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A Halpern-Type Iteration for Solving the Split Feasibility Problem and the Fixed Point Problem of Bregman Relatively Nonexpansive Semigroup in Banach Spaces

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Abstract. We study the split feasibility problem (SFP) involving the fixed point problems (FPP) in the framework of p-uniformly convex and uniformly smooth Banach spaces. We propose a Halpern-type iterative scheme for solving the solution of SFP and FPP of Bregman relatively nonexpansive semigroup. Then we prove its strong convergence theorem of the sequences generated by our iterative scheme under implemented conditions. We finally provide some numerical examples and demonstrate the efficiency of the proposed algorithm. The obtained result of this paper complements many recent results in this direction.

1. Introduction

Throughout in this paper, we let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A which is defined by

$$\langle A^* y, x\rangle := \langle y, Ax\rangle, \tag{1}$$

for all $x \in C$ and $y \in E_2^*$. We consider the following *split feasibility problem* (SFP): find an element

$$\hat{x} \in C \ \text{ such that } \ A\hat{x} \in Q. \tag{2}$$

The set of solutions of problem (2) is denoted by $\Gamma := C \cap A^{-1}(Q) = \{x \in C : Ax \in Q\}$. We assume that Γ is nonempty. Then, we have Γ is a closed and convex subset of E_1. It is clear that $\hat{x}$ is a solution to the split feasibility problem (2) if and only if $\hat{x} \in C$ and $A\hat{x} - P_Q A\hat{x} = 0$. The split feasibility problem originally introduced in Censor and Elfving [11] in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Recently, SFP can also be used to model the intensity-modulated radiation therapy [10, 12–14].

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Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ a bounded linear operator. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. In order to solve the SFP in Hilbert spaces, Byrne [10] introduced the following CQ algorithm: $x_1 \in C$ and

$$x_{n+1} = P_C(x_n - \lambda A^*(I - P_Q)Ax_n),\ n \geq 1, \tag{3}$$

where $\lambda > 0$, P_C and P_Q are the metric projections on C and Q, respectively. It was proved that the sequence $\{x_n\}$ defined by (5.1) converges weakly to a solution of the SFP provided the step-size $\lambda \in (0, \frac{2}{\|A\|^2})$.

In the setting of Banach spaces, Schöpfer et al. [26] first introduced the following algorithm for solving the SFP: $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J^*_{E_1}[J_{E_1}(x_n) - \lambda_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))],\ n \geq 1, \tag{4}$$

where $\{\lambda_n\}$ is a positive sequence, Π_C denotes the generalized projection on E, P_Q is the metric projection on E_2, J_{E_1} is the duality mapping on E_1 and $J^*_{E_1}$ is the duality mapping on E_1^*. It was proved that the sequence $\{x_n\}$ converges weakly to a solution of SFP, under some mild conditions, in p-uniformly convex and uniformly smooth Banach spaces. To be more precisely, the condition that the duality mapping of E_1 is sequentially weak-to-weak-continuous is assumed in [26] (which excludes some important Banach spaces, such as the classical $L_p(2 < p < \infty)$ spaces). Please see some modifications in [27, 28].

Recently, Wang [31] modified the above algorithm (4) and proved its strong convergence for the following multiple-sets split feasibility problem (MSSFP): find $x \in E_1$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, Ax \in \bigcap_{j=1+r}^{r+s} Q_j, \tag{5}$$

where r, s are two given integers, $C_i, i = 1, \ldots, r$ is a closed convex subset in E_1, and $Q_j, j = r+1, \ldots, r+s$, is a closed convex subset in E_2. He defined for each $n \in \mathbb{N}$,

$$T_n(x) = \begin{cases} \Pi_{C_{i(n)}}(x), & 1 \leq i(n) \leq r, \\ J_q^{E_1}[J_p^{E_1}(x) - \lambda_n A^* J_p^{E_2}(Ax - P_{Q_{j(n)}}(Ax))], & r+1 \leq i(n) \leq r+s, \end{cases}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \mod (r+s) + 1,$$

and λ_n satisfies

$$0 < \lambda \leq \lambda_n \leq \Big(\frac{q}{\kappa_q \|A\|^q}\Big)^{\frac{1}{q-1}}, \tag{6}$$

with κ_q a uniform smoothness constant and proposed the following algorithm: For any initial guess x_1, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n \\ D_n = \{w \in E_1 : \Delta_p(y_n, w) \leq \Delta_p(x_n, w)\} \\ E_n = \{w \in E_1 : \langle x_n - w, J_p(x_1) - J_p(x_n) \geq 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_1). \end{cases} \tag{7}$$

Using the idea in the work of Nakajo and Takahashi [23], he proved the following strong convergence theorem in p-uniformly convex Banach spaces which is also uniformly smooth.

Theorem 1.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP* (5) *has a nonempty solution set Ω. Let the sequence $\{x_n\}$ be generated by* (7)*. Then $\{x_n\}$ converges strongly to the Bregman projection of x_1 onto the solution set Ω.*

It is observed that the main advantage of result of Wang [31] is that the weak-to-weak continuity of the duality mapping, assumed in [26] is dispensed with and strong convergence result was achieved.

Recently, Shehu et al. [29] introduced a new iterative scheme for solving the SFP and the fixed point problem of Bregman strongly nonexpansive mappings in the framework of p-uniformly convex real Banach spaces which are also uniformly smooth as follows: $u \in C$, $u_1 \in E_1$ and

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\Big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\Big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)Tx_n\Big)\Big], \ \ \forall n \geq 1, \end{cases} \tag{8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. It was proved that the sequence $\{x_n\}$ and $\{u_n\}$ defined by (8) converge strongly to a solution of the problem under some mild conditions.

