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The Range of Block Hankel Operators

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Abstract. In this note we give a connection between the closure of the range of block Hankel operators acting on the vector-valued Hardy space $H^2_{\mathbb{C}^n}$ and the left coprime factorization of its symbol. Given a subset $F \subseteq H^2_{\mathbb{C}^n}$, we also consider the smallest invariant subspace S^*_F of the backward shift S^* that contains F.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. $\mathcal{B}(\mathcal{H},\mathcal{H})$ is denoted simply by $\mathcal{B}(\mathcal{H})$. A closed subspace $\mathcal{L} \subset \mathcal{H}$ is called an invariant subspace for the operator $T \in \mathcal{B}(\mathcal{H})$ if $T\mathcal{L} \subset \mathcal{L}$. The theory of invariant subspaces of the backward shift operator has enabled important contributions to numerous applications in operator theory and function theory ([6],[13]). Given a subset $F \subseteq H^2_{\mathbb{C}^n}$, the subspace

$$S^*_F := \bigvee \{S^{*n} f : f \in F,\ n \geq 0\},$$

is the smallest invariant subspace of the backward shift S^* that contains F. If $S^*_F \neq H^2_{\mathbb{C}^n}$ then by the Beurling-Lax-Halmos Theorem, there is an inner matrix function $\Theta \in H^2_{M_{n\times m}}$ such that

$$S^*_F = H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^m}. \tag{1}$$

The purpose of this note is to determine the inner matrix function $\Theta \in H^\infty_{M_{n\times m}}$ satisfying (1).

Let us recall the basic properties of unbounded operators ([2]). If $A : \mathcal{H} \to \mathcal{K}$ is a linear operator, then A is also a linear operator from the closure of the domain of A, denoted by $\mathrm{cl}[\mathrm{dom}\, A]$, into $\mathcal{K}$. So we will only consider A such that $\mathrm{dom}\, A$ is dense in $\mathcal{H}$. Then, such an operator A is said to be densely defined. If $A : \mathcal{H} \to \mathcal{K}$ is densely defined, we write $\ker A$ and $\mathrm{ran}\, A$ for the kernel and range of A, respectively. For a set $\mathcal{M}$, $\mathrm{cl}\mathcal{M}$ and $\mathcal{M}^\perp$ respectively denote the closure and orthogonal complement of $\mathcal{M}$. Let $A : \mathcal{H} \to \mathcal{K}$ be densely defined, and let

$$\mathrm{dom}\, A^* = \Big\{ k \in \mathcal{K} : \langle Ah, k\rangle \text{ is a bounded linear functional on } \mathrm{dom} A \Big\}.$$

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Then for each $k \in \mathrm{dom}A^*$, there exists a unique $f \in \mathcal{K}$ such that $\langle Ah, k\rangle = \langle h, f\rangle$ for all $h \in \mathrm{dom}A$. Denote this unique vector f as $f = A^*k$. Thus $\langle Ah, k\rangle = \langle h, A^*k\rangle$ for h in $\mathrm{dom}A$ and k in $\mathrm{dom}A^*$.

We review a few essential facts for Toeplitz operators and Hankel operators, and for that we will use [3], [4], [5], [11] and [12]. For E a Hilbert space, let $L^2_E = L^2_E(\mathbb{T})$ be the set of all E-valued square-integrable measurable functions on the unit circle $\mathbb{T}$ and H^2_E be the corresponding Hardy space. For $f, g \in L^2_E$, the inner product $\langle f,\ g\rangle$ is defined by

$$\langle f,\ g\rangle := \int_{\mathbb{T}} \langle f(z), g(z)\rangle_E dm(z),$$

where m denotes the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Let M_n denote the set of $n \times n$ complex matrices, and let $\mathcal{P}_{\mathbb{C}^n}$ be the set of all polynomials p with value in $\mathbb{C}^n$, which is dense in $H^2_{\mathbb{C}^n}$. For $\Phi \in L^2_{M_n}$, the (unbounded) Hankel operator H_Φ on $H^2_{\mathbb{C}^n}$ and (unbounded) Toeplitz operator T_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$H_\Phi p := JP^\perp(\Phi p) \quad \text{and} \quad T_\Phi p := P(\Phi p) \qquad (p \in \mathcal{P}_{\mathbb{C}^n}),$$

