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Bilateral Set-Valued Stochastic Integral Equations

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Abstract. We investigate bilateral set-valued stochastic integral equations and these equations combine widening and narrowing set-valued stochastic integral equations studied in literature. An existence and uniqueness theorem is established using approximate solutions. In addition stability of the solution with respect to small changes of the initial state and coefficients is established, also we provide a result on boundedness of the solution, and an estimate on a distance between the exact solution and the approximate solution is given. Finally some implications for deterministic set-valued integral equations are presented.

1. Introduction

Set-valued analysis arises naturally in physics, economics, optimization; see e.g. [5, 10] and the references therein. It is fundamental in the theory of set-valued differential equations [24], which are mathematical models of dynamic systems with incomplete information and this theory was used to study properties of solutions for differential inclusions [33]. Set-valued differential equation were considered in [7–9, 13]. Existence of solutions was discussed in [1] while in [2, 3, 6, 14, 16, 18, 32, 35] stability results were provided. Also in the literature there are results using the monotone iterative technique [12], the variation of constants formula [17], monotone flows [25], quasilinearization [34], and periodic solutions were studied in [20]. Set-valued differential equations including equations with causal operators [11, 14, 22], equations with second type Hukuhara derivative [27, 28], equations on time scales [19, 26, 35] were also considered.

In this paper we consider bilateral set-valued integral equations in a stochastic context. More precisely we consider an equation of the form

$$X(t) + (S)\int_0^t F(s, X(s))ds + (S)\int_0^t G(s, X(s))dB(s) = X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))d\tilde{B}(s),$$

where t runs over an interval, X_0 is a set, $F, \tilde{F}, G, \tilde{G}$ are some set-valued stochastic processes, $B, \tilde{B}$ are Brownian motions and all the integrals are set-valued. Unfortuantely this equation cannot easily be reduced to the equation of the above type with only one side. The difficulty lies in the issue of the difference of sets i.e. this difference may not exist. Also each side of the equation has a different effect on the properties of the solution (i.e. a different effect on the behavior of a function whose values are the diameter of the solution at time t); the right-hand side drives an increase in diameter while the integrals on the left forces the diameter

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to decrease. Such an observation with equations only with the left-hand side was explicitly emphasized in [31]. The equation above combines two different types of equations previously investigated in the literature, i.e., widening equations [29] and [30] and narrowing equations [31], so in this case the solutions can change the type of monotonicity of diameter over time.

In this paper we consider the existence of a unique solution under a Lipschitz type condition by first discussing and defining a sequence of approximate set-valued solutions. We also give some estimates on the distance between the approximate and the exact solution. An estimate on the magnitude of the solution is presented which allows us to discuss the solution's boundedness. A justification of low sensitivity of the solution to slight changes in the initial value and coefficients of the equation is also presented. In the last part of the paper, some implications specific to deterministic set-valued equations are discussed which are of interest in their own right [24] and in this setting we show that a certain restrictive condition used for stochastic equations can be replaced by a more convenient one.

2. Preliminaries

In this part of the paper we collect some notions and properties concerning set-valued mappings and integrals; see [29, 31]. It is done for the convenience of th reader.

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a separable Banach space. By the symbol $\mathcal{K}_c^b(\mathcal{X})$ we denote the family of all nonempty closed bounded and convex subsets of $\mathcal{X}$. In $\mathcal{K}_c^b(\mathcal{X})$, the Hausdorff metric $H_{\mathcal{X}}$ is considered

$$H_{\mathcal{X}}(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathcal{X}}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{\mathcal{X}}\right\}.$$

Then $(\mathcal{K}_c^b(\mathcal{X}), H_{\mathcal{X}})$ is a complete metric space (see [21]). Moreover the family of nonempty, closed and convex subsets of a separable and reflexive Banach space $\mathcal{X}$ supplied with the Mosco topology $\tau_{M_{\mathcal{X}}}$ is a Polish topological space. The Mosco topology is metrizable and weaker than the topology $\tau_{H_{\mathcal{X}}}$ generated by the Hausdorff metric $H_{\mathcal{X}}$.

In the set $\mathcal{K}_c^b(\mathcal{X})$ one defines addition and scalar multiplication as follows: for $A, B \in \mathcal{K}_c^b(\mathcal{X})$ and $r \in \mathbb{R}$ we have $A + B = \{a + b : a \in A, b \in B\}$, $rA = \{ra : a \in A\}$. The notion of difference of sets of $A, B \in \mathcal{K}_c^b(\mathcal{X})$ used in this paper is considered as the Hukuhara difference, i.e. $A \ominus B \in \mathcal{K}_c^b(\mathcal{X})$ is a set such that $(A \ominus B) + B = A$. If $A \ominus B$ exists, it is unique.

For the metric $H_{\mathcal{X}}$ and $A, B, C, D \in \mathcal{K}_c^b(\mathcal{X})$ and $\mu \in \mathbb{R}$ the following properties hold

(P1) $H_{\mathcal{X}}(A + B, C + D) \leqslant H_{\mathcal{X}}(A, C) + H_{\mathcal{X}}(B, D)$,

(P2) $H_{\mathcal{X}}(A + C, B + C) = H_{\mathcal{X}}(A, B)$,

(P3) if $A \ominus B$ and $C \ominus D$ exist then $H_{\mathcal{X}}(A \ominus B, C \ominus D) \leqslant H_{\mathcal{X}}(A, C) + H_{\mathcal{X}}(B, D)$.