It is our purpose in this paper to construct an iterative scheme for approximating a solution to split feasibility problems which is also a fixed point of a Bregman relatively nonexpansive semigroup. We also prove its strong convergence of the sequence generated by our scheme in p-uniformly convex real Banach spaces which are uniformly smooth. Our result complements the results of Byrne [10], Schöpfer et al. [26], Wang [31], Shehu et al. [29] and many other recent results in the literature.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and dual space E^* of E. Let $S(E) := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \tfrac{\|x+y\|}{2} : x, y \in S(E), \|x - y\| \geq \epsilon\Big\}.$$

The space E is said to be *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$. Then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex. The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\Big\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S(E)\Big\}.$$

The space E is said to be *uniformly smooth* if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Suppose that $q > 1$, a Banach space E is said to be *q-uniformly smooth* if there exists a $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth. Moreover, we note that a Banach space E is p-uniformly convex if and only if E is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ (see [32]).

Let $p > 1$ be a real number. The *generalized duality mapping* $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\| = \|x\|^{p-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. In particular, $J_p^E = J_2^E$ is called the *normalized duality mapping*.

Here and hereafter, we assume that E is a p-uniformly convex and uniformly smooth, which implies that its dual space, E^* is q-uniformly smooth and uniformly convex. In this situation, it is known that the generalized duality mapping J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^*. Moreover, if E is uniformly smooth then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E. (see [1, 17] for more details).

The examples of generalized duality mapping are shown in below:

Example 2.1. ([1]) *Let $x = (x_1, x_2, ...) \in \ell_p$ $(1 < p < \infty)$. Then the generalized duality mapping J_p in ℓ_p is given by*

$$J_p^{\ell_p}(x) = (|x_1|^{p-1}sgn(x_1), |x_2|^{p-1}sgn(x_2), ...).$$

Example 2.2. ([1]) *Let* $f \in L_p([\alpha, \beta])$ $(1 < p < \infty)$. *Then the generalized duality mapping* $J_p^{L_p}$ *is given by*

$$J_p^{L_p}(f)(t) = |f(t)|^{p-1} sgn(f(t)).$$

Definition 2.3. ([9]) *Let* $f : E \to \mathbb{R}$ *be a convex and Gâteaux differentiable function. The function* $\Delta_f : E \times E \to [0, +\infty)$ *defined by*

$$\Delta_f(x, y) := f(y) - f(x) - \langle f'(x), y - x \rangle,$$

is called the Bregman distance with respect to f.

We remark that the Bregman distance Δ_f is not satisfy the well-known properties of a metric because Δ_f is not symmetric and does not satisfy the triangle inequality.

It is well known that the duality mapping J_p^E is the sub-differential of the functional $f_p(\cdot) = \frac{1}{p}\|\cdot\|^p$ for $p > 1$ (see [15]). Then, we have the Bregman distance with respect to f_p that

$$\begin{aligned} \Delta_p(x, y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle J_p^E x, y - x \rangle \\ &= \frac{1}{q}\|x\|^p - \langle J_p^E x, y \rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{q}\|x\|^p - \frac{1}{q}\|y\|^p - \langle J_p^E x - J_p^E y, y \rangle. \end{aligned} \tag{9}$$

If $p = 2$, we get $\partial\Big(\frac{\|x\|^p}{p}\Big) = \partial\Big(\frac{\|x\|^2}{2}\Big) = 2Jx$ for all $x \in E$, where J is the normalized duality mapping. Then the Bregman distance (9) reduce to $\Delta_2(x, y) := \phi(x, y) = \|x\|^2 - 2\langle y, Jx \rangle + \|y\|^2$ for all $x, y \in E$, where ϕ is called the *Lyapunov function* introduced by Alber [2, 3].

Moreover, the Bregman distance has the following important properties:

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_p^E x - J_p^E z \rangle, \tag{10}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_p^E x - J_p^E y \rangle, \tag{11}$$

for all $x, y, z \in E$. For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [26]):

$$\tau\|x - y\|^p \le \Delta_p(x, y) \le \langle x - y, J_p^E x - J_p^E y \rangle, \tag{12}$$

where $\tau > 0$ is some fixed number.

Definition 2.4. *Let E be a real Banach space. A one parameter family* $\mathcal{S} = \{T(t) : t \ge 0\}$ *from E into E is said to be a* nonexpansive semigroup *if it satisfies the following conditions:*

(S1) $T(0)x = x$ *for all* $x \in E$;

(S2) $T(s + t) = T(s)T(t)$ *for all* $s, t \ge 0$;

(S3) for each $x \in E$ *the mapping* $t \mapsto T(t)x$ *is continuous;*

(S4) for each $t \ge 0$, $T(t)$ *is nonexpansive,* i.e.,

$$\|T(t)x - T(t)y\| \le \|x - y\|, \ \forall x, y \in E.$$

Remark 2.5. *We denote by* $F(\mathcal{S})$ *the set of all common fixed points of* $\mathcal{S}$, *i.e.,* $F(\mathcal{S}) = \{x \in C : T(t)x = x, t \ge 0\} = \bigcap_{t \ge 0} F(T(t))$.

The theory of semigroup is very important in theory of differential equations. Let $E = \mathbb{R}^n$ and let $\mathcal{L}(E)$ be the space of all bounded linear operators on E. Consider the the following initial value problem for a system of homogeneous linear first-order differential equations with constant coefficients:

$$\begin{cases} u_1' = \alpha_{11}u_1 + \alpha_{12}u_2 + ... + \alpha_{1n}u_n, \;\; u_1(0) = x_1 \\ u_2' = \alpha_{21}u_1 + \alpha_{22}u_2 + ... + \alpha_{2n}u_n, \;\; u_2(0) = x_2 \\ \vdots \\ u_n' = \alpha_{n1}u_1 + \alpha_{n2}u_2 + ... + \alpha_{nn}u_n, \;\; u_n(0) = x_n. \end{cases} \tag{13}$$

which can be written in a matrix form as

$$\begin{cases} \mathbf{u}'(t) = A\mathbf{u}(t), \; t \geq 0, \\ \mathbf{u}(0) = \mathbf{x}, \end{cases} \tag{14}$$

where $A \in \mathcal{L}(E)$ is bounded linear operator. In this case, $A = (\alpha_{ij})$ is an $n \times n$ matrix with $\alpha_{ij} \in \mathbb{R}$ for $i, j = 1, 2, ..., n$ and $\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ is a given initial vector with $x_i \in \mathbb{R}$ for all $i = 1, 2, ..., n$. It is well-known that the problem (14) has a unique solution given by explicit formula $\mathbf{u}(t) = e^{tA}\mathbf{x}, \; t \geq 0$, where e^{tA} is a matrix exponential of the linear differential system (14) defined by

