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Uniquely Shift-Transitive Graphs of Valency 5

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Abstract. An automorphism σ of a finite simple graph Γ is a shift, if for every vertex $v \in V(\Gamma)$, σv is adjacent to v in Γ. The graph Γ is shift-transitive, if for every pair of vertices $u, v \in V(\Gamma)$ there exists a sequence of shifts $\sigma_1, \sigma_2, ..., \sigma_k \in \mathrm{Aut}(\Gamma)$ such that $\sigma_1\sigma_2...\sigma_k u = v$. If, in addition, for every pair of adjacent vertices $u, v \in V(\Gamma)$ there exists exactly one shift $\sigma \in \mathrm{Aut}(\Gamma)$ sending u to v, then Γ is uniquely shift-transitive. The purpose of this paper is to prove that, if Γ is a uniquely shift-transitive graph of valency 5 and S_Γ is the set of shifts of Γ then $\langle S_\Gamma \rangle$, the subgroup generated by S_Γ is an Abelian regular subgroup of $\mathrm{Aut}(\Gamma)$.

1. Introduction

Throughout this paper, groups are finite and graphs are simple, finite, connected and undirected. For graph and group-theoretic concepts not defined here, we refer the reader to [1] and [4]. We start by recalling some notations and definitions from [2] and [5]: If u and v are two adjacent vertices in graph Γ, we write $u \sim v$. Let G be a group and S a subset of G that is closed under inverses and does not contain the identity. The *Cayley graph* $\Gamma = \mathrm{Cay}(G, S)$ with connection set S is the graph whose vertex set is G, two vertices u, v being joined by an edge if $uv^{-1} \in S$. A *quasi-Abelian Cayley graph* is a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, where S is the union of conjugacy classes in G. An automorphism σ of a graph Γ is a *shift*, if for every vertex $v \in V(\Gamma)$, we have $\sigma v \sim v$.

We call a graph Γ *shift-transitive* if for every pair of vertices $u, v \in V(\Gamma)$, there exists a sequence of shifts $\sigma_1, \sigma_2, ..., \sigma_k \in \mathrm{Aut}(\Gamma)$, such that $\sigma_1\sigma_2...\sigma_k u = v$. If, in addition, for every pair of adjacent vertices $u, v \in V(\Gamma)$ there exists exactly one (respectively, at least one) shift $\sigma \in \mathrm{Aut}(\Gamma)$ sending u to v, then Γ is *uniquely shift-transitive* (respectively, *strongly shift-transitive*).

Since uniquely shift-transitive graphs are strongly shift-transitive and strongly shift-transitivity implies vertex transitivity, we find that if Γ is a uniquely shift-transitive graph, then it is vertex-transitive. So Γ is regular and the size of S_Γ, which is the set of shifts of Γ, and the valency of Γ are equal.

In [3] the authors investigate these concepts in some standard graph products and the following two questions are posed in [2].

Question 1.1. *Is every uniquely shift-transitive Cayley graph isomorphic with a Cayley graph of an Abelian group?*

Question 1.2. *Does there exist a uniquely shift-transitive non-Cayley graph?*

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Our motivation for this paper is to give an answer to the Question 1.1 without assuming that Γ is a Cayley graph, and a partial answer to the Question 1.2.

In Section 2 we give some propositions that will be used in Section 3 and finally in Section 3 we prove the following main result.

Theorem 1.3. *Every uniquely shift-transitive graph Γ of valency 5 is isomorphic with a Cayley graph of an Abelian group.*

2. Preliminaries

In this section we prove some propositions to show that in a uniquely shift-transitive graph of valency 5 the shifts commute with each other.

Remark 2.1. *If Γ is a uniquely shift-transitive graph and $\alpha, \beta \in \mathrm{Aut}(\Gamma)$ are two shifts such that $\alpha v = \beta v$ for some $v \in V(\Gamma)$, then $\alpha = \beta$. Also if Γ is a uniquely shift-transitive graph of valency 5 and $S_\Gamma = \{\alpha, \beta, \gamma, \delta, \eta\}$ is the set of shifts of Γ then $|V(\Gamma)| \geq 8$ and since $\langle S_\Gamma\rangle$ acts transitively on $V(\Gamma)$, so $|\langle S_\Gamma\rangle| \geq 8$.*

Proposition 2.2. *([2, Proposition 4.1]) Let $\Gamma = \mathrm{Cay}(G, S)$ be a quasi-Abelian Cayley graph of a non-Abelian group G, Then Γ is not uniquely shift-transitive.*

Proposition 2.3. *Let Γ be a uniquely shift-transitive graph of valency 5 and $S_\Gamma = \{\alpha, \beta, \gamma, \delta, \eta\}$ be the set of shifts of Γ. Moreover assume that $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \eta^2 = \mathrm{id}$, where by id we mean the identity permutation. Then*

(1) *If $\alpha\beta\alpha = \gamma$ and $\beta\alpha\beta = \gamma$ then $\alpha\delta\alpha \neq \delta$.*

(2) *If $\alpha\beta\alpha = \gamma$ and $\beta\alpha\beta = \delta$ then $\beta\gamma\beta \neq \gamma$.*

(3) *If $\alpha\beta\alpha = \gamma$, $\beta\alpha\beta = \delta$ and $\beta\gamma\beta = \eta$ then $\alpha\delta\alpha \neq \eta$.*

