $M/\mathfrak{m}M$ is one-dimensional over $R/\mathfrak{m}$, i.e., $M/\mathfrak{m}M$ is a cyclic R-module. □

We recall that an R-module M is said to be a *multiplication* module (see [5] and [8]) if every submodule N of M is of the form IM for some ideal I of R. Let N be a submodule of an R-module M and let $\mathfrak{p}$ be a prime ideal of R. Then the *saturation of N with respect to* $\mathfrak{p}$ is the contraction of $N_{\mathfrak{p}}$ in M and designated by $S_{\mathfrak{p}}(N)$. It is known that $S_{\mathfrak{p}}(N) = \{e \in M \,|\, se \in N \text{ for some } s \in R \setminus \mathfrak{p}\}$. For more details we refer the reader to [13]. The notion of saturation of ideals (or submodules) has been appeared in many literature, such as [3, 13, 19].

Lemma 2.6. *Let M be an R-module and let S be a multiplicatively closed subset of R. Then*

$$X_{S^{-1}M} = \{S^{-1}P \,|\, P \in X_M \text{ and } S^{-1}P \neq S^{-1}M\}.$$

Proof. See [9, Theorem 2.10] or [18, Theorem 1.5]. □

Theorem 2.1. *Consider the following statements for an R-module M.*

1. *M is a multiplication module;*
2. *For every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$;*
3. *M belongs to $\mathcal{A}(R)$;*
4. *$M_{\mathfrak{p}}$ belongs to $\mathcal{A}(R_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p}$ of R;*
5. *M has at most one strongly prime submodule P with $(P : M) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R.*

Then (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5).

Proof. (1) ⇒ (2) This is clear. (2) ⇒ (3) Let N and L be two submodules of M. Then by (2), there exist ideals I and J of R such that $V(N) \cup V(L) = V(IM) \cup V(JM) = V(IJM)$ (see [16, Corollary 3.2]). Thus, M is a module with a Zariski topology. Now the result follows from Lemma 2.1. (3) ⇒ (4) Let P be a strongly prime submodule of $M_{\mathfrak{p}}$. Then by Lemma 2.6, there exists a strongly prime submodule Q of M such that $Q^e := Q_{\mathfrak{p}} = P$. Now let N_1 and N_2 be strongly semiprime submodules of $M_{\mathfrak{p}}$ with $N_1 \cap N_2 \subseteq P$. Then $N_1 \cap M$ and $N_2 \cap M$ are strongly semiprime submodules of M with $(N_1 \cap M) \cap (N_2 \cap M) = (N_1 \cap N_2) \cap M \subseteq P \cap M$. By Lemma 2.6, $P \cap M \in X_M$. Hence, $N_1 \cap M \subseteq P \cap M$ or $N_2 \cap M \subseteq P \cap M$. It follows that $N_1 = (N_1 \cap M)^e \subseteq (P \cap M)^e = P$ or $N_2 \subseteq P$. Thus, $M_{\mathfrak{p}}$ belongs to $\mathcal{A}(R_{\mathfrak{p}})$. (4) ⇒ (5) Let $\mathfrak{p}$ be any prime ideal of R such that there exists a strongly prime submodule P of M with $(P : M) = \mathfrak{p}$. Then $\mathfrak{p}M \subseteq P$. It follows that $(\mathfrak{p}M)^e \subseteq P^e \subseteq M_{\mathfrak{p}}$. By Proposition 2.5, $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is cyclic so that $P^e = (\mathfrak{p}M)^e$ or $P^e = M_{\mathfrak{p}}$. Hence, by [13, Result 2] we have $P = S_{\mathfrak{p}}(P) = P^e \cap M = (\mathfrak{p}M)^e \cap M = S_{\mathfrak{p}}(\mathfrak{p}M)$, or $P = S_{\mathfrak{p}}(M) = M$. Thus, $P = S_{\mathfrak{p}}(\mathfrak{p}M)$. It follows that $P = S_{\mathfrak{p}}(\mathfrak{p}M)$ is the unique strongly prime submodule of M with $(P : M) = \mathfrak{p}$. □

In next theorem we present new examples of modules with the Zariski topology. Let M be an R-module. For every $x \in M$, we define $c(x)$ as the intersection of all ideals I of R such that $x \in IM$. A module M is called a *content* R-module if, for every $x \in M$, $x \in c(x)M$. Every free module, or more generally, every projective module, is a content module [21, p.51]. An R-module M is a content module if and only if for every family $\{A_i | i \in J\}$ of ideals of R, $(\cap_{i\in J} A_i)M = \cap_{i\in J}(A_i M)$ (see [21, p.51]). The equivalence above implies that every faithful multiplication module is also a content module [8, p.758, Theorem 1.6]. Let $\mathcal{B}(R)$ denote the set of all R-modules M such that either $X_M = \emptyset$ or for every strongly prime submodule P of M there exists an ideal I of R such that $P = IM$. In Section 3, we show that $\mathcal{B}(R)$ is non-empty and we investigate properties of elements of $\mathcal{B}(R)$ in details.