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots,$$

where $A^0 = I$ is the identity matrix. Note that the family of matrixes (operators) $\{T(t) := e^{tA} : t \geq 0\}$ is (uniformly continuous) semigroup on E (see [7]). Then, we can write the solution of the problem (14) as $\mathbf{u}(t) = T(t)\mathbf{x}, \; t \geq 0$.

Example 2.6. *Solve the following initial value problem:*

$$\mathbf{u}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{u}(t), \;\; \mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{15}$$

Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. *It is not hard to show that* $T(t) := e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, *which satisfies the semigroup properties. Then, we have the solution of* (15) *is* $\mathbf{u}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

A point $z \in C$ called an *asymptotic fixed point* of T, if there exists a sequence $\{x_n\}$ in C which converges weakly to z such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denoteby $\widehat{F}(T)$ by the set of asymptotic fixed points of T.

We now give the following definition:

Definition 2.7. *A one-parameter family* $\mathcal{S} = \{T(t)\}_{t\geq 0} : C \to E$ *is said to be a* Bregman relatively nonexpansive semigroup *if it satisfies* $(S1), (S2), (S3)$ *and the following conditions:*

(a) $F(\mathcal{S})$ *is nonempty;*

(b) $F(\mathcal{S}) = \widehat{F}(\mathcal{S})$;

(c) $\Delta_p(T(t)x, z) \leq \Delta_p(x, z), \;\; \forall x \in C, \; z \in F(\mathcal{S})$ *and* $t \geq 0$.

Using an idea in [4, 5, 8], we define the following concept:

Definition 2.8. *A continuous operator semigroup* $\mathcal{S} = \{T(t)\}_{t\geq 0} : C \to E$ *is said to be* uniformly asymptotically regular *(in short, u.a.r.) if for all* $s \geq 0$ *and any bounded subset* B *of* C *such that*

$$\lim_{t\to\infty} \sup_{x\in B} \|J_p^E(T(t)x) - J_p^E\big(T(s)T(t)x\big)\| = 0.$$

Definition 2.9. *A Bregman relatively nonexpansive semigroup $\mathcal{S} = \{T(t)\}_{t\geq 0} : C \to E$ is said to be a* uniformly Lipschitzian mapping *if there exists a bounded measurable function $L(t) : (0,\infty) \to [0,\infty)$ such that*

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \ \forall x, y \in C.$$

Recall that the metric projection from E onto C, denote by $P_C x$, satisfying the property

$$\|x - P_C x\| \leq \inf_{y\in C} \|x - y\|, \ \ \forall x \in E.$$

It is well known that $P_C x$ is the unique minimizer of the norm distance. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle J_p^E(x - P_C x), y - P_C x\rangle \leq 0, \ \ \forall y \in C. \tag{16}$$

Similarly, one can define the Bregman projection from E onto C, denote by Π_C, satisfying the property

$$\Delta_p\Big(x, \Pi_C(x)\Big) = \inf_{y\in C} \Delta_p(x, y), \ \ \forall x \in E. \tag{17}$$

Lemma 2.10. ([28]) *Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and let $x \in E$. Then the following assertions hold:*

(i) $z = \Pi_C x$ *if and only if* $\langle J_p^E(x) - J_p^E(z), y - z\rangle \leq 0$, $\forall y \in C$.

(ii) $\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y)$, $\forall y \in C$.

Lemma 2.11. [34] *Let $1 < q \leq 2$ and E be a Banach space. Then the following are equivalent.*

(i) E is q-uniformly smooth.

(ii) There is a constant $\kappa_q > 0$ such that for all $x, y \in E$

$$\|x - y\|^q \leq \|x\|^q - q\langle y, j_q(x)\rangle + \kappa_q\|y\|^q. \tag{18}$$

Remark 2.12. *The constant κ_q satisfying* (18) *is called the q-*uniform smoothness coefficient *of E.*

The following Lemma can be obtained from Theorem 2.8.17 of [1] (see also Lemma 5 of [20]).

Lemma 2.13. *Let $p > 1$, $r > 0$ and E be a Banach space. Then the following statements are equivalent:*

(i) E is uniformly convex;

(ii) There exists a strictly increasing convex function $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_r^*(0) = 0$ such that

$$\Big\|\sum_{k=1}^{N} \alpha_k x_k\Big\|^p \leq \sum_{k=1}^{N} \alpha_k\|x_k\|^p - \alpha_i\alpha_j g_r^*(\|x_i - x_j\|),$$

for all $i, j \in \{1, 2, ..., N\}$, $x_k \in B_r := \{x \in E : \|x\| \leq r\}$, $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$, where $k \in \{1, 2, ..., N\}$.

Lemma 2.14. ([28]) *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Thus, for all $z \in E$, we have*

$$\Delta_p\Big(J_q^{E^*}\Big(\sum_{i=1}^{N} t_i J_p^E(x_i)\Big), z\Big) \leq \sum_{i=1}^{N} t_i\Delta_p(x_i, z),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^{N} t_i = 1$.

The following lemmas, can be found in [28, 29].