Let $(Z, \mathcal{Z}, \mu)$ be a measure space. A set-valued mapping $F\colon Z \to \mathcal{K}_c^b(\mathcal{X})$ is called $\mathcal{Z}$-measurable (or set-valued random variable) if it satisfies:

$$\{z \in Z : F(z) \cap O \neq \emptyset\} \in \mathcal{Z} \ \text{ for every open set } \ O \subset \mathcal{X}.$$

A set-valued random variable F is L^p-integrally bounded ($p \geqslant 1$), if $z \mapsto H_{\mathcal{X}}(F(z), \{0\})$ belongs to $L^p(Z, \mathcal{Z}, \mu; \mathbb{R})$.

Define $I = [0, T]$, where $T < \infty$, and by β_I we denote the Borel σ-algebra of subsets of I. By $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in I}, P)$ we denote a complete filtered probability space satisfying the usual hypotheses, i.e. $\{\mathcal{A}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all P-null sets. Let $\mathcal{N}$ denote the σ-algebra of the nonanticipating elements in $I \times \Omega$, i.e.

$$\mathcal{N} = \{A \in \beta_I \otimes \mathcal{A} : A^t \in \mathcal{A}_t \ \text{ for every } \ t \in I\},$$

where $A^t = \{\omega : (t, \omega) \in A\}$.

Let $\{B(t)\}_{t\in I}$ be an $\{\mathcal{A}_t\}$-Brownian motion. By λ we denote the Lebesgue measure on (I, β_I). Consider the space

$$L^2_{\mathcal{N}}(\lambda \times P) := L^2(I \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d).$$

Then for every $f \in L^2_{\mathcal{N}}(\lambda \times P)$ and $\tau, t \in I$, $\tau < t$ the Itô stochastic integral $\int_\tau^t f(s)dB(s)$ exists and one has $\int_\tau^t f(s)dB(s) \in L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$.

Let $F\colon I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be a set-valued stochastic process, i.e. a family of $\mathcal{A}$-measurable set-valued mappings $F(t,\cdot)\colon \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$, $t \in I$. We call F nonanticipating if $F(\cdot,\cdot)$ is an $\mathcal{N}$-measurable set-valued mapping. Let us define the set

$$S^2_{\mathcal{N}}(F, \lambda \times P) := \{f \in L^2_{\mathcal{N}}(\lambda \times P) : f \in F, \quad \lambda \times P\text{-a.e.}\}.$$

If $F\colon I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ is nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded, then by the Kuratowski and Ryll-Nardzewski Selection Theorem (see [23]) it follows that $S^2_{\mathcal{N}}(F, \lambda \times P) \neq \emptyset$. For such a set-valued stochastic process F we can define the set-valued stochastic Itô trajectory integral. Namely, for $\tau, t \in I$, $\tau < t$, by this integral we mean the set

$$(S)\int_\tau^t F(s)dB(s) := \left\{ \int_\tau^t f(s)dB(s) : f \in S^2_{\mathcal{N}}(F, \lambda \times P) \right\}.$$

From this definition we have $\int_\tau^t F(s)dB(s) \subset L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$. For the set-valued stochastic process F we can also define the following set denoted by $(S)\int_\tau^t F(s)ds$ and called the set-valued stochastic Aumann trajectory integral

$$(S)\int_\tau^t F(s)ds := \left\{ \int_\tau^t f(s)ds : f \in S^2_{\mathcal{N}}(F, \lambda \times P) \right\}.$$

Obviously $(S)\int_\tau^t F(s)ds \subset L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$.

The following properties of stochastic trajectory integrals (see e.g. [29, 30]) are useful in studying set-valued stochastic integral equations.

Lemma 2.1. *Let $F, G\colon I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded set-valued stochastic processes. Let $\tau, t \in I$, $\tau < t$. Then*

$$H^2_{L^2}\left((S)\int_\tau^t F(s)dB(s), (S)\int_\tau^t G(s)dB(s) \right) \leqslant \int_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F, G)ds \times dP,$$

and

$$H^2_{L^2}\left((S)\int_\tau^t F(s)ds, (S)\int_\tau^t G(s)ds \right) \leqslant (t-\tau) \int_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F, G)ds \times dP.$$

Lemma 2.2. *Under assumptions of Lemma 2.1, the mappings*

$$[\tau, T] \ni t \mapsto (S)\int_\tau^t F(s)dB(s) \in \mathcal{K}^b_c(L^2), \quad [\tau, T] \ni t \mapsto (S)\int_\tau^t F(s)ds \in \mathcal{K}^b_c(L^2)$$

are H_{L^2}-continuous.

3. Main Results

For abbreviation, we write L^2 instead of $L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ and L^2_t instead of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ where $t \in I$. Let Θ, θ denote the zero elements in L^2 and $\mathbb{R}^d$, respectively. In this part of the paper we assume that the σ-algebras $\mathcal{A}$, $\mathcal{A}_t$ are separable with respect to the probability measure P. This way the spaces L^2 and L^2_t are separable.