Theorem 2.2. *Let M be an R-module.*

1. *If M is content and belongs to $\mathcal{B}(R)$, then $M \in \mathcal{A}(R)$.*
2. *If M is free, then $M \in \mathcal{A}(R)$ if and only if M is cyclic.*
3. *If M is projective, then $M \in \mathcal{A}(R)$ if and only if M is locally cyclic.*
4. *Let $\dim R = 0$ and R has only finitely many prime ideals. If M has at most one strongly prime submodule P with $(P : M) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R, then $M \in \mathcal{A}(R)$.*
5. *If $S\mathrm{rad}(N) = \sqrt{(N : M)}M$, for each submodule N of M, then $M \in \mathcal{A}(R)$.*

Proof. (1) Let L be a submodule of M and let $S\mathrm{rad}(L) = \bigcap_{\lambda\in\Lambda} P_\lambda$, where P_λ is a strongly prime submodule of M such that $\mathfrak{p}_\lambda := (P_\lambda : M)$ for each $\lambda \in \Lambda$. By assumption, for each $\lambda \in \Lambda$, $P_\lambda = \mathfrak{p}_\lambda M$. Since M is a content module, we have

$$S\mathrm{rad}(L) = \bigcap_{\lambda\in\Lambda} P_\lambda = \bigcap_{\lambda\in\Lambda}(\mathfrak{p}_\lambda M) = (\bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda)M = (\bigcap_{\lambda\in\Lambda}(P_\lambda : M))M$$
$$= ((\bigcap_{\lambda\in\Lambda} P_\lambda) : M)M = (S\mathrm{rad}(L) : M)M.$$

Therefore, $V(L) = V(S\mathrm{rad}(L)) = V((S\mathrm{rad}(L) : M)M)$ and the result follows from Theorem 2.1.

(2) If M is a cyclic R-module, then it is a multiplication module (see [5]). Hence, Theorem 2.1 implies that $M \in \mathcal{A}(R)$. Conversely, suppose that $M \in \mathcal{A}(R)$. Let $\{f_\lambda \mid \lambda \in \Lambda\}$ be a basis for M. Let $\mathfrak{m}$ be any maximal ideal of R. Then $M/\mathfrak{m}M$ is a free $R/\mathfrak{m}$-module with basis $\{f_\lambda + \mathfrak{m}M \mid \lambda \in \Lambda\}$. But $M/\mathfrak{m}M$ is cyclic by Proposition 2.5. Thus, M is cyclic.

(3) Let $M \in \mathcal{A}(R)$ and $\mathfrak{p}$ be any prime ideal of R. By Theorem 2.1 the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ belongs to $\mathcal{A}(R_\mathfrak{p})$. Since M is projective, $M_\mathfrak{p}$ is free and so that $M_\mathfrak{p}$ is cyclic by (2). Thus M is locally cyclic. The converse follows from [16, Theorem 4.1].

(4) Assume that $M \notin \mathcal{A}(R)$. Then it follows that there exist strongly semiprime submodules S_1 and S_2 of M and a strongly prime submodule P of M such that $S_1 \cap S_2 \subseteq P$ but $S_1 \nsubseteq P$ and $S_2 \nsubseteq P$. Put $S_1 = \bigcap_{i\in I} P_i$ and $S_2 = \bigcap_{j\in J} Q_j$, where P_i and Q_j are strongly prime submodules of M for each $i \in I$ and $j \in J$. Since $S_1 \nsubseteq P$, we have $P_i \nsubseteq P$ for every $i \in I$. Similarly, $Q_i \nsubseteq P$ for every $j \in J$. On the other hand, we have $(S_1 : M) \subseteq (P : M)$ or $(S_2 : M) \subseteq (P : M)$, because $S_1 \cap S_2 \subseteq P$. Thus $\bigcap_{i\in I}(P_i : M) \subseteq (P : M)$ or $\bigcap_{j\in J}(Q_j : M) \subseteq (P : M)$. By assumption, both I and J are finite index sets. Consequently, there exists $i_0 \in I$ or $j_0 \in J$ such that $(P_{i_0} : M) = (P : M)$ or $(Q_{j_0} : M) = (P : M)$, since $\dim R = 0$. By assumption, $P_{i_0} = P$ or $Q_{j_0} = P$, which is a contradiction.