Lemma 2.15. *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Let* $V_p : E^* \times E \to [0, +\infty)$ *defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \ \ \forall x \in E, \ \ x^* \in E^*.$$

Then the following assertions hold:

(i) V_p *is nonnegative and convex in the first variable;*

(ii) $\Delta_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \ \ \forall x \in E, \ \ x^* \in E^*.$

(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \ \ \forall x \in E, \ \ x^*, y^* \in E^*.$

Following the proof line as in Proposition 2.5 of [22], we obtain the following result:

Lemma 2.16. *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Suppose that* $x \in E$ *and* $\{x_n\}$ *is a sequence in E. If* $\{\Delta_p(x_n, x)\}$ *is bounded, so is the sequence* $\{x_n\}$ *is bounded.*

Lemma 2.17. *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are bounded sequences in E. Then the following assertions are equivalent:*

(a) $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$;

(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. For the implication $(a) \implies (b)$. Suppose that $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$. From (12), we have

$$0 \leq \tau\|x_n - y_n\|^p \leq \Delta_p(x_n, y_n),$$

where $\tau > 0$ is some fixed number. It follows that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

For the converse implication $(b) \implies (a)$, we assume that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. From (12), we observe that

$$\begin{aligned} 0 \leq \Delta_p(x_n, y_n) &\leq \langle x_n - y_n, J_p^E x_n - J_p^E y_n\rangle \\ &\leq \|x_n - y_n\|\|J_p^E x_n - J_p^E y_n\| \\ &\leq \|x_n - y_n\|M, \end{aligned}$$

where $M = \sup_{n\geq 1}\{\|x_n\|^{p-1}, \|y_n\|^{p-1}\}$. It follows that $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$. This completes the proof. □

Lemma 2.18. ([33]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \ \ \forall n \geq 1,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that* $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.19. ([21]) *Let* $\{a_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists an increasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \ \text{ and } \ a_k \leq a_{m_k+1}.$$

In fact, m_k *is the largest number n in the set* $\{1, 2, ..., k\}$ *such that the condition* $a_n \leq a_{n+1}$ *holds.*

In what follows, we shall use the following notations:
• $x_n \to x$ mean that $\{x_n\}$ converges strongly to x;
• $x_n \rightharpoonup x$ mean that $\{x_n\}$ converges weakly to x.

Lemma 2.20. *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$. Then, we have*

$$\Delta_p\Big(J_q^{E^*}\Big(\sum_{k=1}^{N} \alpha_k J_p^E(x_k)\Big), z\Big) \leq \sum_{k=1}^{N} \alpha_k \Delta_p(x_k, z) - \alpha_i\alpha_j g_r^*\Big(\|J_p^E(x_i) - J_p^E(x_j)\|\Big),$$

for all $i, j \in \{1, 2, ..., N\}$.

Proof. Since p-uniformly convex, hence it is uniformly convex. From Lemmas 2.13 and 2.14, we have

$$\begin{aligned}
& \Delta_p\Big(J_q^{E^*}\Big(\sum_{k=1}^{N} \alpha_k J_p^E(x_k)\Big), z\Big) \\
= \;& V_p\Big(\sum_{k=1}^{N} \alpha_k J_p^E(x_k), z\Big) \\
= \;& \frac{1}{q}\Big\|\sum_{k=1}^{N} \alpha_k J_p^E(x_k)\Big\|^q - \Big\langle \sum_{k=1}^{N} \alpha_k J_p^E(x_k), z\Big\rangle + \frac{1}{p}\|z\|^p \\
\leq \;& \frac{1}{q}\sum_{k=1}^{N} \alpha_k \|J_p^E(x_k)\|^q - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) - \Big\langle \sum_{k=1}^{N} \alpha_k J_p^E(x_k), z\Big\rangle + \frac{1}{p}\|z\|^p \\
= \;& \frac{1}{q}\sum_{k=1}^{N} \alpha_k \|J_p^E(x_k)\|^q - \sum_{k=1}^{N} \alpha_k \langle J_p^E(x_k), z\rangle + \frac{1}{p}\|z\|^p - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) \\
= \;& \sum_{k=1}^{N} \alpha_k \Delta_p(x_k, z) - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),
\end{aligned}$$

for all $i, j \in \{1, 2, ..., N\}$. This completes the proof. □

3. Main Results

Theorem 3.1. *Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces and let C and Q be a nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be adjoint of A. Let $\mathcal{S} = \{T(t)\}_{t\geq 0}$ be a u.a.r. Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mapping of C into E_1 with a bounded measurable function $L(t) : (0, \infty) \to [0, \infty)$ such that $F(\mathcal{S}) := \bigcap_{h\geq 0} F(T(h)) \neq \emptyset$. Suppose that $F(\mathcal{S}) = \widehat{F}(\mathcal{S})$ and $F(\mathcal{S}) \cap \Gamma \neq \emptyset$. For given $u \in E_1$, let $\{u_n\}$ be a sequence generated by $u_1 \in C$ and*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\Big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\Big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1 - \alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + (1 - \beta_n)T(t_n)x_n\Big)\Big], \quad \forall n \geq 1, \end{cases} \tag{19}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, $\{t_n\}$ is a real positive divergent sequence and $\{\lambda_n\}$ is real positive sequence which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \leq \beta_n \leq b < 1$;

(C3) $0 < c \leq \lambda_n \leq d < \left(\frac{q}{\kappa_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $x^* = \Pi_{F(\mathcal{S})\cap\Gamma}u$.