Proof. (1) : Suppose, in contrary, that $\alpha\delta\alpha = \delta$. Since $\alpha\beta\alpha = \gamma$, we have $\alpha\eta\alpha = \eta$. Now consider the shift $\beta\delta\beta$. If $\beta\delta\beta = \eta$, then we have

$$\eta = \alpha\eta\alpha = \alpha\beta\delta\beta\alpha = \gamma\alpha\delta\alpha\gamma = \gamma\delta\gamma = \gamma\beta\eta\beta\gamma = \beta\alpha\eta\alpha\beta = \beta\eta\beta = \delta,$$

which is a contradiction. Thus $\beta\delta\beta = \delta$ and so $\beta\eta\beta = \eta$. Therefore we have the following equalities:

$$\alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \alpha\eta\alpha = \eta, \beta\alpha\beta = \gamma,\ \beta\delta\beta = \delta,\ \beta\eta\beta = \eta, \delta\eta\delta = \eta. \tag{2.1}$$

Let $G = \langle S_\Gamma\rangle$, $H = \langle \alpha, \beta, \gamma, \delta, \eta \mid \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \eta^2 = \mathrm{id},\ \alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \alpha\eta\alpha = \eta,\ \beta\alpha\beta = \gamma,\ \beta\delta\beta = \delta,\ \beta\eta\beta = \eta,\ \delta\eta\delta = \eta\rangle$, $M = \langle\ \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = \mathrm{id},\ \alpha\beta\alpha = \gamma,\ \beta\alpha\beta = \gamma\rangle$ and $N = \langle\ \delta, \eta \mid \delta^2 = \eta^2 = \mathrm{id},\ \delta\eta = \eta\delta\rangle$. Then by Equation 2.1 we have $M \trianglelefteq H$, $N \trianglelefteq H$, $M \cap N = \{\mathrm{id}\}$ and $H = MN$. Thus $H = M \times N$. Since $M \cong S_3$ and $N \cong C_2^2$, so $H \cong S_3 \times C_2^2$ and G is isomorphic to a quotient of $S_3 \times C_2^2$. Since $|V(\Gamma)|$ divides $|G|$ and $|G|$ divides $|H| = 24$, we find that $|G| = 8$ or 12 or 24. If G has order 8 or 12 we have $|V(\Gamma)| = |G|$ which means that G acts regularly on $V(\Gamma)$. So Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2, Γ is not uniquely shift-transitive, which is a contradiction. If $|G| = 24$ then $G \cong S_3 \times C_2^2$ and

$$G = \{\mathrm{id},\ \alpha,\ \beta,\ \gamma\beta,\ \gamma,\ \beta\gamma,\ \delta,\ \alpha\delta,\ \eta,\ \alpha\eta,\ \beta\delta,\ \gamma\beta\delta,\ \gamma\delta,\ \beta\gamma\delta,$$
$$\eta\delta,\ \alpha\eta\delta,\ \beta\eta,\ \gamma\beta\eta,\ \gamma\eta\alpha,\ \gamma\eta,\ \eta\beta\delta,\ \gamma\eta\beta\delta,\ \alpha\eta\beta\delta,\ \gamma\eta\delta\}.$$

Now note that the stabiliser of a vertex v in G is a core-free subgroup of $S_3 \times C_2^2$, which has order 1 or 2. Recall that a subgroup H of a group G is called *core-free* if $\bigcap_{g\in G} H^g = 1$. Moreover if H is a core-free subgroup of G then the largest normal subgroup of G which is contained in H is 1. If G_v has order 2 then $G_v = \{\mathrm{id}, \theta\}$ where $\theta^2 = \mathrm{id}$. Thus,

$$\theta \in \{\alpha,\ \beta,\ \gamma,\ \delta,\ \eta,\ \alpha\delta,\ \alpha\eta,\ \beta\delta,\ \beta\eta,\ \gamma\delta,\ \gamma\eta,\ \eta\delta,\ \alpha\eta\delta,\ \beta\eta\delta,\ \gamma\eta\delta\}.$$

Since the shifts fix no vertices so $\theta \notin \{\alpha, \beta, \gamma, \delta, \eta\}$. Also if $\theta = \alpha\delta$ then $\alpha\delta v = v$. Thus $\alpha v = \delta v$ which implies $\alpha = \delta$, a contradiction. So $\theta \notin \{\alpha\delta, \alpha\eta, \beta\delta, \beta\eta, \gamma\delta, \gamma\eta, \eta\delta\}$. Therefore G_v is one of $\{\mathrm{id}, \alpha\eta\delta\}$, $\{\mathrm{id}, \beta\eta\delta\}$ or $\{\mathrm{id}, \gamma\eta\delta\}$ and Γ has order 12. Without loss of generality we may assume that $G_v = \{\mathrm{id}, \alpha\eta\delta\}$. Thus $\alpha\eta\delta v = v$ and so $\eta\delta v = \alpha v$. Therefore by Equation 2.1 we find that Γ has vertex set,

$$V(\Gamma) = \{v,\ \alpha v,\ \beta v,\ \gamma\beta v,\ \gamma v,\ \beta\gamma v,\ \delta v,\ \alpha\delta v,\ \gamma\delta v,\ \beta\gamma\delta v,\ \beta\delta v,\ \gamma\beta\delta v\}.$$

In this graph,

$$\sigma = \delta\eta = (v\ \alpha v)(\beta v\ \gamma\beta v)(\gamma v\ \beta\gamma v)(\delta v\ \alpha\delta v)(\gamma\delta v\ \beta\gamma\delta v)(\beta\delta v\ \gamma\beta\delta v),$$

is a shift different from α, β, γ, δ and η. This is a contradiction with the unique shift-transitivity of Γ.

Assume that G_v has order 1. Then Γ has order 24 and G acts regularly on $V(\Gamma)$. Hence Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2, Γ is not uniquely shift-transitive, which is a contradiction.

So in the above cases, we obtain a contradiction. Therefore the proof of Part (1) is complete.