Let $F, \tilde{F}, G, \tilde{G} \colon I \times \Omega \times \mathcal{K}^b_c(L^2) \to \mathcal{K}^b_c(\mathbb{R}^d)$ and $X_0 \in \mathcal{K}^b_c(L^2_0)$ be given. By a bilateral set-valued stochastic integral equation we mean the following relation in the metric space $(\mathcal{K}^b_c(L^2), H_{L^2})$:

$$\begin{aligned} &X(t) + (S)\int_0^t F(s, X(s))ds + (S)\int_0^t G(s, X(s))dB(s) \\ &= X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))d\tilde{B}(s) \quad \text{for} \quad t \in I. \end{aligned} \tag{3.1}$$

Notice that if $F \equiv \{0\}$ and $G \equiv \{0\}$ then the equation written above takes the form

$$X(t) = X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))d\tilde{B}(s) \quad \text{for} \quad t \in I, \tag{3.2}$$

which is a natural extension of classical single-valued stochastic integral equations [4, 15] to a set-valued framework. Such equations were studied in [29], for example. Also, if $\tilde{F} \equiv \{0\}$ and $\tilde{G} \equiv \{0\}$ then (3.1) reduces to

$$X(t) + (S)\int_0^t F(s, X(s))ds + (S)\int_0^t G(s, X(s))dB(s) = X_0 \quad \text{for} \quad t \in I. \tag{3.3}$$

This equation, investigated in [31], is quite different from (3.2). It also includes classical single-valued stochastic integral equations but in a rather different way than in (3.2). The main difference between (3.2) and (3.3) can be seen from the geometrical properties of their solutions. Namely, solutions X to equation (3.2) possess the property that the function $t \mapsto \operatorname{diam}(X(t))$ is nondecreasing, while solutions X to equation (3.3) satisfy the property that $t \mapsto \operatorname{diam}(X(t))$ is nonincreasing. Hence we call (3.2) and (3.3) the widening equation and the narrowing equation, respectively. These equations can be useful in the mathematical description of stochastic dynamics of real life phenomena when some additional nonstochastic uncertainties of initial values or imprecise parameter values are taken into account.

For example, consider a situation when a microbiologist grows a population of microorganisms in a limited area. Suppose that the number of individuals can depend on random factors and stochastic noises and the microbiologist has the ability to control the population growth by changing the doses of food. In such a setting, the number of individuals at the instant $t \in I$, denoted by $x(t)$, is random and can be described by a controlled stochastic integral equation

$$x(t) = x_0 + \int_0^t f(x(s), u(s))ds + \int_0^t g(x(s), u(s))dB(s), \quad t \in I, \ P\text{-a.e.}, \tag{3.4}$$

where $x_0 \colon \Omega \to \mathbb{R}$ is the initial number of individuals, $f \colon \mathbb{R}^2 \to \mathbb{R}$ denotes the drift coefficient, $g \colon \mathbb{R}^2 \to \mathbb{R}$ is a diffusion coefficient, u is a feeding strategy, $u \in U$, and U is a set of controls. Assuming that $x(t) \in L^2(\Omega, \mathcal{A}, P; \mathbb{R})$ for $t \in I$, equation (3.4) can be transformed to an equation in the space $L^2(\Omega, \mathcal{A}, P; \mathbb{R})$, i.e. to the equation

$$x(t) = x_0 + \int_0^t \bar{f}(s, x(s), u)ds + \int_0^t \bar{g}(s, x(s), u)dB(s), \quad t \in I,$$

where the coefficients $\bar{f}, \bar{g} \colon I \times \Omega \times L^2 \times U \to \mathbb{R}$ are defined as

$$\bar{f}(s, \omega, \xi, u) := f(\xi(\omega), u(s, \omega)) \ \text{ and } \ \bar{g}(s, \omega, \xi, u) := g(\xi(\omega), u(s, \omega)).$$

In most cases, a microbiologist cannot accurately determine x_0. Suppose the microbiologist only knows that x_0 is an $\mathcal{A}_0$-measurable random variable whose values are bounded by a fixed number $a > 0$. In this way, in the presence of additional nonstochastic uncertainty, the initial number of individuals can be viewed as the following set

$$X_0 := \{x_0 \in L_0^2 : 0 \leqslant x_0 \leqslant a\} \in \mathcal{K}_c^b(L_0^2).$$

The dynamics of the uncertain number of individuals $X(t)$ can be described by the set-valued stochastic integral equation

$$X(t) = X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))dB(s), \ \ t \in I, \tag{3.5}$$

where $\tilde{F}, \tilde{G} \colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R})$ are defined as

$$\tilde{F}(s, \omega, A) := \overline{co}\left(\bigcup_{\xi\in A}\bigcup_{u\in U} \bar{f}(s, \omega, \xi, u)\right) \text{ and } \tilde{G}(s, \omega, A) := \overline{co}\left(\bigcup_{\xi\in A}\bigcup_{u\in U} \bar{g}(s, \omega, \xi, u)\right);$$

here $\overline{co}(B)$ denotes the closed convex hull of the set B. Now we observe that equation (3.5) is a set-valued stochastic integral equation of the type (3.2). Hence for its solution X, which denotes an uncertain number of individuals, one has that $\operatorname{diam}(X(t))$ starts from $\operatorname{diam}(X_0)$ and cannot decrease. The number $\operatorname{diam}(X(t))$ can be interpreted by the microbiologist as a level of uncertainty of the number of individuals. It seems natural that in the case of a large number of microorganisms, uncertainty should be greater than in the case of a small number. However, if one wants to achieve the dynamics of the number of individuals with decreasing uncertainty, one should replace equation (3.2) with equation (3.3) and with

$$F(s, \omega, A) := \overline{co}\left(\bigcup_{\xi\in A}\bigcup_{u\in U} \bar{f}(s, \omega, \xi, u)\right) \text{ and } G(s, \omega, A) := \overline{co}\left(\bigcup_{\xi\in A}\bigcup_{u\in U} \bar{g}(s, \omega, \xi, u)\right).$$