Proof. We first show that $\{x_n\}$ is bounded. Set $x_n = \Pi_C v_n$, where

$$v_n = J_q^{E_1^*}\Big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\Big)$$

for all $n \geq 1$. Let $v \in F(\mathcal{S}) \cap \Gamma$. From (16), we observe that

$$\begin{aligned} & \langle J_p^{E_2}(Au_n - P_Q(Au_n)), Au_n - Av\rangle \\ = \; & \langle J_p^{E_2}(Au_n - P_Q(Au_n)), Au_n - P_Q(Au_n)\rangle + \langle J_p^{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Av\rangle \\ = \; & \|Au_n - P_Q(Au_n)\|^p + \langle J_p^{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Av\rangle \\ \geq \; & \|Au_n - P_Q(Au_n)\|^p. \end{aligned} \tag{20}$$

It follows from Lemma 2.11 and (20) that

$$\begin{aligned} \Delta_p(x_n, v) &\leq \Delta_p(v_n, v) \\ &= \Delta_p\Big(J_q^{E_1^*}[J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))], v\Big) \\ &= \frac{1}{q}\|J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\|^q - \langle J_p^{E_1}(u_n), v\rangle \\ &\quad + \lambda_n\langle J_p^{E_2}(Au_n - P_Q(Au_n)), Av\rangle + \frac{1}{p}\|v\|^p \\ &\leq \frac{1}{q}\|J_p^{E_1}(u_n)\|^q - \lambda_n\langle Au_n, J_p^{E_2}(Au_n - P_Q(Au_n))\rangle + \frac{\kappa_q(\lambda_n\|A\|)^q}{q}\|J_p^{E_2}(Au_n - P_Q(Au_n))\|^q \\ &\quad - \langle J_p^{E_1}(u_n), v\rangle + \lambda_n\langle J_p^{E_2}(Au_n - P_Q(Au_n)), Av\rangle + \frac{1}{p}\|v\|^p \\ &= \frac{1}{q}\|u_n\|^p - \langle J_p^{E_1}(u_n), v\rangle + \frac{1}{p}\|v\|^p + \lambda_n\langle J_p^{E_2}(Au_n - P_Q(Au_n)), Av - Au_n\rangle \\ &\quad + \frac{\kappa_q(\lambda_n\|A\|)^q}{q}\|J_p^{E_2}(Au_n - P_Q(Au_n))\|^q \\ &= \Delta_p(u_n, v) + \lambda_n\langle J_p^{E_2}(Au_n - P_Q(Au_n)), Av - Au_n\rangle + \frac{\kappa_q(\lambda_n\|A\|)^q}{q}\|J_p^{E_2}(Au_n - P_Q(Au_n))\|^q \\ &\leq \Delta_p(u_n, v) - \Big(\lambda_n - \frac{\kappa_q(\lambda_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p. \end{aligned} \tag{21}$$

By (C3), we get that

$$\Delta_p(x_n, v) \leq \Delta_p(u_n, v).$$

Now, we set

$$y_n = J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)J_p^{E_1}(T(t_n)x_n))$$

for all $n \geq 1$. From Lemma 2.20, we have

$$\begin{aligned} \Delta_p(y_n, v) &= \Delta_p(J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)J_p^{E_1}(T(t_n)x_n)), v) \\ &\leq \beta_n\Delta_p(x_n, v) + (1-\beta_n)\Delta_p(T(t_n)x_n, v) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\|) \\ &\leq \Delta_p(x_n, v) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\|) \qquad (22) \\ &\leq \Delta_p(x_n, v) \qquad (23) \end{aligned}$$

It follows from (23) that

$$\begin{aligned}
\Delta_p(x_{n+1}, v) &\leq \Delta_p(u_{n+1}, v)\\
&\leq \Delta_p\Big(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)), v\Big)\\
&\leq \alpha_n\Delta_p(u, v) + (1-\alpha_n)\Delta_p(y_n, v)\\
&\leq \alpha_n\Delta_p(u, v) + (1-\alpha_n)\Delta_p(x_n, v)\\
&\leq \max\{\Delta_p(u, v), \Delta_p(x_n, v)\}\\
&\ \vdots\\
&\leq \max\{\Delta_p(u, v), \Delta_p(x_1, v)\}. \qquad (24)
\end{aligned}$$

Hence, $\{\Delta_p(x_n, v)\}$ is bounded, which implies by Lemma 2.16 that $\{x_n\}$ is bounded.

Let $u_{n+1} = \Pi_C z_n$, where $z_n = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)\Big]$ for all $n \geq 1$. From Lemma 2.15 and (22), we have

$$\begin{aligned}
&\Delta_p(x_{n+1}, v)\\
\leq\ & \Delta_p(u_{n+1}, v)\\
\leq\ & \Delta_p(z_n, v)\\
=\ & V_p\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n), v\Big)\\
\leq\ & V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(v), v)) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle\\
=\ & V_p(\alpha_n J_p^{E_1}(v) + (1-\alpha_n)J_p^{E_1}(y_n), v) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle\\
\leq\ & \alpha_n V_p(J_p^{E_1}(v), v) + (1-\alpha_n)V_p(J_p^{E_1}(y_n), v) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle\\
=\ & \alpha_n\Delta_p(v, v) + (1-\alpha_n)\Delta_p(y_n, v) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle\\
\leq\ & (1-\alpha_n)[\Delta_p(x_n, v) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\|)]\\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle\\
\leq\ & (1-\alpha_n)\Delta_p(x_n, v) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\|)]\\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle \qquad (25)\\
\leq\ & (1-\alpha_n)\Delta_p(x_n, v) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle. \qquad (26)
\end{aligned}$$

Next, we will divide the proof into two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n, v)\}_{n=n_0}^{\infty}$ is nonincreasing. By the boundedness of $\{\Delta_p(x_n, v)\}_{n=1}^{\infty}$, we have $\{\Delta_p(x_n, v)\}_{n=1}^{\infty}$ is convergent. Furthermore, we have

$$\Delta_p(x_n, v) - \Delta_p(x_{n+1}, v) \to 0 \text{ as } n \to \infty.$$

Then, from (25) and $(C2)$, we have

$$\begin{aligned}
0 &\leq a(1-b)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\|)\\
&\leq \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\|)\\
&\leq \Delta_p(x_n, v) - \Delta_p(x_{n+1}, v) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - v\rangle \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which implies by the property of g_r^* that

$$\lim_{n\to\infty}\|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\| = 0. \qquad (27)$$

Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, then

$$\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = 0. \qquad (28)$$

From Lemma 2.17, we also have

$$\lim_{n\to\infty} \Delta_p(T(t_n)x_n, x_n) = 0. \tag{29}$$

Since $\{T(t)\}_{t\geq 0}$ is uniformly Lipschitzian with a bounded measurable function $L(t)$. Then, we have from (28) that

$$\begin{aligned} \|T(t)T(t_n)x_n - T(t)x_n\| &\leq L(t)\|T(t_n)x_n - x_n\| \\ &\leq \sup_{t\geq 0}\{L(t)\}\|T(t_n)x_n - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Again, since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1. Then, we also have

$$\lim_{n\to\infty} \|J_p^{E_1}\Big(T(t)T(t_n)x_n\Big) - J_p^{E_1}\Big(T(t)x_n\Big)\| = 0. \tag{30}$$

For each $t \geq 0$, we note that

$$\begin{aligned} \|J_p^{E_1}(x_n) - J_p^{E_1}\Big(T(t)x_n\Big)\| &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}\Big(T(t_n)x_n\Big)\| + \|J_p^{E_1}\Big(T(t_n)x_n\Big) - J_p^{E_1}\Big(T(t)T(t_n)x_n\Big)\| \\ &\quad + \|J_p^{E_1}\Big(T(t)T(t_n)x_n\Big) - J_p^{E_1}\Big(T(t)x_n\Big)\| \\ &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}\Big(T(t_n)x_n\Big)\| + \|J_p^{E_1}\Big(T(t)T(t_n)x_n\Big) - J_p^{E_1}\Big(T(t)x_n\Big)\| \\ &\quad + \sup_{x\in\{x_n\}} \|J_p^{E_1}\Big(T(t_n)x\Big) - J_p^{E_1}\Big(T(t)T(t_n)x\Big)\|. \end{aligned}$$

Since $\{T(t)\}_{t\geq 0}$ is a u.a.r. Bregman relatively nonexpansive semigroup with $\lim_{n\to\infty} t_n = \infty$, from (27) and (30), we get

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}\Big(T(t)x_n\Big)\| = 0.$$

Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*. Then, we get that

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0, \ \ \forall t \geq 0. \tag{31}$$

By the reflexivity of a Banach space and the boundedness of $\{x_n\}$, without loss of generality, we may assume that $x_{n_i} \rightharpoonup z \in C$ as $i \to \infty$. From (30), we obtain $z \in F(\mathcal{S}) = \widehat{F}(\mathcal{S})$. Next, we show that $z \in \Gamma$. From (21) and $(C2)$, we have

$$\begin{aligned} 0 &\leq c\Big(1 - \frac{\kappa_q d^{q-1}\|A\|^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \\ &\leq \Big(\lambda_n - \frac{\kappa_q(\lambda_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \\ &\leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*) \\ &\leq \alpha_{n-1}\Delta_p(u, x^*) + \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{32}$$

Since $v_n = J_q^{E_1^*}\Big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\Big)$ for all $n \geq 1$, it follows that

$$\begin{aligned} 0 \leq \|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| &\leq \lambda_n\|A^*\|\|J_p^{E_2}(Au_n - P_Q(Au_n))\| \\ &\leq \Big(\frac{q}{\kappa_q\|A\|^q}\Big)^{\frac{1}{q-1}}\|A^*\|\|Au_n - P_Q(Au_n)\|^{p-1}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| = 0. \tag{33}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, then

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{34}$$

By Lemma 2.10 (*ii*) and (24), we have

$$\begin{aligned} \Delta_p(v_n, x_n) &= \Delta_p(v_n, \Pi_C v_n) \le \Delta_p(v_n, x^*) - \Delta_p(x_n, x^*) \\ &\le \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*) \\ &\le \alpha_{n-1}\Delta_p(u, x^*) + \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*) \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

By Lemma 2.17, we get that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{35}$$

From (33) and (35), we obtain that

$$\|x_n - u_n\| \le \|v_n - u_n\| + \|v_n - x_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{36}$$

Since $x_{n_i} \rightharpoonup z \in C$ and from (36), we also get that $u_{n_i} \rightharpoonup z \in C$. From (16), we have

$$\begin{aligned} \|(I - P_Q)Az\|^p &= \langle J_p^{E_2}(Az - P_Q(Az)), Az - P_Q(Az)\rangle \\ &= \langle J_p^{E_2}(Az - P_Q(Az)), Az - Au_{n_i}\rangle + \langle J_p^{E_2}(Az - P_Q(Az)), Au_{n_i} - P_Q(Au_{n_i})\rangle \\ &\quad + \langle J_p^{E_2}(Az - P_Q(Az)), P_Q(Au_{n_i}) - P_Q(Az)\rangle \\ &\le \langle J_p^{E_2}(Az - P_Q(Az)), Az - Au_{n_i}\rangle + \langle J_p^{E_2}(Az - P_Q(Az)), Au_{n_i} - P_Q(Au_{n_i})\rangle. \end{aligned} \tag{37}$$

Since A is continuous, we have $Au_{n_i} \rightharpoonup Az$ as $i \to \infty$. From (32), we obtain

$$\|(I - P_Q)Az\| = 0,$$

that is $Az = P_Q(Az)$, this shows that $Az \in Q$. Hence, we get $z \in \Omega := F(\mathcal{S}) \cap \Gamma$.