(2) : Suppose, to the contrary, that $\beta\gamma\beta = \gamma$. Thus $\beta\eta\beta = \eta$. Since $\beta\alpha\beta = \delta$ and $\alpha\beta\alpha = \gamma$ so $\alpha\delta\alpha = \alpha\beta\alpha\beta\alpha = \gamma\beta\alpha = \beta\gamma\alpha = \beta\alpha\beta = \delta$ and $\alpha\eta\alpha = \eta$. Therefore we have the following relations between the shifts of Γ:

$$\alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \alpha\eta\alpha = \eta,\ \beta\alpha\beta = \delta,\ \beta\gamma\beta = \gamma,\ \beta\eta\beta = \eta. \tag{2.2}$$

Let $G = \langle S_\Gamma \rangle$, $H = \langle \alpha, \beta, \gamma, \delta, \eta \mid \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \eta^2 = \mathrm{id},\ \alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \alpha\eta\alpha = \eta,\ \beta\alpha\beta = \delta,\ \beta\gamma\beta = \gamma,\ \beta\eta\beta = \eta\rangle$, $M = \langle \alpha, \beta, \gamma, \delta \mid \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \mathrm{id},\ \alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \beta\alpha\beta = \delta, \beta\gamma\beta = \gamma\rangle$ and $N = \langle \eta \mid \eta^2 = \mathrm{id}\rangle$, then by Equation 2.2 we have $M \trianglelefteq H$, $N \trianglelefteq H$, $M \cap N = \{\mathrm{id}\}$ and $H = MN$. So $H = M \times N$. Since $M \cong D_8$ and $N \cong C_2$ thus $H \cong D_8 \times C_2$ and G is isomorphic to a quotient of $D_8 \times C_2$.

By Remark 2.1, we find that $|G| = 8$ or 16. If $|G| = 8$, then $|V(\Gamma)| = 8$ and G acts regularly on $V(\Gamma)$. So in this case Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2, Γ is not uniquely shift-transitive, which is a contradiction. If $|G| = 16$ then $G \cong D_8 \times C_2$ and

$$G = \{\mathrm{id},\ \beta,\ \alpha,\ \gamma\alpha,\ \gamma,\ \beta\gamma,\ \delta,\ \gamma\delta,\ \eta,\ \beta\eta,\ \eta\gamma,\ \beta\eta\gamma,\ \alpha\eta,\ \gamma\alpha\eta,\ \delta\eta,\ \gamma\delta\eta\}.$$

Now the stabiliser of a vertex v in G is a core-free subgroup of $D_8 \times C_2$, and so it has order 1 or 2. If G_v has order 2 then $G_v = \{id, \theta\}$, where $\theta^2 = \mathrm{id}$. Thus,

$$\theta \in \{\alpha,\ \beta,\ \gamma,\ \delta,\ \eta,\ \beta\gamma,\ \alpha\eta,\ \beta\eta,\ \gamma\eta,\ \delta\eta,\ \alpha\delta,\ \beta\eta\gamma\}.$$

By a similar argument as Part(1), $\theta \notin \{\alpha, \beta, \gamma, \delta, \eta, \beta\gamma, \alpha\eta, \beta\eta, \gamma\eta, \delta\eta, \alpha\delta\}$. If $\theta = \beta\eta\gamma$ then $G_v \trianglelefteq G$, which is a contradiction because G_v is a core-free subgroup of G.

If G_v has order 1 then, G acts regularly on $V(\Gamma)$. So Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2, Γ is not uniquely shift-transitive, which is a contradiction. Therefore the proof of Part (2) is complete.

(3) : Suppose, by way of contradiction, that $\alpha\delta\alpha = \eta$. We have the following relations between the shifts of Γ:

$$\alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \eta, \beta\alpha\beta = \delta,\ \beta\gamma\beta = \eta, \tag{2.3}$$

Let $G = \langle S_\Gamma \rangle$, $a = \beta\alpha$ and $b = \alpha$. Then by Equation 2.3, we have $a^5 = b^2 = (ba)^2 = \mathrm{id}$. Thus $G \cong D_{10}$ and,

$$G = \{\mathrm{id},\ \alpha,\ \beta,\ \beta\alpha,\ \gamma,\ \gamma\alpha,\ \delta,\ \delta\alpha,\ \eta,\ \eta\alpha\}.$$

Now the stabiliser of a vertex in G is a core-free subgroup of D_{10}, so it has order 1 or 2. If it has order 2 then Γ has order 5 which is impossible. If it has order 1 then Γ has order 10 and G acts regularly on $V(\Gamma)$. So Γ is a quasi-Abelian Cayley graph of non-Abelian group G with connection set S_Γ. By Proposition 2.2 Γ is not uniquely shift-transitive which is a contradiction. □

Proposition 2.4. *Let Γ be a uniquely shift-transitive graph of valency 5 and $S_\Gamma = \{\alpha, \beta, \gamma, \delta, \delta^{-1}\}$ be the set of shifts of Γ, such that $\alpha^2 = \beta^2 = \gamma^2 = \mathrm{id}$. Then*

(1) *If* $\alpha\beta\alpha = \gamma, \alpha\delta\alpha = \delta$ *and* $\beta\alpha\beta = \gamma$*, then* $\beta\delta\beta \neq \delta$.

(2) *If* $\alpha\beta\alpha = \gamma, \alpha\delta\alpha = \delta^{-1}, \beta\alpha\beta = \gamma$ *and* $\beta\delta\beta = \delta^{-1}$*, then* $\delta^{-1}\alpha\delta \neq \beta$.