(5) Let N be a submodule of M. If $V(N) = \emptyset$, then $V(N) = V(RM)$. Otherwise, $V(N) = V(S\mathrm{rad}(N)) = V(\sqrt{(N : M)}M)$. In both cases, the result follows from Theorem 2.1. □

In the next section we will provide another example of a module with a Zariski topology (see Theorem 3.2). In the sequel, we investigate a topological property of X_M. A topological space X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition. Recall that a ring has Noetherian spectrum if and only if the ascending chain condition (ACC) for radical ideals holds [20]. In the next theorem, we generalize this fact to modules that belong to $\mathcal{A}(R)$. Let M be an R-module and Y be a subset of X_M. Then the intersection of all elements in Y is denoted by $\mathfrak{I}(Y)$.

intersections and form an open base; (4) each irreducible closed subset of Y has a generic point.

Theorem 2.3. *Let M be an R-module.*

1. *If $M \in \mathcal{A}(R)$, then X_M is a Noetherian topological space if and only if the ACC holds for strongly prime radical submodules of M.*
2. *If $M \in \mathcal{A}(R)$ and for every submodule N of M there exists a finitely generated submodule L of N such that $S\mathrm{rad}(N) = S\mathrm{rad}(L)$, then X_M is a Noetherian topological space.*
3. *Let R be a Noetherian ring and M be an R-module such that for every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$. Then X_M is a Noetherian topological space.*
4. *If $S\mathrm{rad}(N) = \sqrt{(N : M)}M$, for each submodule N of M, and $\mathrm{Spec}(R)$ is Noetherian, then X_M is a Noetherian topological space.*

Proof. (1) Suppose the ACC for strongly prime radical submodules of M holds. Let $V(N_1) \supseteq V(N_2) \supseteq \cdots$ be a descending chain of closed subsets of X_M, where $N_i \leq M$. Then $\mathfrak{I}(V(N_1)) \subseteq \mathfrak{I}(V(N_2)) \subseteq \cdots$ is an ascending chain of strongly prime radical submodules $\mathfrak{I}(V(N_i)) = S\mathrm{rad}(N_i)$ of M. So, by assumption there exists $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $\mathfrak{I}(V(N_k)) = \mathfrak{I}(V(N_{k+i}))$. Now, we infer that

$$V(N_k) = V(\mathfrak{I}(V(N_k))) = V(\mathfrak{I}(V(N_{k+i}))) = V(N_{k+i}).$$

Hence, the first chain is stationary, i.e., X_M is a Noetherian space.

Conversely, we suppose that X_M is a Noetherian topological space. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of strongly prime radical submodules of M. Thus $N_i = \mathfrak{I}(V(N_i)) = S\mathrm{rad}(N_i)$. Hence $V(N_1) \supseteq V(N_2) \supseteq \cdots$ is a descending chain of closed subsets of X_M. By assumption there is $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $V(N_k) = V(N_{k+i})$. Therefore

$$N_k = S\mathrm{rad}(N_k) = \mathfrak{I}(V(N_k)) = \mathfrak{I}(V(N_{k+i})) = S\mathrm{rad}(N_{k+i}) = N_{k+i}.$$

(2) Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of strongly prime radical submodules of M, and let $N = \bigcup_i N_i$. By assumption, there exists a finitely generated submodule L of N such that $S\mathrm{rad}(N) = S\mathrm{rad}(L)$. Hence there exists a positive integer n such that $L \subseteq N_n$. Then

$$S\mathrm{rad}(N) = S\mathrm{rad}(L) \subseteq S\mathrm{rad}(N_n) \subseteq S\mathrm{rad}(N),$$

so that $N_n = N_{n+1} = N_{n+2} = \cdots$. Thus, M satisfies ACC on strongly prime radical submodules. Hence, by (1), X_M is a Noetherian topological space.

(3) We note that Theorem 2.1 shows that M belongs to $\mathcal{A}(R)$. By [7, p.97, Proposition 9], it is enough for us to show that every open subset of X is quasi-compact. Let H be an open subset of M and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be an open covering of H. Then there are submodules N and N_λ of M such that $H = X \setminus V(N)$ and $E_\lambda = X \setminus V(N_\lambda)$ for each $\lambda \in \Lambda$ and

$$H \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda = X \setminus \bigcap_{\lambda\in\Lambda} V(N_\lambda).$$

By assumption, for each $\lambda \in \Lambda$, we may set $V(N_\lambda) = V(J_\lambda M)$, where J_λ is an ideal of R. Then

$$H \subseteq X \setminus V(\sum_{\lambda\in\Lambda} J_\lambda M) = X \setminus V((\sum_{\lambda\in\Lambda} J_\lambda)M).$$

Since R is a Noetherian ring, there exists a finite subset Λ' of Λ such that $H \subseteq \bigcup_{\lambda\in\Lambda'} E_\lambda$. Hence, H is quasi-compact. Therefore X is a Noetherian space.