where P and $P^\perp$ denote the orthogonal projections that map from $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively, and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ onto $L^2_{\mathbb{C}^n}$, given by $(Jg)(z) := \overline{z}I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ ($I_n :=$ the $n \times n$ identity matrix). For $\Phi \in L^2_{M_{n\times m}}$, we write

$$\widetilde{\Phi}(z) \equiv \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ is called an *inner* if Θ is an isometric a.e. on $\mathbb{T}$. The following basic relations can be easily derived from the definition:

$$T^*_\Phi = T_{\Phi^*},\ \ H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{2}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi},\ \ H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}); \tag{3}$$

$$H^*_\Phi H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H^*_\Phi H_{\Theta^*} H^*_{\Theta^*} H_\Phi \quad (\Theta \in H^\infty_{M_n} \text{ is inner},\ \Phi \in L^\infty_{M_n}). \tag{4}$$

The *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := T_{zI_n}.$$

The following fundamental result known as the Beurling-Lax-Halmos theorem is useful in the sequel.

The Beurling-Lax-Halmos Theorem. ([7], [11]) A nonzero subspace M of $H^2_{\mathbb{C}^n}$ is invariant for the shift operator S on $H^2_{\mathbb{C}^n}$ if and only if $M = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$. Furthermore, Θ is unique up to a unitary constant right factor. That is, if $M = \Delta H^2_{\mathbb{C}^r}$ (for Δ an inner function in $H^\infty_{M_{n\times r}}$), then $m = r$ and $\Theta = \Delta W$, where W is a unitary matrix mapping $\mathbb{C}^m$ onto $\mathbb{C}^m$.

As is customarily done, we say that two matrix functions A and B are *equal* if they are equal up to a unitary constant right factor. If $\Phi \in L^\infty_{M_n}$, then by (3), $\ker H_\Phi$ is an invariant subspace of the shift operators on $H^2_{\mathbb{C}^n}$. Thus, if $\ker H_\Phi \neq \{0\}$, then the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function $\Theta \in H^\infty_{M_{n\times m}}$.

A function $\phi \in L^2$ is said to be of a *bounded type* if there are functions $\psi_1, \psi_2 \in H^\infty$ such that $\phi = \frac{\psi_1}{\psi_2}$ a.e. on $\mathbb{T}$. For a matrix-valued function $\Phi \equiv \left[\phi_{ij}\right] \in L^2_{M_{n\times m}}$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type. For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. We would remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero, then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero, then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$ ([3], [7]).

From now on, for notational convenience, we write

$$I_\omega := \omega I_n \quad (\omega \in H^2).$$

Let $\Phi \in L^2_{M_n}$ with Φ be of bounded type. Then it is well known ([9]) that Φ can be represented as

$$\Phi = I_\theta^* A \qquad (A \in H^2_{M_n},\ \theta \text{ is inner}). \tag{5}$$

In (5), I_θ and A need not be left coprime. If $\Omega = \text{left-g.c.d.}\,\{I_\theta, A\}$, then $I_\theta = \Omega\Omega_\ell$ and $A = \Omega A_\ell$ for some inner matrix Ω_ℓ and A_ℓ in $H^2_{M_n}$. Therefore we can write

$$\Phi = \Omega_\ell^* A_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{6}$$

In this case, $\Omega_\ell^* A_\ell$ is called the *left coprime factorization* of Φ, and we write

$$\Phi = \Omega_\ell^* A_\ell \ \ \text{(left coprime)}.$$

Similarly, we can write

$$\Phi = A_r \Omega_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{7}$$

In this case, $A_r\Omega_r^*$ is called the *right coprime factorization* of Φ, and we write

$$\Phi = A_r \Omega_r^* \ \ \text{(right coprime)}.$$

Our main theorem is now stated as:

Theorem 1.1. *Let $F \in H^2_{M_n}$ be such that F^* is of a bounded type. Then in view of (6), we may write*

$$F^* = \Theta^* A \quad (\textit{left coprime}).$$

Then

$$cl\ ran\, H_{F^*} = \mathcal{H}(\widetilde{\Theta}).$$

2. The Proof of Main Theorem

In this section we give a proof of Theorem 1.1. We recall the inner-outer factorization of vector-valued functions. Let D and E be Hilbert spaces. If F is a function with values in $\mathcal{B}(E, D)$ such that $F(\cdot)e \in H^2_D$ for each $e \in E$, then F is called a strong H^2-function. The strong H^2-function F is called an *inner* function if $F(\cdot)$ is an isometric operator from D into E. Write $\mathcal{P}_E$ for the set of all polynomials with values in E. Then the function $Fp = \sum_{k=0}^n \widehat{Fp}(k) z^k$ belongs to H^2_D. The strong H^2-function F is called *outer* if $\text{cl}\, F \cdot \mathcal{P}_E = H^2_D$. We then have an analogue of the scalar inner-outer factorization Theorem. Note that every $F \in H^2_{M_n}$ is a strong H^2-function.

Lemma 2.1. *([11]) Every strong H^2-function F with values in $\mathcal{B}(E, D)$ can be expressed in the form*

$$F = F_i F_e,$$

where F_e is an outer function with values in $\mathcal{B}(E, D')$ and F_i is an inner function with values in $\mathcal{B}(D', D)$, for some Hilbert space D'.

For $\phi = \left[\phi_1, \phi_2, \cdots, \phi_n\right]^t \in L^2_{\mathbb{C}^n}$, we write

$$\overline{\phi} := \left[\overline{\phi}_1, \overline{\phi}_2, \cdots, \overline{\phi}_n\right]^t \quad \text{and} \quad \breve{\phi} := \left[\widetilde{\overline{\phi}}_1, \widetilde{\overline{\phi}}_2, \cdots, \widetilde{\overline{\phi}}_n\right]^t.$$

Then it is easy to show that

$$S^*\overline{g} = J\breve{g} \quad \text{if } g \in \left(H^2_{\mathbb{C}^n}\right)^\perp. \tag{8}$$

Lemma 2.2. *Let $f \equiv [f_1, f_2, \cdots, f_n]^t \in H^2_{\mathbb{C}^n}$. Then,*

$$S^*_f = cl\ ranH_{\overline{z}\breve{f}}.$$

Proof. For each $n \in \mathbb{N}$, it follows from (8) that

$$\begin{aligned} S^{*n} f &= S^*\left(P(\overline{z}^{n-1} f)\right) \\ &= S^*\overline{\left((I-P)(z^{n-1}\overline{f})\right)} \\ &= J(I-P)(z^{n-1}\breve{f}) \\ &= H_{\overline{z}\breve{f}} z^n. \end{aligned}$$

Thus,

$$S^*_f = \bigvee \{S^{*n} f : n \geq 0\} = \text{cl ran} H_{\overline{z}\breve{f}},$$

which gives the result. □

For an inner matrix function $\Theta \in H^\infty_{M_{n\times m}}$, we write $\mathcal{H}(\Theta) := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^m}$. It is easy to show that [11]:

$$f \in \mathcal{H}(\Theta) \Longleftrightarrow \Theta^* f \in (H^2_{\mathbb{C}^n})^\perp. \tag{9}$$

We now recall the notion of the reduced minimum modulus([1], [10]). The reduced minimum modulus of operators measures the closedness for the range of operators. If $T \in \mathcal{B}(\mathcal{H})$ then the *reduced minimum modulus* of T is defined by

$$\gamma(T) = \begin{cases} \inf\left\{\|Tx\| : \ \text{dist}\,(x, \ker T) = 1\right\} & \text{if } T \neq 0 \\ 0 & \text{if } T = 0. \end{cases}$$