(3) *If* $\alpha\beta\alpha = \beta, \alpha\gamma\alpha = \gamma$ *and* $\beta\gamma\beta = \gamma$*, then* $\delta^{-1}\alpha\delta \neq \beta$.

Proof. (1) : Suppose as a contradiction that $\beta\delta\beta = \delta$. We number the equalities as follows:

$$\alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \beta\alpha\beta = \gamma,\ \beta\delta\beta = \delta \tag{2.4}$$

Let $G = \langle S_\Gamma \rangle$, $|\delta| = n \geq 3$, $H = \langle \alpha,\ \beta,\ \gamma,\ \delta \mid \alpha^2 = \beta^2 = \gamma^2 = \delta^n = \mathrm{id},\ \alpha\beta\alpha = \gamma,\ \alpha\delta\alpha = \delta,\ \beta\alpha\beta = \gamma,\ \beta\delta\beta = \delta\rangle$, $M = \langle \alpha,\ \beta,\ \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = \mathrm{id},\ \alpha\beta\alpha = \gamma,\ \beta\alpha\beta = \gamma\rangle$ and $N = \langle \delta \mid \delta^n = \mathrm{id}\rangle$. Then by Equation 2.4, we have $M \trianglelefteq H$, $N \trianglelefteq H$, $M \cap N = \{\mathrm{id}\}$ and $H = MN$. So $H \cong M \times N$.

But $M \cong S_3$ and $N \cong C_n$. Thus $H \cong S_3 \times C_n$ and G is isomorphic to a quotient of $S_3 \times C_n$. Since $\delta \in G$ so n divides $|G|$. Thus $|G| = n, 2n, 3n$ or $6n$. If $|G| = n$ then $G = \langle\delta\rangle$ is a cyclic group, which is a contradiction. If $|G| = 6n$ then $G \cong S_3 \times C_n$ and

$$G = \{\mathrm{id},\ \alpha,\ \beta,\ \gamma,\ \gamma\beta,\ \beta\gamma,\ \delta,\ \alpha\delta,\ \beta\delta,\ \gamma\beta\delta,\ \gamma\delta,\ \beta\gamma\delta, \cdots,\ \delta^{n-1},\ \alpha\delta^{n-1},\ \beta\delta^{n-1},\ \gamma\beta\delta^{n-1},\ \gamma\delta^{n-1},\ \beta\gamma\delta^{n-1}\}.$$

Here, a core-free subgroup has order at most 3. The stabiliser of a vertex v in G is a core-free subgroup of $S_3 \times C_n$, so it has order at most 3. Note that the shifts fix no vertices. If G_v has order 3 then G_v is one of $\{\mathrm{id},\ \delta^k,\ \delta^{2k}\}$ or $\{\mathrm{id},\ \gamma\beta\delta^k,\ \beta\gamma\delta^{2k}\}$ where $3k = n$.

If $G_v = \{\mathrm{id},\ \delta^k,\ \delta^{2k}\}$ then $G_v \trianglelefteq G$, which is a contradiction, because G_v is a core-free subgroup of G.

Let $G_v = \{\mathrm{id},\ \gamma\beta\delta^k,\ \beta\gamma\delta^{2k}\}$ then $\delta^k v = \beta\gamma v$. In this case,

$$V(\Gamma) = \{v,\ \alpha v,\ \beta v,\ \gamma\beta v,\ \gamma v,\ \beta\gamma v,\ \delta v,\ \alpha\delta v,\ \beta\delta v,\ \gamma\beta\delta v,\ \gamma\delta v,\ \beta\gamma\delta v,\ \cdots,\ \delta^{k-1}v,\ \alpha\delta^{k-1}v,\ \beta\delta^{k-1}v,\ \gamma\beta\delta^{k-1}v,\ \gamma\delta^{k-1}v,\ \beta\gamma\delta^{k-1}v\}$$

and

$$\sigma = \alpha\delta^k = (v\ \beta v\ \gamma\beta v\ \alpha v\ \beta\gamma v\ \gamma v)(\delta v\ \beta\delta v\ \gamma\beta\delta v\ \alpha\delta v\ \beta\gamma\delta v\ \gamma\delta v)\cdots(\delta^{k-1}v\ \beta\delta^{k-1}v\ \gamma\beta\delta^{k-1}v\ \alpha\delta^{k-1}v\ \beta\gamma\delta^{k-1}v\ \gamma\delta^{k-1}v)$$

is a shift not in $\{\alpha,\ \beta,\ \gamma,\ \delta,\ \delta^{-1}\}$, which contradicts the unique shift-transitivity of Γ.

If G_v has order 2 then G_v is one of $\{\mathrm{id}, \delta^k\}$, $\{\mathrm{id}, \alpha\delta^k\}$, $\{\mathrm{id}, \beta\delta^k\}$ or $\{\mathrm{id}, \gamma\delta^k\}$ where $2k = n$. If $G_v = \{\mathrm{id}, \delta^k\}$ then $G_v \trianglelefteq G$ which is a contradiction. Without loss of generality we may assume that $G_v = \{\mathrm{id}, \alpha\delta^k\}$. By using Equation 2.4, we obtain:

$$V(\Gamma) = \{v,\ \alpha v,\ \beta v,\ \gamma\beta v,\ \gamma v,\ \beta\gamma v,\ \delta v,\ \alpha\delta v,\ \beta\delta v,\ \gamma\beta\delta v,\ \gamma\delta v,\ \beta\gamma\delta v, \cdots,\ \delta^{k-1}v,\ \alpha\delta^{k-1}v,\ \beta\delta^{k-1}v,\ \gamma\beta\delta^{k-1}v,\ \gamma\delta^{k-1}v,\ \beta\gamma\delta^{k-1}v\}.$$

In this case

$$\sigma = \delta^k = (v\ \alpha v)(\beta v\ \gamma\beta v)(\gamma v\ \beta\gamma v)(\delta v\ \alpha\delta v)(\beta\delta v\ \gamma\beta\delta v)(\gamma\delta v\ \beta\gamma\delta v)\cdots(\delta^{k-1}v\ \alpha\delta^{k-1}v)(\beta\delta^{k-1}v\ \gamma\beta\delta^{k-1}v)(\gamma\delta^{k-1}v\ \beta\gamma\delta^{k-1}v)$$

is a shift not in $\{\alpha,\ \beta,\ \gamma,\ \delta,\ \delta^{-1}\}$, which is a contradiction.