(4) We note that according to Theorem 2.2, M belongs to $\mathcal{A}(R)$. Let $V(N_1) \supseteq V(N_2) \supseteq \cdots$ be a descending chain of closed subsets of X_M. Then, we have $S\mathrm{rad}(N_1) \subseteq S\mathrm{rad}(N_2) \subseteq \cdots$. For each $i \in \mathbb{N}$, there is a set Λ_i, such that $S\mathrm{rad}(N_i) = \bigcap_{\lambda\in\Lambda_i} P_\lambda$, where for every $\lambda \in \Lambda_i$, $P_\lambda \in V(N_i)$. We may assume that $\Lambda_i \neq \emptyset$ for each $i \in \mathbb{N}$. If for each $i \in \mathbb{N}$ there exists some λ_i such that $\lambda_i \in \Lambda_i$ and $(P_{\lambda_i} : M) = (0)$, then $\sqrt{(0)}M = S\mathrm{rad}(N_1) = S\mathrm{rad}(N_2) = \cdots$ and we are done. If not, suppose that j is the largest index such that $(P_\lambda : M) = (0)$ for some $\lambda \in \Lambda_j$ (if there is no such index, put $j = 0$). Hence, $\sqrt{(0)}M = S\mathrm{rad}(N_1) = S\mathrm{rad}(N_2) = \cdots = S\mathrm{rad}(N_j)$. By assumption, for each $k > j$ and $\lambda \in \Lambda_k$, $S\mathrm{rad}(N_\lambda) = \sqrt{(N_\lambda : M)}M$. Since $\mathrm{Spec}(R)$ is Noetherian, the ascending chain $\sqrt{(N_k : M)} \subseteq \sqrt{(N_{k+1} : M)} \subseteq \cdots$ of radical ideals must be stationary (see [20]). This implies that there exists $h \geq k$ such that $V(N_h) = V(N_{h+1}) = \cdots$. Thus X_M is a Noetherian topological space. $\square$

Corollary 2.7. *Let M be an R-module such that $M \in \mathcal{A}(R)$. If M is Noetherian or Artinian, then X_M is a Noetherian topological space.*

Proof. Use Theorem 2.3 and [17, p.1367, Proposition 18]. □

3. Modules Belong to $\mathcal{B}(R)$

The Part (1) of Theorem 2.2 is a motivation for us to investigate some properties of modules in $\mathcal{B}(R)$. One can easily show that if $M \in \mathcal{B}(R)$, then $P = (P : M)M$ for every strongly prime submodule P of M.

Proposition 3.1. *Let $M \in \mathcal{B}(R)$ and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then there is at most one strongly prime submodule P of M such that $(P : M) = \mathfrak{p}$.*

Proof. We may assume that X_M is non-empty. Let P and Q be two strongly prime submodules of M such that $(P : M) = (Q : M) = \mathfrak{p}$. Since $M \in \mathcal{B}(R)$,

$$P = (P : M)M = (Q : M)M = Q,$$

as desired. □

The following result is a direct consequence of Lemma 2.1, Theorem 2.2(4) and Proposition 3.1.

Corollary 3.2. *Let R be a zero dimensional ring with only finitely many prime ideals and M be an R-module such that $M \in \mathcal{B}(R)$. Then M is a module with Zariski topology.*

An R-module M is called *weak multiplication* if every prime submodule P of M is of the form IM for some ideal I of R (see [1] and [4]). By definition, if M is a weak multiplication R-module, then $M \in \mathcal{B}(R)$. However, in the next example we show that $\mathcal{B}(R)$ is strictly larger than the class of weak multiplication modules.

Example 3.3. *We claim that the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \left(\bigoplus_{i\in I} \frac{\mathbb{Z}}{p_i\mathbb{Z}}\right)$, where $\{p_i\}_{i\in I}$ is the set of all prime integers, belongs to $\mathcal{B}(\mathbb{Z})$ and M is not a weak multiplication module. For, it is easy to see that* $\operatorname{Max}(M) = \{pM \,|\, p$ *is a prime integer*$\}$ *and the set of all prime submodules of M is*

$$\operatorname{Max}(M) \cup \left\{(0) \oplus \left(\bigoplus_{i\in I} \frac{\mathbb{Z}}{p_i\mathbb{Z}}\right)\right\}.$$