Next, we show that $\{x_n\}$ converges strongly to $\Pi_{F(\mathcal{S})\cap\Gamma}u$. From Lemma 2.14 and (29), we have

$$\begin{aligned} \Delta_p(y_n, x_n) &= \Delta_p(J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)J_p^{E_1}(T(t_n)x_n)), x_n) \\ &\le \beta_n\Delta_p(x_n, x_n) + (1-\beta_n)\Delta_p(T(t_n)x_n, x_n) \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta_p(z_n, x_n) &= \Delta_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)), x_n) \\ &\le \alpha_n\Delta_p(u, x_n) + (1-\alpha_n)\Delta_p(y_n, x_n) \to 0 \ \text{ as } \ n \to \infty, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{38}$$

Now, let $x^* := \Pi_{F(\mathcal{S})\cap\Gamma}u$. Since $\{x_n\}$ is bounded, also we can obtain

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^*\rangle = \lim_{i\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^*\rangle.$$

From (38) and Lemma 2.10, we get that

$$\begin{aligned} \limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_n - x^*\rangle &= \limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^*\rangle \\ &= \lim_{i\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^*\rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z - x^*\rangle \le 0. \end{aligned} \tag{39}$$

Using (C1), (26) and (39), we can conclude that $\Delta_p(x_n, x^*) \to 0$ as $n \to \infty$ by Lemma 2.18. Hence $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\Delta_p(x_{n_i}, x^*) < \Delta_p(x_{n_i+1}, x^*)$ for all $i \in \mathbb{N}$. By Lemma 2.19, Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\Delta_p(x_{m_k}, x^*) \leq \Delta_p(x_{m_k+1}, x^*) \text{ and } \Delta_p(x_k, x^*) \leq \Delta_p(x_{m_k+1}, x^*).$$

Then we have

$$\begin{aligned}
0 &\leq \lim_{k\to\infty} \Big(\Delta_p(x_{m_k+1}, x^*) - \Delta_p(x_{m_k}, x^*)\Big) \\
&\leq \limsup_{n\to\infty} \Big(\Delta_p(x_{n+1}, x^*) - \Delta_p(x_n, x^*)\Big) \\
&\leq \limsup_{n\to\infty} \Big(\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(x_n, x^*) - \Delta_p(x_n, x^*)\Big) \\
&= \limsup_{n\to\infty} \alpha_n\Big(\Delta_p(u, x^*) - \Delta_p(x_n, x^*)\Big) = 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \Big(\Delta_p(x_{m_k+1}, x^*) - \Delta_p(x_{m_k}, x^*)\Big) = 0. \tag{40}$$

By following the method of proof line as in Case 1, we can show that

$$\lim_{k\to\infty} \|x_{m_k} - T(t)x_{m_k}\| = 0, \quad \forall t \geq 0.$$

and

$$\lim_{k\to\infty} \|Au_{m_k} - P_Q(Au_{m_k})\| = 0.$$

Furthermore, we can show that

$$\limsup_{k\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_{m_k} - x^*\rangle \leq 0.$$

Again from (26), we have

$$\Delta_p(x_{m_k+1}, x^*) \leq (1-\alpha_{m_k})\Delta_p(x_{m_k}, x^*) + \alpha_{m_k}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_{m_k} - x^*\rangle,$$

which implies that

$$\alpha_{m_k}\Delta_p(x_{m_k}, x^*) \leq \Delta_p(x_{m_k}, x^*) - \Delta_p(x_{m_k+1}, x^*) + \alpha_{m_k}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_{m_k} - x^*\rangle.$$

Since $\Delta_p(x_{m_k}, x^*) \leq \Delta_p(x_{m_k+1}, x^*)$ and $\alpha_{m_k} > 0$, we get that

$$\Delta_p(x_{m_k}, x^*) \leq \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_{m_k} - x^*\rangle.$$

Hence, $\lim_{k\to\infty} \Delta_p(x_{m_k}, x^*) = 0$. From (40), we also have $\lim_{k\to\infty} \Delta_p(x_{m_k+1}, x^*) = 0$ and hence

$$\limsup_{k\to\infty} \Delta_p(x_k, x^*) \leq \lim_{k\to\infty} \Delta_p(x_{m_k+1}, x^*) = 0.$$

It follows that $x_k \to x^*$ as $k \to \infty$. Therefore, from the above two cases, we conclude that $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = \Pi_{F(\mathcal{S})\cap\Gamma}u$. This completes the proof. □

4. Convergence Theorems for a Family of Mappings

In this section, we apply our main result to a countable family of nonexpasive mappings.

Definition 4.1. ([6]) *Let C be a subset of a real p-uniformly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the* AKTT-condition *if, for any bounded subset B of C,*

$$\sum_{n=1}^{\infty} \sup_{z\in B}\{\|J_p^E(T_{n+1}z) - J_p^E(T_n z)\|\} < \infty.$$

As in [30], we can prove the following Proposition.

Proposition 4.2. *Let C be a nonempty, closed and convex subset of a real p-uniformly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Suppose that for any bounded subset B of C. Then there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in B, \tag{41}$$

and

$$\lim_{n\to\infty} \sup_{z\in B} \|J_p^E(Tz) - J_p^E(T_n z)\| = 0.$$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (41) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3. *Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces and let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of Bregman relatively nonexpansive mappings on C into E_1 such that $F(T_n) = \widehat{F}(T_n)$ for all $n \geq 1$. Suppose that $\bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma \neq \emptyset$. For given $u \in E_1$, let $\{u_n\}$ be a sequence generated by $u_1 \in C$ and*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\Big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\Big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}\Big(T_n x_n\Big)\Big], \quad \forall n \geq 1, \end{cases} \tag{42}$$

where $\{\alpha_n\}$ is sequence in $(0,1)$ and $\{\lambda_n\}$ is real positive sequence which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \leq \lambda_n \leq b < \left(\frac{q}{\kappa_q \|A\|^q}\right)^{\frac{1}{q-1}}$.

In addition, if $(\{T_n\}, T)$ satisfies the AKTT-condition. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $x^* = \Pi_{\bigcap_{n=1}^{\infty} F(T_n)\cap\Gamma} u$.