Now assume G_v has order 1. Then G acts regularly on $V(\Gamma)$, and Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2 Γ is not uniquely shift-transitive which is a contradiction. Hence in the above two cases, we obtain a contradiction.

If $|G| = 2n$ then $G \cong H/R$ where $R \trianglelefteq H$ and $|R| = 3$. Since

$$H = \{x\delta^t \mid x \in \{\mathrm{id}, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\}, 0 \leq t \leq n-1\},$$

so the elements of order 3 in H are $\gamma\beta, \beta\gamma, \delta^k, \delta^{2k}, \gamma\beta\delta^k, \beta\gamma\delta^k, \gamma\beta\delta^{2k}, \beta\gamma\delta^{2k}$ where $3k = n$. Note that the last six elements are exist whenever 3 divides n. This implies that R is one of $A_1 = \{\mathrm{id},\ \beta\gamma,\ \gamma\beta\}, A_2 = \{\mathrm{id},\ \delta^k,\ \delta^{2k}\}, A_3 =$

$\{\mathrm{id}, \gamma\beta\delta^k, \beta\gamma\delta^{2k}\}$ or $A_4 = \{\mathrm{id}, \beta\gamma\delta^k, \gamma\beta\delta^{2k}\}$. It is easy to see that only A_1 and A_2 are normal subgroups of H. Suppose first that $R = A_1$. Then $H/R = \{R, \alpha R, \delta R, \alpha\delta R, \cdots, \delta^{n-1}R, \alpha\delta^{n-1}R\}$. In this case $G \cong H/R$ is an Abelian group which is a contradiction. If $R = A_2$, then

$$H/R = \{x\delta^t R \mid x \in \{\mathrm{id}, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\},\ 0 \le t \le k-1\}.$$

Let $M_1 = \{R, \alpha R, \beta R, \gamma R, \beta\gamma R, \gamma\beta R\}$ and $N_1 = \{R, \delta R, \delta^2 R, \cdots, \delta^{k-1}R\}$. Then $M_1 \trianglelefteq H/R$, $N_1 \trianglelefteq H/R$ $M_1 \cap N_1 = \{R\}$ and $M_1N_1 = H/R$. Hence $H/R \cong M_1 \times N_1$. Since $M_1 \cong S_3$ and $N_1 \cong C_k$, we conclude that $G \cong H/R \cong S_3 \times C_k$. Now by a similar argument as in case $|G| = 6n$ we obtain that Γ is not uniquely shift-transitive which is a contradiction.

Let $|G| = 3n$. Then $G \cong H/R$ where $R \trianglelefteq H$ and $|R| = 2$. An easy calculation shows elements of order 2 in H are $\alpha, \beta, \gamma, \delta^m, \alpha\delta^m, \beta\delta^m, \gamma\delta^m$ where $n = 2m$. Thus the only normal subgroup of order 2 in H is $\{\mathrm{id}, \delta^m\}$. In this case we have:

$$H/R = \{x\delta^t R \mid x \in \{\mathrm{id}, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\},\ 0 \le t \le m-1\}$$

Let $M = \{R, \alpha R, \beta R, \gamma R, \gamma\beta R, \beta\gamma R\}$ and $N = \{\delta^i R \mid 0 \le i \le m-1\}$. Then $M \trianglelefteq H/R$, $N \trianglelefteq H/R$ $M \cap N = \{R\}$ and $MN = H/R$. Hence $H/R \cong M \times N$. Since $M \cong S_3$ and $N \cong C_m$, we have $G \cong H/R \cong S_3 \times C_m$. Now a similar argument as in case $|G| = 6n$ we shows that Γ is not uniquely shift-transitive which is a contradiction. This complete the proof of (1).

(2) Assume, to the contrary, that $\delta^{-1}\alpha\delta = \beta$. Since Γ is uniquely shift-transitive, we have the following relations between the shifts of Γ:

$$\alpha\beta\alpha = \gamma, \alpha\delta\alpha = \delta^{-1}, \beta\alpha\beta = \gamma, \beta\delta\beta = \delta^{-1}, \delta^{-1}\alpha\delta = \beta, \delta^{-1}\beta\delta = \gamma, \delta^{-1}\gamma\delta = \alpha. \tag{2.5}$$

Let $G = \langle S_\Gamma \rangle, a = \delta$ and $b = \beta$. Then by Equation 2.5, we have $a^6 = b^2 = (ba)^2 = \mathrm{id}$. So G is a quotient of D_{12}. Since $|G| \ge 8$ we conclude that $G \cong D_{12} \cong S_3 \times C_2$ and

$$G = \{\mathrm{id}, \alpha, \beta, \gamma, \delta, \gamma\beta, \beta\gamma, \gamma\delta, \delta^{-1}, \beta\delta^{-1}, \beta\delta, \alpha\beta\delta\},$$

Here, a core-free subgroup has order at most 2. The stabiliser of a vertex in G is a core-free subgroup of $S_3 \times C_2$, so it has order 1 or 2. It follows that Γ has order 6 or 12. If Γ has order 6 then Γ is a complete graph, which is not uniquely shift-transitive. When Γ has order 12, Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2 Γ is not uniquely shift-transitive, which is a contradiction.