Then the function $t \mapsto \operatorname{diam}(X(t))$ is nonincreasing. However, if one wants to keep the uncertainty at a certain fixed level through an appropriate control strategy or if one wants this uncertainty in the time set to be the same as at the initial moment, neither the widening equation (3.2) nor the narrowing equation (3.3) would be adequate. Considering bilateral set-valued stochastic integral equations (3.1) gives a possibility to handle such situations. Returning to the example of microorgamisms growth, one might study the separate influences of births and deaths of individuals on the number of individuals. Suppose that a dift of births is described by $\tilde{F}$ and diffusion by $\tilde{G}$ and the volatility is driven by the Brownian motion $\tilde{B}$, simultaneously assume that a drift of deaths is described by F and with a diffusion part G is driven by the Brownian motion B. Notice that in general case the Brownian motions $\tilde{B}$ and B can be as the model needs. They can be equal or correlated either independent (and in (3.1) we consider $\tilde{B}$ and B as freely chosen). Here, in this example, it is reasonable to assume that Brownian motions are independent. Obviously, births force the number of individuals to increase and as before to increase a level of uncertainty $\operatorname{diam}(X(t))$, while deaths decrease the number of individuals and force $\operatorname{diam}(X(t))$ to be decreasing. Hence we arrive exactly to the bilateral equation (3.1). The set-valued solution $t \mapsto X(t)$ can give information on approximate dynamics of the population growth $t \mapsto x(t)$.

This paper includes both the widening and narrowing properties and allows us to have solutions with varying diameter of their values. It also motivates a future new research direction in the field of set-valued integral equations in the stochastic and deterministic context. For example with this formulation it may be possible in the future to consider periodic solutions to set-valued equations.

We begin our formal study. First we say what is meant by a solution to (3.1).

Definition 3.1. *By a global solution to* (3.1) *we mean a H_{L^2}-continuous set-valued mapping $X \colon I \to \mathcal{K}_c^b(L^2)$ that satisfies* (3.1) *for every $t \in I$. A global solution $X \colon I \to \mathcal{K}_c^b(L^2)$ to* (3.1) *is unique if $X(t) = Y(t)$ for every $t \in I$ where $Y \colon I \to \mathcal{K}_c^b(L^2)$ is any solution of* (3.1).

Let $J = [0, \tilde{T}] \subset I = [0, T]$, where $\tilde{T} < T$.

Definition 3.2. *A set-valued mapping $X\colon J \to \mathcal{K}_c^b(L^2)$ is said to be a local solution to* (3.1) *if it is H_{L^2}-continuous and satisfies* (3.1) *for $t \in J$.*

The uniqueness of a local solution is defined in an obvious way. We begin the analysis of the bilateral set-valued stochastic integral equations with the existence and uniqueness of a solution to equation (3.1).

Suppose that the coefficients of equation (3.1), i.e. $F, \tilde{F}, G, \tilde{G}\colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$, satisfy

(H1) the set-valued mappings $F(\cdot,\cdot,\cdot), \tilde{F}(\cdot,\cdot,\cdot), G(\cdot,\cdot,\cdot), \tilde{G}(\cdot,\cdot,\cdot)\colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ are $\mathcal{N} \times \beta(\tau_{M_{L^2}})$-measurable, where $\beta(\tau_{M_{L^2}})$ is the Borel σ-algebra induced by the Mosco topology $\tau_{M_{L^2}}$,

(H2) there exist $K_F, K_{\tilde{F}}, K_G, K_{\tilde{G}} \in L^2(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e. for every $A, B \in \mathcal{K}_c^b(L^2)$

$$H^2_{\mathbb{R}^d}\Big(F(t,\omega,A), F(t,\omega,B)\Big) \leqslant K_F(t,\omega) H^2_{L^2}(A,B),$$

$$H^2_{\mathbb{R}^d}\Big(\tilde{F}(t,\omega,A), \tilde{F}(t,\omega,B)\Big) \leqslant K_{\tilde{F}}(t,\omega) H^2_{L^2}(A,B),$$

$$H^2_{\mathbb{R}^d}\Big(G(t,\omega,A), G(t,\omega,B)\Big) \leqslant K_G(t,\omega) H^2_{L^2}(A,B),$$

$$H^2_{\mathbb{R}^d}\Big(\tilde{G}(t,\omega,A), \tilde{G}(t,\omega,B)\Big) \leqslant K_{\tilde{G}}(t,\omega) H^2_{L^2}(A,B),$$

(H3) there exist $C_F, C_{\tilde{F}}, C_G, C_{\tilde{G}} \in L^1(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}\Big(F(t,\omega,\{\Theta\}), \{\theta\}\Big) \leqslant C_F(t,\omega),$$

$$H^2_{\mathbb{R}^d}\Big(\tilde{F}(t,\omega,\{\Theta\}), \{\theta\}\Big) \leqslant C_{\tilde{F}}(t,\omega),$$

$$H^2_{\mathbb{R}^d}\Big(G(t,\omega,\{\Theta\}), \{\theta\}\Big) \leqslant C_G(t,\omega),$$

$$H^2_{\mathbb{R}^d}\Big(\tilde{G}(t,\omega,\{\Theta\}), \{\theta\}\Big) \leqslant C_{\tilde{G}}(t,\omega),$$