Proof. By following the method of proof in Theorem 3.1, we can prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. To this end, it suffices to show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1. Then, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 4.2, we observe that

$$\begin{aligned} \|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \|J_p^{E_1}(T_n x_n) - J_p^{E_1}(Tx_n)\| \\ &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_n x) - J_p^{E_1}(Tx)\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, it follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

4.1. Some applications

In this section, we give an application of Theorem 3.1 to the convexly constrained linear inverse problem in the framework of p-uniformly convex and uniformly smooth Banach spaces.

Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces and let C and Q be a nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $k \in E_2$. The *convexly constrained linear inverse problem* [18] is to find

$$\hat{x} \in C \text{ such that } A\hat{x} = k. \tag{43}$$

It is well known that the problem (43) is equivalent to the following minimization problem:

$$\min_{x\in C} \frac{1}{2}\|Ax-k\|^2.$$

The set of solutions of problem (43) is denoted by $\Gamma = \{x \in C : x = A^{-1}k\}$. A classical method for solving problem (43) the well-known projected Landweber method (see [19]) which is defined by the following iteration:

$$x_{n+1} = P_C(x_n - \lambda A^*(Ax_n - k)), \tag{44}$$

where A^* is the adjoint operator of A and $0 < \lambda < \frac{2}{\|A\|^2}$. It is proved that the projected Landweber iteration (44) converges weakly to a solution of problem (43). To obtain strong convergence, Eicke [18] introduced the so-called damped projection method. In what follows, we present an iterative algorithm with strong convergence, for approximating solutions of problem (43) which is also a fixed point problem of a Bregman relatively nonexpansive semigroup.

Setting $P_Q(Au_n) = k$ in Theorem 3.1, we obtain the following result.

Theorem 4.4. *Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces and let C and Q be a nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be adjoint of A. Let $\mathcal{S} = \{T(t)\}_{t\geq 0}$ be a u.a.r. Bregman relatively nonexpansive semigroup and uniformly of Lipschitzian mappings on C into E_1 with a bounded measurable function $L(t) : (0,\infty) \to [0,\infty)$ such that $F(\mathcal{S}) := \bigcap_{h\geq 0} F(T(h)) \neq \emptyset$. Suppose that $F(\mathcal{S}) = \widehat{F}(\mathcal{S})$ and $F(\mathcal{S}) \cap \Gamma \neq \emptyset$. For given $u \in E_1$, let $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by $u_1 \in E_1$ and*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\Big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - k)\Big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)T(t_n)x_n\Big)\Big], \quad \forall n \geq 1, \end{cases} \tag{45}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{t_n\}$ is a real positive divergent sequence and $\{\lambda_n\}$ is real positive sequence which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(C2) $0 < a \leq \beta_n \leq b < 1$;

(C3) $0 < c \leq \lambda_n \leq d < \left(\frac{q}{\kappa_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $x^* = \Pi_{F(\mathcal{S})\cap\Gamma} u$.

5. Numerical Examples

In this section, we present some numerical experiments to support our main Theorem 3.1.

Example 5.1. *Let* $E_1 = E_2 = \mathbb{R}^3$, $C = \{x = (a,b,c)^T \in \mathbb{R}^3 : a^2 + b^2 + c^2 \leq 1\}$ *and* $Q = \{y = (p,q,r)^T \in \mathbb{R}^3 : 2p + q - r \geq -1\}$. *For each* $t \geq 0$, *let* $T(t) : C \to \mathbb{R}^3$ *be defined by* $T(t)x = e^{-t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2t & -\sin 2t \\ 0 & \sin 2t & \cos 2t \end{pmatrix} x$, *where* $x \in C$. *We can check that* $T(t)$ *is a u.a.r. Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mapping with* $\Delta_p(x,y) = \|x - y\|^2$. *We aim to find* $x^* \in C$ *such that* $Ax^* \in Q$, *where* $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 3 & 4 \end{pmatrix}$ *and also* x^* *is a common fixed point of* $T(t)$.

Choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.1$, $\lambda_n = 0.5$ and $t_n = n$ for all $n \in \mathbb{N}$. The stopping criterion is defined by $E_n = \|u_{n+1} - u_n\| < 10^{-4}$. For points u and u_1 randomly, the numerical experiment is reported in Table 1 and the error E_n is demonstrated in Figure 1, respectively.

Choice 1	$u = (0.5, 0.5, 0.5)^T$ $u_1 = (1, -2, 1)^T$	No. of Iter. cpu (Time)	162 0.096795
Choice 2	$u = (0.6, 0, 0.8)^T$ $u_1 = (0.5, 0.7, 1)^T$	No. of Iter. cpu (Time)	121 0.067907
Choice 3	$u = (1, 0, 0)^T$ $u_1 = (-2, 2, 1)^T$	No. of Iter. cpu (Time)	180 0.099143
Choice 4	$u = (-0.2, 0.1, -0.2)^T$ $u_1 = (-1, 3, 0)^T$	No. of Iter. cpu (Time)	187 0.108388

Table 1: The numerical experiment in Example 5.1

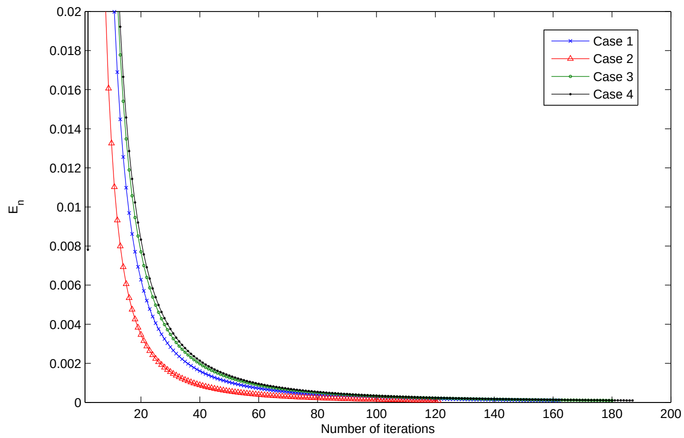

Figure 1: The convergence behavior of E_n in Example 5.1

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