(3) : Suppose, for a proof by contradiction, that $\delta^{-1}\alpha\delta = \beta$. Since Γ is uniquely shift-transitive, we have the following equalities:

$$\alpha\beta\alpha = \beta, \alpha\gamma\alpha = \gamma, \alpha\delta\alpha = \delta^{-1}, \beta\gamma\beta = \gamma, \beta\delta\beta = \delta^{-1},$$
$$\delta^{-1}\alpha\delta = \beta, \delta^{-1}\beta\delta = \alpha, \delta^{-1}\gamma\delta = \gamma.$$

A similar argument as in Part (2) of Proposition 2.3 shows that $G \cong D_8 \times C_2$ and we find again a contradiction. □

Proposition 2.5. *Suppose that Γ is a uniquely shift-transitive graph of valency 5 and $S_\Gamma = \{\alpha, \beta, \beta^{-1}, \gamma, \gamma^{-1}\}$ be the set of shifts of Γ. If $\alpha^2 = \mathrm{id}$, $\alpha\beta\alpha = \beta$, $\alpha\gamma\alpha = \gamma$ and $\beta^{-1}\gamma\beta = \gamma^{-1}$, then $\gamma^{-1}\beta\gamma \neq \beta^{-1}$.*

Proof. Suppose that the statement is not true, i.e. $\gamma^{-1}\beta\gamma = \beta^{-1}$. Let $G = \langle S_\Gamma \rangle$ and $H = \langle \alpha, \beta, \gamma \mid \alpha^2 = \mathrm{id}, \alpha\beta\alpha = \beta, \alpha\gamma\alpha = \gamma, \beta^{-1}\gamma\beta = \gamma^{-1}, \gamma^{-1}\beta\gamma = \beta^{-1} \rangle$. Set $M = \langle \beta, \gamma \mid \beta^{-1}\gamma\beta = \gamma^{-1}, \gamma^{-1}\beta\gamma = \beta^{-1} \rangle$ and $N = \langle \alpha \mid \alpha^2 = \mathrm{id} \rangle$. Then these relations imply, $M \trianglelefteq G$, $N \trianglelefteq G$, $M \cap N = \{\mathrm{id}\}$ and $H = MN$. Thus $H = M \times N$. But $H \cong Q_8, N \cong C_2$. So $H \cong Q_8 \times C_2$ and G is a quotient of $Q_8 \times C_2$. Since $|G| \ge 8$ so $|G| = 8$ or 16. If $|G| = 8$ then $|V(\Gamma)| = 8$ and G acts regularly on $V(\Gamma)$. So Γ is a quasi-Abelian Cayley graph of a non-Abelian group G with connection set S_Γ. By Proposition 2.2 Γ is not uniquely shift-transitive, which is a contradiction.

If $|G| = 16$ then $G \cong Q_8 \times C_2$ and

$$G = \{id, \alpha, \beta, \gamma, \alpha\beta, \gamma\beta, \beta^{-1}, \beta^2, \gamma^{-1}, \gamma^{-1}\beta, \alpha\beta^{-1}, \alpha\beta^2, \alpha\gamma^{-1}, \alpha\gamma^{-1}\beta, \alpha\gamma, \alpha\gamma\beta\}.$$

The only core-free subgroup of this group is the identity, so Γ is a quasi-Abelian Cayley graph on G with the connection set S_Γ. By Proposition 2.2 Γ is not uniquely shift-transitive which is a contradiction. □

3. Uniquely Shift-Transitive Graphs of Valency 5

Theorem 3.1. *Let Γ be a uniquely shift-transitive graph of valency 5 and $S_\Gamma = \{\alpha, \beta, \gamma, \delta, \eta\}$ be the set of distinct shifts of Γ. Then $\langle S_\Gamma \rangle$, is an Abelian group.*

Proof. Since the inverse of a shift is a shift, we must only consider the three following cases:

(1) : $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \eta^2 = \text{id}$.

(2) : $\alpha^2 = \beta^2 = \gamma^2 = \delta\eta = \text{id}$.

(3) : $\alpha^2 = \beta\gamma = \delta\eta = \text{id}$.

First we consider Case (1). In this case it is enough to prove that $\alpha\beta = \beta\alpha$. We will prove this by contradiction. Suppose $\alpha\beta \neq \beta\alpha$. It is obvious that the conjugate of a shift is also a shift, so $\alpha\beta\alpha \neq \alpha$ is a shift of Γ. Since $\alpha\beta\alpha \neq \beta$ so $\alpha\beta\alpha \in \{\gamma, \delta, \eta\}$. Let

$$\alpha\beta\alpha = \gamma \tag{3.1}$$

Consider the shift $\beta\alpha\beta$. Then $\beta\alpha\beta \in \{\gamma, \delta, \eta\}$. First assume that

$$\beta\alpha\beta = \gamma \tag{3.2}$$

then consider the shift $\alpha\delta\alpha \neq \alpha, \beta, \gamma$. By Part(1) of Proposition 2.3, $\alpha\delta\alpha = \delta$, which is impossible. Therefore

$$\alpha\delta\alpha = \eta. \tag{3.3}$$

By Equations 3.1, 3.2 and 3.3 we have:

$$\beta\delta\beta = \eta. \tag{3.4}$$

Now consider the shift $\delta\alpha\delta$ which is neither α nor δ.