It is easy to see that $\gamma(T) > 0$ if and only if $T(\mathcal{H}_0)$ is closed for each closed subspace $\mathcal{H}_0$ of $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ is a nonzero operator, then we can see that $\gamma(T) = \inf\left(\sigma(|T|) \setminus \{0\}\right)$, where $|T|$ denotes $(T^*T)^{\frac{1}{2}}$. Thus we have that $\gamma(T) = \gamma(T^*)$ ([8]). For $\mathcal{X}$ a closed subspace of $H^2_{\mathbb{C}^n}$, $P_{\mathcal{X}}$ denotes the orthogonal projection from $H^2_{\mathbb{C}^n}$ onto $\mathcal{X}$.

Lemma 2.3. *([9]) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:*

(i) *Φ is of bounded type;*

(ii) *$\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;*

(iii) *$\Phi = A\Theta^*$ (right coprime).*

Lemma 2.4. *Let $\Theta, \Delta \in H^2_{M_n}$ be inner functions. Then*

(a) *$H_{\Theta^*}(\Delta H^2_{\mathbb{C}^n})$ is closed.*

(b) *If Θ and Δ are left coprime, then $H_{\Theta^*}(\Delta H^2_{\mathbb{C}^n}) = \mathcal{H}(\widetilde{\Theta})$.*

Proof. If $\Theta \in M_n$, then Θ is a unitary matrix, and hence $\widetilde{\Theta} \in M_n$ is a unitary matrix. Thus, $H_{\Theta^*}(\Delta H^2_{\mathbb{C}^n}) = \{0\} = \mathcal{H}(\widetilde{\Theta})$. This gives the result. Let $\Theta \notin M_n$. Since Θ is an inner function, by (4), we have $H^*_{\Theta^*}H_{\Theta^*} = P_{\mathcal{H}(\Theta)}$, so that $|H_{\Theta^*}| = P_{\mathcal{H}(\Theta)} \neq 0$. Thus $\gamma(H_{\Theta^*}) = \inf\big(\sigma(|H_{\Theta^*}|) \setminus \{0\}\big) = 1$, and hence $H_{\Theta^*}(\Delta H^2_{\mathbb{C}^n})$ is closed. This proves (a). Suppose Θ and Δ are left coprime inner functions. Then $\Theta H^2_{\mathbb{C}^n} \bigvee \Delta H^2_{\mathbb{C}^n} = H^2_{\mathbb{C}^n}$. Thus,

$$\mathcal{A} \equiv \left\{\Theta h_1 + \Delta h_2 : h_1, h_2 \in H^2_{\mathbb{C}^n}\right\}$$

is dense in $H^2_{\mathbb{C}^n}$. On the other hand, it follows from Lemma 2.3 that $\ker H_{\Theta^*} = \Theta H^2_{\mathbb{C}^n}$, and hence $\text{cl}\, H_{\Theta^*}(\mathcal{A}) = H_{\Theta^*}(\Delta H^2_{\mathbb{C}^n})$. Since $\mathcal{H}(\widetilde{\Theta}) = \left(\ker H^*_{\Theta^*}\right)^{\perp} = \text{ran}\, H_{\Theta^*}$, it follows that

$$\mathcal{H}(\widetilde{\Theta}) = H_{\Theta^*}(\text{cl}\mathcal{A}) \subseteq \text{cl}\, H_{\Theta^*}(\mathcal{A}) = H_{\Theta^*}(\Delta H^2_{\mathbb{C}^n}) \subseteq \text{ran}\, H_{\Theta^*} = \mathcal{H}(\widetilde{\Theta}),$$

which gives (b). □

Proof of Theorem 1.1. Let $p \in \mathcal{P}_{\mathbb{C}^n}$ be arbitrary. Write $p_1 \equiv P_{\mathcal{H}(\Theta)}Ap$. Then it follows from (9) that