(H4) there exists $\tilde{T} \in (0, T]$ such that the sequence $\{X_n\}_{n=0}^\infty$ described by

$$X_0(t) = X_0, \quad t \in J := [0, \tilde{T}],$$

and for $n = 1, 2, \dots$

$$\begin{aligned} X_n(t) \;&=\; \left[X_0 + (S)\int_0^t \tilde{F}(s, X_{n-1}(s))ds + (S)\int_0^t \tilde{G}(s, X_{n-1}(s))d\tilde{B}(s)\right] \\ &\ominus \left[(S)\int_0^t F(s, X_{n-1}(s))ds + (S)\int_0^t G(s, X_{n-1}(s))dB(s)\right], \quad t \in J \end{aligned}$$

can be defined, i.e. the Hukuhara differences exist.

Assumptions (H2) and (H3), i.e. the Lipschitz condition and the boundedness condition, are formulated with integrable stochastic processes $K_F, K_{\tilde{F}}, K_G, K_{\tilde{G}}, C_F, C_{\tilde{F}}, C_G, C_{\tilde{G}}$ and this is more general than considering these processes to be constant. Condition (H4) may look restrictive. However, if one takes a closer look at equation (3.1) one sees that it is extremely important and natural. Note (3.1) can be rewritten as

$$\begin{aligned} X(t) \;&=\; \left[X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))d\tilde{B}(s)\right] \\ &\ominus \left[(S)\int_0^t F(s, X(s))ds + (S)\int_0^t G(s, X(s))dB(s)\right] \quad \text{for } t \in I. \end{aligned} \tag{3.6}$$

Now it is easy to see that the nature of the equation under consideration requires the existence of Hukuhara's differences. Note condition (H4) was not needed in the study of the widening equations i.e. there were no Hukuhara's differences in the formulation of the equation. However here this condition is necessary.

In the proof of the existence of a unique local solution to (3.1), the sequence $\{X_n\}_{n=0}^{\infty}$ will be used. The mappings X_n, $n = 0, 1, 2, \ldots$, shall approximate the exact solution.

Lemma 3.3. *Assume that* $F, \tilde{F}, G, \tilde{G} \colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ *satisfy hypotheses (H1)-(H4). Then each* $X_n \colon J \to \mathcal{K}_c^b(L^2)$ *is a well-defined* H_{L^2}*-continuous set-valued mapping.*

Proof. First, notice that $X_0(\cdot)$ is a well-defined mapping, since it is constantly equal to X_0, $X_0 \in \mathcal{K}_c^b(L_0^2)$. Next, using the measurability condition (H1), we can infer that the set-valued mappings

$$F(\cdot, \cdot, X_0), \tilde{F}(\cdot, \cdot, X_0), G(\cdot, \cdot, X_0), \tilde{G}(\cdot, \cdot, X_0) \colon I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$$

are nonanticipating. It can be checked, using (H2) and (H3), that the following inequalities hold $\lambda \times P$-a.e.

$$H_{\mathbb{R}^d}^2(F(t,\omega,X_0), \{\theta\}) \leqslant 2K_F(t,\omega) H_{L^2}^2(X_0, \{\Theta\}) + 2C_F(t,\omega),$$

$$H_{\mathbb{R}^d}^2(\tilde{F}(t,\omega,X_0), \{\theta\}) \leqslant 2K_{\tilde{F}}(t,\omega) H_{L^2}^2(X_0, \{\Theta\}) + 2C_{\tilde{F}}(t,\omega),$$

$$H_{\mathbb{R}^d}^2(G(t,\omega,X_0), \{\theta\}) \leqslant 2K_G(t,\omega) H_{L^2}^2(X_0, \{\Theta\}) + 2C_G(t,\omega),$$

$$H_{\mathbb{R}^d}^2(\tilde{G}(t,\omega,X_0), \{\theta\}) \leqslant 2K_{\tilde{G}}(t,\omega) H_{L^2}^2(X_0, \{\Theta\}) + 2C_{\tilde{G}}(t,\omega).$$

Thus $F(\cdot, \cdot, X_0)$, $\tilde{F}(\cdot, \cdot, X_0)$, $G(\cdot, \cdot, X_0)$ and $\tilde{G}(\cdot, \cdot, X_0)$ are $L_{\mathcal{N}}^2(\lambda \times P)$-integrally bounded. Next we claim that the set-valued stochastic trajectory integrals in the formulation of $X_1(t)$ are well-defined and are elements of the set $\mathcal{K}_c^b(L_t^2)$. Since $X_0 \in \mathcal{K}_c^b(L_0^2) \subset \mathcal{K}_c^b(L_t^2)$ and it is assumed that the Hukuhara differences in (H4) exist, we obtain that $X_1(t) \in \mathcal{K}_c^b(L_t^2)$ for every $t \in J$. Moreover, the mapping $t \mapsto X_1(t)$ is H_{L^2}-continuous from Lemma 2.2. Since the Mosco topology $\tau_{M_{L^2}}$ is is weaker than the topology generated by the Hausdorff metric H_{L^2}, the mapping $t \mapsto X_1(t)$ is continuous with respect to topology $\tau_{M_{L^2}}$ as well. Hence the set-valued mappings $(t,\omega) \mapsto F(t,\omega,X_1(t))$, $(t,\omega) \mapsto \tilde{F}(t,\omega,X_1(t))$, $(t,\omega) \mapsto G(t,\omega,X_1(t))$ and $(t,\omega) \mapsto \tilde{G}(t,\omega,X_1(t))$ are nonanticipating. Observing that