If $\delta\alpha\delta = \beta$, then by Equations 3.1, 3.2, 3.3 and 3.4 we have:
$\delta\gamma\delta = \delta(\beta\alpha\beta)\delta = (\delta\beta)\alpha(\beta\delta) = \alpha\delta\alpha\delta\alpha = \alpha(\delta\alpha\delta)\alpha = \alpha\beta\alpha = \gamma$. So $\gamma\delta\gamma = \delta$.

On the other hand $\gamma\delta\gamma = \alpha\beta\alpha\delta\alpha\beta\alpha = \alpha\beta\eta\beta\alpha = \alpha\delta\alpha = \eta$. Thus $\delta = \eta$ which is a contradiction.

If $\delta\alpha\delta = \gamma$, then by Equations 3.1, 3.2 and 3.4 we have:
$\delta\beta\delta = \delta(\alpha\gamma\alpha)\delta = (\delta\alpha)\gamma(\alpha\delta) = \gamma\delta\gamma\delta\gamma = \gamma\alpha\gamma = \alpha\beta\gamma = \alpha\alpha\beta = \beta$. So $\beta\delta\beta = \delta = \eta$, which is another contradiction.

If $\delta\alpha\delta = \eta$, then $\delta\beta\delta \in \{\beta, \gamma\}$. If $\delta\beta\delta = \beta$ then by Equation 3.4 we obtain $\beta\delta\beta = \delta = \eta$, which is a contradiction. Finally if $\delta\beta\delta = \gamma$ then by Equations 3.1, 3.3 and 3.4 we have:
$\eta = \delta\alpha\delta = (\alpha\eta\alpha)\alpha\delta = \alpha\eta\delta = \alpha(\beta\delta\beta)\delta = \alpha\beta\gamma = \alpha\alpha\beta = \beta$ which is another contradiction.
So in either case we have a contradiction and consequently Equation 3.2 can not arise.
Now let we have:

$$\beta\alpha\beta = \delta. \tag{3.5}$$

Then $\beta\gamma\beta \in \{\gamma, \eta\}$. By Part(2) of Proposition 2.3, the equation $\beta\gamma\beta = \gamma$ can not arise.

Therefore

$$\beta\gamma\beta = \eta. \tag{3.6}$$

Now consider the shift $\alpha\delta\alpha \in \{\delta, \eta\}$.

If $\alpha\delta\alpha = \delta$ then $\alpha\eta\alpha = \eta$ and by these equalities and Equations 3.1, 3.5 and 3.6 we obtain
$\delta = \beta\alpha\beta = \beta\eta\alpha\eta\beta = \beta\eta(\beta\delta\beta)\eta\beta = (\beta\eta\beta)\delta(\beta\eta\beta) = \gamma\delta\gamma = (\alpha\beta\alpha)\delta(\alpha\beta\alpha) = \alpha\beta(\alpha\delta\alpha)\beta\alpha = \alpha\beta\delta\beta\alpha = \alpha\alpha\alpha = \alpha$ which is a contradiction. The second case cannot arise by Part(3) of Proposition 2.3. So we find that Equation 3.5 can not occur.

By a similar argument we can show that the equality $\beta\alpha\beta = \eta$ is impossible. So the Equation 3.1 can not occur. (For cases $\alpha\beta\alpha = \delta$ or $\alpha\beta\alpha = \eta$ the proof is similar). Thus $\beta\alpha = \alpha\beta$ and the proof is complete.

Proof of theorem in Case (2): In this case $S_\Gamma = \{\alpha, \beta, \gamma, \delta, \delta^{-1}\}$. It is sufficient to prove $\alpha\beta = \beta\alpha$ and $\alpha\delta = \delta\alpha$.

First we prove $\alpha\beta = \beta\alpha$: Consider the shift $\alpha\beta\alpha$. Since $\alpha\beta\alpha$ is of order two so $\alpha\beta\alpha = \beta$ or γ. If $\alpha\beta\alpha = \beta$ then the proof is complete. So we suppose

$$\alpha\beta\alpha = \gamma. \tag{3.7}$$

Therefore

$$\beta\alpha\beta = \gamma. \tag{3.8}$$

Since $\alpha\delta\alpha$ and δ have the same order, we have $\alpha\delta\alpha = \delta$ or δ^{-1}

First assume that

$$\alpha\delta\alpha = \delta. \tag{3.9}$$

Then $\beta\delta\beta \in \{\delta, \delta^{-1}\}$.

If $\beta\delta\beta = \delta^{-1}$ then $\gamma\delta\gamma \in \{\delta, \delta^{-1}\}$.

If $\gamma\delta\gamma = \delta$ then by Equations 3.8 and 3.9 we conclude that
$\delta = \gamma\delta\gamma = (\alpha\beta\alpha)\delta(\alpha\beta\alpha) = \alpha\beta(\alpha\delta\alpha)\beta\alpha = \alpha\beta\delta\beta\alpha = \alpha\delta^{-1}\alpha = \delta^{-1}$, which is a contradiction.

If $\gamma\delta\gamma = \delta^{-1}$ then by Equations 3.8 and 3.9 we obtain
$\delta^{-1} = \gamma\delta\gamma = (\beta\alpha\beta)\delta(\beta\alpha\beta) = \beta\alpha(\beta\delta\beta)\alpha\beta = \beta\alpha\delta^{-1}\alpha\beta = \beta\delta^{-1}\beta = \delta$, which is again a contradiction.

The second case cannot arise by Part(1) of Proposition 2.4. From these contradictions, we conclude that Equation 3.9 can not occur.