Hence, M is not weak multiplication. To show that M belongs to $\mathcal{B}(\mathbb{Z})$, it is enough for us to show that $L := (0) \oplus (\bigoplus_{i\in I} \mathbb{Z}/p_i\mathbb{Z})$ is not a strongly prime submodule. Let $x, y \in M \setminus L$. Therefore,

$$(((0) \oplus (\bigoplus_{i\in I} \mathbb{Z}/p_i\mathbb{Z}) + \mathbb{Z}x) :_{\mathbb{Z}} M)y = ((\mathbb{Z}x \oplus (\bigoplus_{i\in I} \mathbb{Z}/p_i\mathbb{Z})) :_{\mathbb{Z}} M)y = (\mathbb{Z}x :_{\mathbb{Z}} \mathbb{Q})y = (0).$$

This implies that $(L + \mathbb{Z}x : M)y = (0) \subseteq L$. Hence, L is not a strongly prime submodule of M and so $X_M =$ $\operatorname{Max}(M) =$ $\{pM \,|\, p$ *is a prime integer*$\}$. *This shows that $M \in \mathcal{B}(\mathbb{Z})$.*

The next theorem indicates a local property.

Theorem 3.1. *Let M be an R-module. Then $M \in \mathcal{B}(R)$ if and only if $M_{\mathfrak{p}} \in \mathcal{B}(R_{\mathfrak{p}})$ for every prime (or maximal) ideal $\mathfrak{p}$ of R.*

Proof. Let $M \in \mathcal{B}(R)$ and P be a strongly prime submodule of $M_{\mathfrak{p}}$ where $\mathfrak{p}$ is a prime ideal of R. According to Lemma 2.6, there exists a strongly prime submodule Q of M such that $Q_{\mathfrak{p}} = P$. Since $M \in \mathcal{B}(R)$, there is an ideal I of R such that $Q = IM$. Therefore, $P = Q_{\mathfrak{p}} = I_{\mathfrak{p}}M_{\mathfrak{p}}$.

Conversely, let P be a strongly prime submodule of M and let $\mathfrak{m}$ be an arbitrary maximal ideal of R. If $P_{\mathfrak{m}} \neq M_{\mathfrak{m}}$, then by Lemma 2.6, $P_{\mathfrak{m}}$ is a strongly prime submodule of $M_{\mathfrak{m}}$, so $P_{\mathfrak{m}} = (P_{\mathfrak{m}} : M_{\mathfrak{m}})M_{\mathfrak{m}}$. Since every strongly prime submodule is prime, it follows from [12, Corollary 1] that

$$\left(\frac{P}{(P:M)M}\right)_{\mathfrak{m}} = \frac{P_{\mathfrak{m}}}{(P:M)_{\mathfrak{m}}M_{\mathfrak{m}}} = \frac{P_{\mathfrak{m}}}{(P_{\mathfrak{m}}:M_{\mathfrak{m}})M_{\mathfrak{m}}} = (0).$$

If $P_{\mathfrak{m}} = M_{\mathfrak{m}}$, then $(P_{\mathfrak{m}} : M_{\mathfrak{m}}) = R_{\mathfrak{m}}$. Hence,

$$\left(\frac{P}{(P:M)M}\right)_{\mathfrak{m}} = \frac{P_{\mathfrak{m}}}{(P:M)_{\mathfrak{m}}M_{\mathfrak{m}}} = \frac{M_{\mathfrak{m}}}{M_{\mathfrak{m}}} = (0).$$

Therefore $P = (P : M)M$. This shows that $M \in \mathcal{B}(R)$. □

Corollary 3.4. *Let $M \in \mathcal{B}(R)$ and $\mathfrak{p}$ be a minimal prime ideal of R. Then $M_{\mathfrak{p}} \in \mathcal{A}(R_{\mathfrak{p}})$.*

Proof. By Theorem 3.1 we have $M_{\mathfrak{p}} \in \mathcal{B}(R_{\mathfrak{p}})$. Hence, Corollary 3.2 implies that $M_{\mathfrak{p}} \in \mathcal{A}(R_{\mathfrak{p}})$. □

We recall that an R-module is called *uniserial* if its submodules are linearly ordered by inclusion (see [23]). Obviously, any uniserial module belongs to $\mathcal{A}(R)$. An R-module M is called *distributive* if the lattice of its submodules is distributive, i.e., $A \cap (B + C) = (A \cap B) + (A \cap C)$ and $A + (B \cap C) = (A + B) \cap (A + C)$ for all submodules A, B and C of M (see [5]). Theorem 3.1 enables us to show that every distributive module belongs to both $\mathcal{B}(R)$ and $\mathcal{A}(R)$.