$$H_{F^*}p = J(I-P)(\Theta^*Ap) = J(\Theta^*p_1) = \overline{z}\widetilde{\Theta}\breve{p}_1,$$

which implies that $\widetilde{\Theta}^* H_{F^*}p \in (H^2_{\mathbb{C}^n})^{\perp}$. Thus, by again (9), $H_{F^*}p \in \mathcal{H}(\widetilde{\Theta})$, so that

$$\text{cl ran}\, H_{F^*} \subseteq \mathcal{H}(\widetilde{\Theta}).$$

For the converse inclusion, let $h \in \ker H^*_{F^*}$ be arbitrary. Since $A \in H^2_{M_n}$ is a strong H^2-function, by Lemma 2.1, we can write

$$A = A_i A_e\,,$$

where $A_i \in H^2_{M_{n\times m}}$ is inner and $A_e \in H^2_{M_{m\times n}}$ is outer. Then we have that (cf. [11, p.44])

$$\text{cl}\, A\mathcal{P}_{\mathbb{C}^n} = A_i H^2_{\mathbb{C}^n}. \tag{10}$$

For each $p \in \mathcal{P}_{\mathbb{C}^n}$, we have

$$0 = \left\langle p,\ H^*_{F^*}h\right\rangle = \left\langle J(I-P)\Theta^*Ap,\ h\right\rangle = \left\langle \Theta^*Ap,\ Jh\right\rangle.$$

Thus, it follows from (10) that

$$\left\langle H_{\Theta^*}(A_if),\ h\right\rangle = \left\langle \Theta^*A_if,\ Jh\right\rangle = 0 \quad \text{for all } f \in H^2_{\mathbb{C}^n}. \tag{11}$$

On the other hand, since Θ and A are left coprime, Θ and A_i are left coprime. Thus, it follows from Lemma 2.4 and (11) that $\ker H^*_{F^*} \subseteq \left(H_{\Theta^*}(A_iH^2)\right)^{\perp} = \widetilde{\Theta}H^2_{\mathbb{C}^n}$, so that

$$\mathcal{H}(\widetilde{\Theta}) \subseteq \left(\ker H^*_{F^*}\right)^{\perp} = \text{cl ran}\, H_{F^*}.$$

This completes the proof. □

For $F = \{f_1, f_2, f_3, \cdots, f_m\} \subset H^2_{\mathbb{C}^n}$ $(m \leq n)$, let

$$\Phi_F \equiv \overline{z}[\breve{f}_1, \breve{f}_2, \cdots, \breve{f}_m, 0, \cdots, 0] \in H^2_{M_n}.$$

We then have:

Corollary 2.5. *Let $F \equiv \{f_1, f_2, \cdots, f_m\} \subset H^2_{\mathbb{C}^n}$ $(m \le n)$ be such that $\overline{f}_i$ is of bounded type for each i. Then in view of (6), we may write*

$$\Phi_F = \Theta^* A \quad (\textit{left coprime}).$$

Then

$$S_F^* = \mathcal{H}(\widetilde{\Theta}).$$

Proof. It follows from Lemma 2.2 and Theorem 1.1 that

$$S_F^* = \bigvee_{k=1}^{m} \mathrm{ran} H_{\overline{z}\breve{f}_k} = \mathrm{cl}\ \mathrm{ran}\, H_{\Phi_F} = \mathcal{H}(\widetilde{\Theta}).$$

This completes the proof. □

Remark 2.6. *Suppose $F \equiv \{f_1, f_2, \cdots, f_N\} \subset H^2_{\mathbb{C}^n}$ $(N > n)$ be such that $\overline{f}_i$ is of a bounded type for each i. Let*

$$\Phi_F \equiv \overline{z}\begin{bmatrix} \breve{f}_1 & \breve{f}_2 & \cdots & \breve{f}_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \Theta^* A \in H^2_{M_N} \qquad (\textit{left coprime}).$$