$$H_{\mathbb{R}^d}^2(F(t,\omega,X_1(t)), \{\theta\}) \leqslant 2K_F(t,\omega) \sup_{t \in J} H_{L^2}^2(X_1(t), \{\Theta\}) + 2C_F(t,\omega),$$

$$H_{\mathbb{R}^d}^2(\tilde{F}(t,\omega,X_1(t)), \{\theta\}) \leqslant 2K_{\tilde{F}}(t,\omega) \sup_{t \in J} H_{L^2}^2(X_1(t), \{\Theta\}) + 2C_{\tilde{F}}(t,\omega),$$

$$H_{\mathbb{R}^d}^2(G(t,\omega,X_1(t)), \{\theta\}) \leqslant 2K_G(t,\omega) \sup_{t \in J} H_{L^2}^2(X_1(t), \{\Theta\}) + 2C_G(t,\omega),$$

$$H_{\mathbb{R}^d}^2(\tilde{G}(t,\omega,X_1(t)), \{\theta\}) \leqslant 2K_{\tilde{G}}(t,\omega) \sup_{t \in J} H_{L^2}^2(X_1(t), \{\Theta\}) + 2C_{\tilde{G}}(t,\omega)$$

and $\sup_{t \in J} H_{L^2}^2\big(X_1(t), \{\Theta\}\big) < \infty$, we get that $(t,\omega) \mapsto F(t,\omega,X_1(t))$, $(t,\omega) \mapsto \tilde{F}(t,\omega,X_1(t))$, $(t,\omega) \mapsto G(t,\omega,X_1(t))$ and $(t,\omega) \mapsto \tilde{G}(t,\omega,X_1(t))$ are $L_{\mathcal{N}}^2(\lambda \times P)$-integrally bounded. This allows us to infer that that X_2 is well-defined and H_{L^2}-continuous. Proceeding recursively we see that every mapping X_n is well-defined and H_{L^2}-continuous. □

Theorem 3.4. *Let the assumptions of Lemma 3.3 be satisfied. Then equation* (3.1) *has a unique (possibly local) solution.*

Proof. From Lemma 3.3 we see that each X_n is continuous with respect to the metric H_{L^2}. Consider the space $C(J, \mathcal{K}_c^b(L^2))$ endowed with the supremum metric. We now show that $\{X_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this metric space.

Applying properties (P3), (P2) and (P1) we have for $t \in J$ that

$$\begin{aligned} H_{L^2}^2(X_1(t), X_0(t)) &= H_{L^2}^2\Big(\Big[X_0 + (S)\int_0^t \tilde{F}(s,X_0)ds + (S)\int_0^t \tilde{G}(s,X_0)d\tilde{B}(s)\Big] \\ &\quad \ominus \Big[(S)\int_0^t F(s,X_0)ds + (S)\int_0^t G(s,X_0)dB(s)\Big], X_0\Big) \\ &\leqslant 2H_{L^2}^2\Big(X_0 + (S)\int_0^t \tilde{F}(s,X_0)ds + (S)\int_0^t \tilde{G}(s,X_0)d\tilde{B}(s), X_0\Big) \\ &\quad +2H_{L^2}^2\Big((S)\int_0^t F(s,X_0)ds + (S)\int_0^t G(s,X_0)dB(s), \{\Theta\}\Big) \\ &\leqslant 4H_{L^2}^2\left((S)\int_0^t \tilde{F}(s,X_0)ds, \{\Theta\}\right) + 4H_{L^2}^2\left((S)\int_0^t \tilde{G}(s,X_0)d\tilde{B}(s), \{\Theta\}\right) \\ &\quad + 4H_{L^2}^2\left((S)\int_0^t \tilde{F}(s,X_0)ds, \{\Theta\}\right) + 4H_{L^2}^2\left((S)\int_0^t G(s,X_0)dB(s), \{\Theta\}\right). \end{aligned}$$

Using Lemma 2.1 we get

$$\begin{aligned} H_{L^2}^2(X_1(t), X_0(t)) &\leqslant 4t\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(\tilde{F}(s,X_0),\{\theta\})ds\times dP + 4\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(\tilde{G}(s,X_0),\{\theta\})ds\times dP \\ &\quad + 4t\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(F(s,X_0),\{\theta\})ds\times dP + 4\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(G(s,X_0),\{\theta\})ds\times dP \\ &\leqslant 8t\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(\tilde{F}(s,X_0),\tilde{F}(s,\{\Theta\}))ds\times dP + 8t\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(\tilde{F}(s,\{\Theta\}),\{\theta\})ds\times dP \\ &\quad + 8\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(\tilde{G}(s,X_0),\tilde{G}(s,\{\Theta\}))ds\times dP + 8\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(\tilde{G}(s,\{\Theta\}),\{\theta\})ds\times dP \\ &\quad + 8t\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(F(s,X_0),F(s,\{\Theta\}))ds\times dP + 8t\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(F(s,\{\Theta\}),\{\theta\})ds\times dP \\ &\quad + 8\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(G(s,X_0),G(s,\{\Theta\}))ds\times dP + 8\int_{[0,t]\times\Omega} H_{\mathbb{R}^d}^2(G(s,\{\Theta\}),\{\theta\})ds\times dP \end{aligned}$$