Theorem 3.2. *If M is a distributive R-module, then $M \in \mathcal{A}(R) \cap \mathcal{B}(R)$.*

Proof. Let P be a strongly prime submodule of a distributive R-module M such that $\mathfrak{p} := (P : M)$. Let N and L be two arbitrary submodules of M such that $N \cap L \subseteq P$. Then by Lemma 2.6, P_p is strongly prime, and $N_{\mathfrak{p}} \cap L_{\mathfrak{p}} = (N \cap L)_{\mathfrak{p}} \subseteq P_{\mathfrak{p}}$. Since $M_{\mathfrak{p}}$ is distributive and $R_{\mathfrak{p}}$ is a local ring, by [22, Corollary 1], $M_{\mathfrak{p}}$ is uniserial. Therefore, either $N_{\mathfrak{p}} \subseteq P_{\mathfrak{p}}$ or $L_{\mathfrak{p}} \subseteq P_{\mathfrak{p}}$. Consequently either $N \subseteq N_{\mathfrak{p}} \cap M \subseteq P_{\mathfrak{p}} \cap M = P$ or $L \subseteq P$. This shows that $M \in \mathcal{A}(R)$.

We show that $M \in \mathcal{B}(R)$. By Theorem 3.1, it is enough for us to show that for each prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, $M_{\mathfrak{p}} \in \mathcal{B}(R_{\mathfrak{p}})$. Let $P \in X_{M_{\mathfrak{p}}}$ and $m \in P$. Since P is a proper submodule of $M_{\mathfrak{p}}$, there is an element $m' \in M_{\mathfrak{p}} \setminus P$. By assumption, $M_{\mathfrak{p}}$ is uniserial. Hence, $P \subseteq m'R_{\mathfrak{p}}$. This implies that $rm' = m \in P$ for some $r \in R_{\mathfrak{p}}$. Since P is strongly prime, we infer that $r \in (P :_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$. Thus $m \in (P :_{R_{\mathfrak{p}}} M_{\mathfrak{p}})M_{\mathfrak{p}}$. This yields that $P = (P :_{R_{\mathfrak{p}}} M_{\mathfrak{p}})M_{\mathfrak{p}}$. We conclude that $M_{\mathfrak{p}} \in \mathcal{B}(R_{\mathfrak{p}})$. □

Question: Is the converse of Theorem 3.2 true?

Recall that if R is an integral domain with quotient field K, the *rank* of an R-module M (*rank* M or *rank*$_R$ M) is defined to be the maximal number of elements of M linearly independent over R. Indeed, the rank of M is equal to the dimension of the vector space KM over K, that is *rank* M = *rank*$_K$ KM (see [14]). The next proposition is a generalization of [4, Proposition 2.4]. It is shown in [4, Proposition 2.4] that if M is a weak multiplication module over an integral domain, then the following statements hold.

1. if M is a nonzero torsion-free module, then *rank* $M = 1$.
2. if M is a torsion module, then *rank* $M = 0$.

Proposition 3.5. *Let R be an integral domain and $M \in \mathcal{B}(R)$. Then*

1. *If M is a nonzero torsion-free module, then rank $M = 1$.*
2. *If M is a torsion module, then rank $M = 0$.*

Proof. (1) First let R be a field and $0 \neq M \in \mathcal{B}(R)$ be a vector space. If $rank\, M > 1$, then let $0 \neq V$ be a maximal subspace of M. By [18, Proposition 1.3], V is a strongly prime submodule of M, and since $M \in \mathcal{B}(R)$, $V = IM$, where I is an ideal of the field R. So $I = 0$ or $I = R$, which is a contradiction. Hence $rank\, M \leq 1$, and since $0 \neq M$, then $rank\, M = 1$.

Now in the general case, if M is a nonzero torsion-free R-module, then $KM \neq 0$, where K is the quotient field of R. By Theorem 3.1, $KM \in \mathcal{B}(K)$ is a K-vector space, and as we mentioned above, $rank_K\, KM = 1$. Hence, $rank\, M = rank_K\, KM = 1$.

(2) Suppose that M is a torsion module. Then $KM = 0$ and therefore $rank\, M = rank_K\, KM = 0$. □

Let M be an R-module and consider the following map.

$$\begin{aligned} f_M :X_M &\longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M)) \\ P &\longmapsto (P :_R M)/\mathrm{Ann}(M) \end{aligned}$$

Then f_M is called the *natural map* of X_M. Obviously, if M is a multiplication R-module, then $M \in \mathcal{B}(R)$. We are going to show that the converse is true if M is finitely generated. According to [9, Corollary 4.4] every finitely generated module has the surjective natural map. However, the converse is not true in general. For example, consider the faithful $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$. It is easy to see that

$$X_M = \{pM \mid p \text{ is a prime integer}\} \cup \{(0) \oplus \mathbb{Z}(p^\infty)\}.$$

This implies that for every prime ideal $\mathfrak{p}$ of $\mathbb{Z}$, there exists a strongly prime submodule $P := \mathfrak{p}M$ of M such that $(P : M) = \mathfrak{p}$, i.e. M has the surjective natural map. We note that M is not finitely generated.