Then, it follows from Corollary 2.5 that

$$S_F^* \bigoplus 0|_{\mathbb{C}^{N-n}} = \mathcal{H}(\widetilde{\Theta}).$$

Example 2.7. *Let a and c be nonzero complex numbers and $f = [az, cb_\alpha]^t$ $(b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z},\ 0 < |\alpha| < 1)$. Put*

$$\Phi = \begin{bmatrix} a\overline{z}^2 & 0 \\ c\overline{z}b_\alpha(\overline{z}) & 0 \end{bmatrix}.$$

Observe that for $x, y \in H^2$,

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} \in \ker H_{\widetilde{\Phi}} &\Longleftrightarrow \overline{a}b_\alpha x + \overline{c}zy \in z^2 b_\alpha H^2 \\ &\Longrightarrow \begin{cases} \overline{a}b_\alpha(0)x(0) = 0 \\ \overline{c}\alpha y(\alpha) = 0. \end{cases} \\ &\Longrightarrow \begin{cases} x = zx_1 \text{ for some } x_1 \in H^2 \\ y = b_\alpha y_1 \text{ for some } y_1 \in H^2. \end{cases} \end{aligned} \tag{12}$$

By (12), we have that $x = zx_1$ and $y = b_\alpha y_1$ for some $x_1, y_1 \in H^2$. We thus have

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} \in \ker H_{\widetilde{\Phi}} &\Longleftrightarrow \overline{a}x_1(0) + \overline{c}y_1(0) = 0 \\ &\Longleftrightarrow x_1(0) = \gamma y_1(0) \quad \left(\gamma := -\frac{\overline{c}}{\overline{a}}\right). \end{aligned} \tag{13}$$

Put

$$\Theta := \frac{1}{\sqrt{1+|\gamma|^2}} \begin{bmatrix} z & 0 \\ 0 & b_\alpha \end{bmatrix} \begin{bmatrix} z & \gamma \\ -\overline{\gamma}z & 1 \end{bmatrix}.$$

Then Θ is inner, and it follows from (13) that

$$\ker H_{\widetilde{\Phi}} = \Theta H^2_{\mathbb{C}^2}.$$

Thus by Lemma 2.3, we have $\widetilde{\Phi} = A\Theta^*$ *(right coprime) and hence* $\Phi = \widetilde{\Theta}^*\widetilde{A}$ *(left coprime). It thus follows from Corollary 2.5 that*

$$S_f^* = \mathcal{H}(\Theta).$$

Corollary 2.8. *Let* $f \in H^2$.

(a) $\overline{f}$ *is not of bounded type if and only if* $S_f^* = H^2$.

(b) If $\overline{f}$ *is of bounded type of the form*

$$f = \theta\overline{a} \quad (\textit{left coprime}),$$

then

$$S_f^* = \begin{cases} \mathcal{H}(z\theta) & \textit{if } a(0) \neq 0 \\ \mathcal{H}(\theta) & \textit{if } a(0) = 0. \end{cases}$$

Proof. Note that $\overline{f}$ is of a bounded type if and only if $\overline{z}\breve{f}$ is of a bounded type. Thus, it follows from Corollary 2.5 that $\overline{f}$ is of a bounded type if and only if $S_f^* \neq H^2$. This proves (a). For (b), let $a(0) \neq 0$. Then, z and a are coprime so that $z\widetilde{\theta}$ and $\widetilde{a}$ are coprime. Thus

$$\overline{z}\breve{f} = \overline{(z\widetilde{\theta})}\widetilde{a} \quad \text{(left coprime)}.$$

It follows from Corollary 2.5 that $S_f^* = \mathcal{H}(z\theta)$. If instead $a(0) = 0$, then we may write $a = za'$ for some $a' \in H^2$ so that

$$\overline{z}\breve{f} = \overline{\widetilde{\theta}}\widetilde{a'} \quad \text{(left coprime)}.$$

Thus, again by Corollary 2.5, we have $S_f^* = \mathcal{H}(\theta)$. This completes the proof. □

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