and by hypotheses (H2) and (H3) we have

$$\begin{aligned} H_{L^2}^2(X_1(t), X_0(t)) &\leqslant 16(t+1)H_{L^2}^2(X_0,\{\Theta\})\int_{[0,t]\times\Omega}(K_{\tilde{F}}(s)+K_{\tilde{G}}(s)+K_F(s)+K_G(s))ds\times dP \\ &\quad + 16(t+1)\int_{[0,t]\times\Omega}(C_{\tilde{F}}(s)+C_{\tilde{G}}(s)+C_F(s)+C_G(s))ds\times dP \\ &\leqslant M_1, \end{aligned}$$

where

$$\begin{aligned} M_1 &= 16(\tilde{T}+1)\Big[H_{L^2}^2(X_0,\{\Theta\})\int_{J\times\Omega}(K_{\tilde{F}}(s)+K_{\tilde{G}}(s)+K_F(s)+K_G(s))ds\times dP \\ &\quad + \int_{J\times\Omega}(C_{\tilde{F}}(s)+C_{\tilde{G}}(s)+C_F(s)+C_G(s))ds\times dP\Big] < \infty. \end{aligned} \tag{3.7}$$

Considering $n \geqslant 2$ we get

$$\begin{aligned}
H^2_{L^2}(X_n(t), X_{n-1}(t)) \;\leqslant\;& 4t \int_{[0,t]\times\Omega} K_{\tilde{F}}(s) H^2_{L^2}(X_{n-1}(s), X_{n-2}(s)) ds \times dP \\
&+ 4 \int_{[0,t]\times\Omega} K_{\tilde{G}}(s) H^2_{L^2}(X_{n-1}(s), X_{n-2}(s)) ds \times dP \\
&+ 4t \int_{[0,t]\times\Omega} K_{F}(s) H^2_{L^2}(X_{n-1}(s), X_{n-2}(s)) ds \times dP \\
&+ 4 \int_{[0,t]\times\Omega} K_{G}(s) H^2_{L^2}(X_{n-1}(s), X_{n-2}(s)) ds \times dP \\
\leqslant\;& \Big[4t \left(\int_{[0,t]\times\Omega} K^2_{\tilde{F}}(s) ds \times dP \right)^{1/2} + 4 \left(\int_{[0,t]\times\Omega} K^2_{\tilde{G}}(s) ds \times dP \right)^{1/2} \\
&+ 4t \left(\int_{[0,t]\times\Omega} K^2_{F}(s) ds \times dP \right)^{1/2} + 4 \left(\int_{[0,t]\times\Omega} K^2_{G}(s) ds \times dP \right)^{1/2} \Big] \\
&\times \left(\int_{[0,t]\times\Omega} H^4_{L^2}(X_{n-1}(s), X_{n-2}(s)) ds \times dP \right)^{1/2}.
\end{aligned}$$

Thus

$$H^4_{L^2}(X_n(t), X_{n-1}(t)) \leqslant M_2 \int_0^t H^4_{L^2}(X_{n-1}(s), X_{n-2}(s)) ds,$$

where

$$\begin{aligned}
M_2 \;=\;& \Big[4\tilde{T} \left(\int_{J\times\Omega} K^2_{\tilde{F}}(s) ds \times dP \right)^{1/2} + 4 \left(\int_{J\times\Omega} K^2_{\tilde{G}}(s) ds \times dP \right)^{1/2} \\
&+ 4\tilde{T} \left(\int_{J\times\Omega} K^2_{F}(s) ds \times dP \right)^{1/2} + 4 \left(\int_{J\times\Omega} K^2_{G}(s) ds \times dP \right)^{1/2} \Big]^2.
\end{aligned} \tag{3.8}$$

This leads us to the conclusion that

$$H_{L^2}(X_n(t), X_{n-1}(t)) \leqslant \left(M_1^2 \frac{(M_2 t)^{n-1}}{(n-1)!} \right)^{1/4} \tag{3.9}$$

and

$$\sup_{t\in J} H_{L^2}(X_n(t), X_{n-1}(t)) \leqslant \left(M_1^2 \frac{(M_2 \tilde{T})^{n-1}}{(n-1)!} \right)^{1/4}.$$

Hence for $m < n$ we have

$$\sup_{t\in J} H_{L^2}(X_n(t), X_m(t)) \leqslant \sum_{k=m}^{n-1} \left(M_1^2 \frac{(M_2 \tilde{T})^{k}}{k!} \right)^{1/4}$$

and this allows us to infer that $\{X_n\}$ is a Cauchy sequence in the set $C(J, \mathcal{K}^b_c(L^2))$ supplied with the supremum metric. Thus there exists $X \in C(J, \mathcal{K}^b_c(L^2))$ such that

$$\sup_{t\in J} H_{L^2}(X_n(t), X(t)) \to 0 \text{ as } n \to \infty.$$

Since $(\mathcal{K}^b_c(L^2_t)), H_{L^2_t}\big)$ is a complete metric space for each $t \in J$ and $X_n(t), X_m(t) \in \mathcal{K}^b_c(L^2_t)$ and

$$H_{L^2_t}(X_n(t), X_m(t)) = H_{L^2}(X_n(t), X_m(t)) \to 0 \text{ as } n, m \to \infty \text{ for every } t \in J,$$

we obtain that $X(t) \in \mathcal{K}^b_c(L^2_t)$ for every $t \in J$.