Modules with surjective natural map have interesting property. For example, the following result is a generalization of Nakayama lemma to the class of all modules with surjective natural map.

Lemma 3.6 (Nakayama's Lemma). *Let M be an R-module with surjective natural map and I be an ideal of R contained in the Jacobson radical of R. If $IM = M$, then $M = 0$.*

Proof. Let $M \neq 0$. Then there is a maximal ideal $\mathfrak{m}$ of R such that $\mathrm{Ann}(M) \subseteq \mathfrak{m}$. By assumption there is s strongly prime submodule P of M such that $(P : M) = \mathfrak{m}$. Thus, $M = IM \subseteq \mathfrak{m}M \subseteq P$, a contradiction. Therefore $M = 0$. □

We recall that by [5, Proposition 5] a finitely generated module is a multiplication module if and only if it is locally cyclic. Thus, the next theorem is a major generalization of [4, Theorem 2.7]. More precisely, it is shown in [4, Theorem 2.7] that every finitely generated weak multiplication module is a multiplication module.

Theorem 3.3. *Let $M \in \mathcal{B}(R)$ be a non-zero R-module with surjective natural map. Then M is locally cyclic.*

Proof. By assumption $\mathrm{Supp}(M)$ is a non-empty set. Let $\mathfrak{p} \in \mathrm{Supp}(M)$. Then it follows from Theorem 3.1 that $M_\mathfrak{p} \in \mathcal{B}(R_\mathfrak{p})$.

We claim that $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module with surjective natural map. Suppose that $\mathrm{Ann}(M_\mathfrak{p}) \subseteq \mathfrak{q} \in \mathrm{Spec}(R_\mathfrak{p})$. Then there is a prime ideal $\mathfrak{q}'$ of R such that $(\mathrm{Ann}(M))_\mathfrak{p} \subseteq \mathrm{Ann}(M_\mathfrak{p}) \subseteq \mathfrak{q}'R_\mathfrak{p} = \mathfrak{q} \subseteq \mathfrak{p}R_\mathfrak{p}$. Taking the contraction of each term of this sequence of ideals in R, we have that $\mathrm{Ann}(M) \subseteq \mathfrak{q}' \subseteq \mathfrak{p}$. Thus, $\mathfrak{q}'$ is a prime ideal of R containing $\mathrm{Ann}(M)$ so that by assumption there is a strongly prime submodule Q of M such that $(Q : M) = \mathfrak{q}'$. By Lemma 2.6, $QR_\mathfrak{p}$ is a strongly prime submodule of $M_\mathfrak{p}$ and [12, p.3742, Corollary 3 to Proposition 1] yields that $(QR_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p}) = \mathfrak{q}'R_\mathfrak{p} = \mathfrak{q}$.

Also, it is easy to see that $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \in \mathcal{B}(R_\mathfrak{p})$. If $\mathfrak{p}M_\mathfrak{p} = M_\mathfrak{p}$, then by Lemma 3.6, $M_\mathfrak{p} = 0$, a contradiction. So $\mathfrak{p}M_\mathfrak{p} \neq M_\mathfrak{p}$ whence by Proposition 3.5 we infer that $rank_{\frac{R_\mathfrak{p}}{\mathfrak{p}R_p}} \frac{M_\mathfrak{p}}{\mathfrak{p}M_\mathfrak{p}} = 1$. Thus $M_\mathfrak{p}$ is a cyclic R_p-module. Therefore M is a locally cyclic R-module. □

Corollary 3.7. *If M is a finitely generated R-module, then the following statements are equivalent:*

1. *M belongs to $\mathcal{A}(R)$;*
2. *M is multiplication;*
3. *M belongs to $\mathcal{B}(R)$;*

Proof. (1) ⇒ (2) Use Proposition 2.5 and [8, Corollary 1.5]. (2) ⇒ (1) Use Theorem 2.1. (2) ⇒ (3) This is true by definition. (3) ⇒ (2) Use Theorem 3.3 and [5, Proposition 5]. □

The next theorem is a generalization of [4, Theorem 2.8]. More precisely, it is shown in [4, Theorem 2.8] that if R is a ring, then the following are equivalent.

1. $\dim R = 0$.
2. For every weak multiplication R-module M, if $S_{(0)}(0) = 0$, then M is cyclic.
3. For every weak multiplication R-module M, if $S_{(0)}(0) = 0$, then M is a multiplication module.