Now assume that

$$\alpha\delta\alpha = \delta^{-1}. \tag{3.10}$$

Consider the shift $\beta\delta\beta$. This shift can be δ or δ^{-1}. If $\beta\delta\beta = \delta$ then $\gamma\delta\gamma = \delta$ or δ^{-1}. If $\gamma\delta\gamma = \delta$ then by Equations 3.9 and 3.10 we have $\delta = \gamma\delta\gamma = (\beta\alpha\beta)\delta(\beta\alpha\beta) = \beta\alpha(\beta\delta\beta)\alpha\beta = \beta\alpha\delta\alpha\beta = \beta\delta^{-1}\beta = \delta^{-1}$ a contradiction. If $\gamma\delta\gamma = \delta^{-1}$ then by Equations 3.7 and 3.10 we have
$\delta^{-1} = \gamma\delta\gamma = (\alpha\beta\alpha)\delta(\alpha\beta\alpha) = \alpha\beta(\alpha\delta\alpha)\beta\alpha = \alpha\beta\delta^{-1}\beta\alpha = \alpha\delta^{-1}\alpha = \delta$, which is a contradiction. So $\beta\delta\beta = \delta^{-1}$.

Since α and $\delta^{-1}\alpha\delta$ have the same order, then $\delta^{-1}\alpha\delta = \alpha, \beta$ or γ. If $\delta^{-1}\alpha\delta = \alpha$ then $\delta = \alpha\delta\alpha = \delta^{-1}$ which is a contradiction. Indeed by Part(2) of Proposition 2.4 the case $\delta^{-1}\alpha\delta = \beta$ is impossible (for case $\delta^{-1}\alpha\delta = \gamma$ the proof is similar). Thus $\alpha\beta = \beta\alpha$. A similar argument shows that $\alpha\gamma = \gamma\alpha$ and $\gamma\beta = \beta\gamma$.

By using Part(3) of Proposition 2.4, we find that $\delta^{-1}\alpha\delta \neq \beta, \gamma$. Hence $\alpha\delta = \delta\alpha$ and the proof in Case(2) is complete.

Proof of theorem in Case(3): In this case $S_\Gamma = \{\alpha, \beta, \beta^{-1}, \gamma, \gamma^{-1}\}$, and it is enough to prove that $\alpha\beta = \beta\alpha$ and $\beta\gamma = \gamma\beta$. Since α and $\beta^{-1}\alpha\beta$ have the same order, we have $\beta^{-1}\alpha\beta = \alpha$ and $\alpha\beta = \beta\alpha$. Hence either $\beta^{-1}\gamma\beta = \gamma$ or $\beta^{-1}\gamma\beta = \gamma^{-1}$. If $\beta^{-1}\gamma\beta = \gamma$ then $\gamma\beta = \beta\gamma$ and the proof is complete. So assume that $\beta^{-1}\gamma\beta = \gamma^{-1}$. In this case $\gamma^{-1}\beta\gamma = \beta^{-1}$ and by Proposition 2.5, such a graph can not exist.
Since we have proved in all cases that the shifts commute with each other, so $\langle S_\Gamma \rangle$ is an Abelian subgroup of $\mathrm{Aut}(\Gamma)$. □

Lemma 3.2. *([1, Lemma 16.3]): Let* Γ *be a connected graph. The automorphism group* $\mathrm{Aut}(\Gamma)$ *has a subgroup* H *which acts regularly on* $V(\Gamma)$ *if and only if* Γ *is a Cayley graph,* $\mathrm{Cay}(H, \Omega)$*, for some set* Ω *generating* H*.*

Theorem 3.3. *(Main Theorem) Every uniquely shift-transitive graph* Γ *of valency 5 is isomorphic with a Cayley graph of an Abelian group.*

Proof. By Theorem 3.1, $\langle S_\Gamma \rangle$ is an Abelian group. Now by [4, Proposition 4.4], $\langle S_\Gamma \rangle$ is regular on $V(\Gamma)$ and by Lemma 3.2, Γ is isomorphic with $\mathrm{Cay}(\langle S_\Gamma \rangle, S_\Gamma)$. So Γ is isomorphic with a Cayley graph of an Abelian group. □

Remark 3.4. *The converse of Theorem 3.3 is not true, because if* Γ *is isomorphic with* C_4 *or* K_n*, then* Γ *is a Cayley graph of an Abelian group, but* Γ *is not uniquely shift-transitive graph. Moreover the converse of Theorem 3.1 is true whenever* Γ *is a strongly shift-transitive graph by the next proposition.*

Proposition 3.5. *Let* Γ *be a strongly shift-transitive graph and* S_Γ *be the set of shifts of* Γ*. If* $\langle S_\Gamma \rangle$ *is an Abelian group, then* Γ *is uniquely shift-transitive.*

Proof. Suppose Γ is not uniquely shift-transitive, so there exist adjacent vertices u and v and distinct shifts α and β of Γ, such that $\alpha u = v = \beta u$. Since $\alpha \neq \beta$ so there exists vertex $x \neq u$ of Γ such that $\alpha x \neq \beta x$. But Γ is shift-transitive, so there exists a sequence of shifts $\sigma_1, \sigma_2, \cdots, \sigma_k \in \mathrm{Aut}(\Gamma)$, such that $\sigma_1\sigma_2\cdots\sigma_k v = x$. Now we have:

$$\alpha x = \alpha\sigma_1\sigma_2\cdots\sigma_k v = \alpha\sigma_1\sigma_2\cdots\sigma_k\beta u = \alpha\beta\sigma_1\sigma_2\cdots\sigma_k u,$$
$$\beta x = \beta\sigma_1\sigma_2\cdots\sigma_k v = \beta\sigma_1\sigma_2\cdots\sigma_k\alpha u = \alpha\beta\sigma_1\sigma_2\cdots\sigma_k u.$$

Thus $\alpha x = \beta x$ which is a contradiction. □

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