Theorem 3.4. *Suppose that R is a ring. Then the following statements are equivalent.*

1. $\dim R = 0$.
2. *Every torsion-free R-module $M \in \mathcal{B}(R)$ is cyclic.*
3. *Every torsion-free R-module $M \in \mathcal{B}(R)$ is multiplication.*

Proof. (1) ⇒ (2) First let R be a field and let $M \in \mathcal{B}(R)$ be a torsion-free R-module. If $M = 0$, then M is cyclic. So, let $0 \neq M$. Hence, M is a nonzero vector space over the field R. According to Proposition 3.5, we have $\mathit{rank}\, M = 1$. That is $M \cong R$, and M is cyclic.

Now we assume that R is an arbitrary ring and $0 \neq M$. It is easy to see that $T(M) = 0$ is a prime submodule of M where $T(M)$ is the torsion submodule of M. By [11, Theorem 1], $M \cong M/(0) = M/T(M)$ is a torsion-free $R/(T(M) : M)$-module. Since $(T(M) : M)$ is a prime ideal of R and $\dim R = 0$, $R/(T(M) : M)$ is a field. So, $M \in \mathcal{B}(R/(T(M) : M))$ is a torsion-free module over the field $R/(T(M) : M)$. As we mentioned above M is a cyclic $R/(T(M) : M)$-module and whence M is a cyclic R-module. (2) ⇒ (3) Every cyclic module is multiplication (see [5]). (3) ⇒ (1) Let $\mathfrak{p}$ be a prime ideal of R. If K is the quotient field of the integral domain $R/\mathfrak{p}$, then by [12, Theorem 1], K as $\frac{R}{\mathfrak{p}}$-module has only one strongly prime submodule, namely (0). So K is a torsion-free $R/\mathfrak{p}$-module and $M \in \mathcal{B}(R/\mathfrak{p})$. By assumption it is a multiplication module. Since $R/\mathfrak{p}$ is a submodule of K, $R/\mathfrak{p} = IK$, where I is a nonzero ideal I of $R/\mathfrak{p}$. Note that $IK = K$. Hence $R/\mathfrak{p} = K$. Therefore $\mathfrak{p}$ is a maximal ideal. Consequently $\dim R = 0$. □

Corollary 3.8. *If R is an integral domain, then the following statements are equivalent.*

1. *R is a field;*
2. *Every R-module $M \in \mathcal{B}(R)$ is cyclic;*
3. *Every R-module $M \in \mathcal{B}(R)$ is a multiplication module.*

Proof. This follows from Theorem 3.4. □

It is shown in [5, Proposition 8] that every finitely generated Artinian multiplication module is cyclic. The following result is a generalization of [5, Proposition 8].

Proposition 3.9. *Let R be an Artinian ring and $M \in \mathcal{B}(R)$. Then M is cyclic.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. Then $M_\mathfrak{p} \in \mathcal{B}(R_\mathfrak{p})$ by Theorem 3.1. Since $R_\mathfrak{p}$ is Artinian, there is some integer $n \in \mathbb{N}$ such that $(\mathfrak{p}R_\mathfrak{p})^n = (0)$. If $\mathfrak{p}M_\mathfrak{p} = M_\mathfrak{p}$, then $M_\mathfrak{p} = (\mathfrak{p}R_\mathfrak{p})^n M_\mathfrak{p} = (0)$. Otherwise, $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ belongs to $\mathcal{B}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ and by Proposition 3.5, $\mathit{rank}_{\frac{R_\mathfrak{p}}{\mathfrak{p}R_p}} \frac{M_\mathfrak{p}}{\mathfrak{p}M_\mathfrak{p}} = 1$. This yields that $\mathfrak{p}M_\mathfrak{p}$ is a maximal submodule of $M_\mathfrak{p}$. If $x \in M_\mathfrak{p} \setminus \mathfrak{p}M_\mathfrak{p}$, then $\mathfrak{p}M_\mathfrak{p} \subset \mathfrak{p}M_\mathfrak{p} + xR_\mathfrak{p} \subseteq M_\mathfrak{p}$, and therefore $\mathfrak{p}M_\mathfrak{p} + xR_\mathfrak{p} = M_\mathfrak{p}$. Thus $(0) = (\mathfrak{p}R_\mathfrak{p})^n \frac{M_\mathfrak{p}}{xR_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p} \frac{M_\mathfrak{p}}{xR_\mathfrak{p}} = \frac{M_\mathfrak{p}}{xR_\mathfrak{p}}$. Therefore, $M_\mathfrak{p} = xR_\mathfrak{p}$. Hence, M is locally cyclic and the result follows from [5, Lemma 3]. □

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