We now show that X is a solution to (3.1). In the case $J = I$, X will be a global solution, otherwise if $J \subsetneq I$, X will be a local solution. Notice that for every $t \in J$ we get

$$\begin{aligned} & H_{L^2}^2\Big(\Big[X_0 + (S)\int_0^t \tilde{F}(s,X(s))ds + (S)\int_0^t \tilde{G}(s,X(s))d\tilde{B}(s)\Big] \ominus \Big[(S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s)\Big], X(t)\Big) \\ & \leqslant \;\; 2R_n(t) + 2H_{L^2}^2(X_n(t),X(t)), \end{aligned}$$

where

$$\begin{aligned} R_n(t) \;\; = \;\; & H_{L^2}^2\Big(\Big[X_0 + (S)\int_0^t \tilde{F}(s,X(s))ds + (S)\int_0^t \tilde{G}(s,X(s))d\tilde{B}(s)\Big] \\ & \ominus \Big[(S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s)\Big], \\ & \Big[X_0 + (S)\int_0^t \tilde{F}(s,X_{n-1}(s))ds + (S)\int_0^t \tilde{G}(s,X_{n-1}(s))d\tilde{B}(s)\Big] \\ & \ominus \Big[(S)\int_0^t F(s,X_{n-1}(s))ds + (S)\int_0^t G(s,X_{n-1}(s))dB(s)\Big]\Big). \end{aligned}$$

Observe that

$$\begin{aligned} R_n(t) \;\; \leqslant \;\; & 2H_{L^2}^2\Big((S)\int_0^t \tilde{F}(s,X_{n-1}(s))ds + (S)\int_0^t \tilde{G}(s,X_{n-1}(s))d\tilde{B}(s), \\ & (S)\int_0^t \tilde{F}(s,X(s))ds + (S)\int_0^t \tilde{G}(s,X(s))d\tilde{B}(s)\Big) \\ & + 2H_{L^2}^2\Big((S)\int_0^t F(s,X_{n-1}(s))ds + (S)\int_0^t G(s,X_{n-1}(s))dB(s), \\ & (S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s)\Big) \\ \leqslant \;\; & 4H_{L^2}^2\Big((S)\int_0^t \tilde{F}(s,X_{n-1}(s))ds, (S)\int_0^t \tilde{F}(s,X(s))ds\Big) \\ & + 4H_{L^2}^2\Big((S)\int_0^t \tilde{G}(s,X_{n-1}(s))d\tilde{B}(s), (S)\int_0^t \tilde{G}(s,X(s))d\tilde{B}(s)\Big) \\ & + 4H_{L^2}^2\Big((S)\int_0^t F(s,X_{n-1}(s))ds, (S)\int_0^t F(s,X(s))ds\Big) \\ & + 4H_{L^2}^2\Big((S)\int_0^t G(s,X_{n-1}(s))dB(s), (S)\int_0^t G(s,X(s))dB(s)\Big) \end{aligned}$$

Furthermore we have

$$\begin{aligned} R_n(t) \;\; \leqslant \;\; & \Big[4t\Big(\int_{[0,t]\times\Omega} K_{\tilde{F}}^2(s)ds \times dP\Big)^{1/2} + 4\Big(\int_{[0,t]\times\Omega} K_{\tilde{G}}^2(s)ds \times dP\Big)^{1/2} \\ & + 4t\Big(\int_{[0,t]\times\Omega} K_F^2(s)ds \times dP\Big)^{1/2} + 4\Big(\int_{[0,t]\times\Omega} K_G^2(s)ds \times dP\Big)^{1/2}\Big] \\ & \times\Big(\int_{[0,t]\times\Omega} H_{L^2}^4(X_{n-1}(s),X(s))ds \times dP\Big)^{1/2}. \end{aligned}$$

As a consequence we obtain

$$R_n^2(t) \leqslant M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X(s))ds \leqslant M_2 \tilde{T} \sup_{t\in J} H_{L^2}^4(X_{n-1}(t), X(t)),$$

where M_2 is like in (3.8). Since $\sup_{t\in J} H_{L^2}^4(X_{n-1}(t), X(t)) \stackrel{n\to\infty}{\longrightarrow} 0$, we have $R_n(t) \stackrel{n\to\infty}{\longrightarrow} 0$ for every $t \in J$. This allows us to infer that

$$\begin{aligned} H_{L^2}^2\Big(&\Big[X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))d\tilde{B}(s)\Big] \\ &\ominus\Big[(S)\int_0^t F(s, X(s))ds + (S)\int_0^t G(s, X(s))dB(s)\Big], X(t)\Big) = 0 \end{aligned}$$

for every $t \in J$, which means that X is a solution (possibly local) to (3.1).

We now show that the solution X is unique. In order to see that, suppose that $X\colon J \to \mathcal{K}_c^b(L^2)$ and $Y\colon J \to \mathcal{K}_c^b(L^2)$ are two solutions to (3.1). Then we can check for $t \in J$ that

$$H_{L^2}^4(X(t), Y(t)) \leqslant M_2 \int_0^t H_{L^2}^4(X(s), Y(s))ds,$$

where M_2 is like in (3.8). Thus, by applying the Gronwall inequality, we get

$$H_{L^2}^4(X(t), Y(t)) = 0 \text{ for every } t \in J.$$

It follows that $X(t) = Y(t)$ for every $t \in J$, which proves uniqueness of the solution X. □

The mapping X_n approximates the exact solution X of (3.1). We now give an estimate on the error between X_n and X. From the result below we see that the convergence of $\{X_n\}$ to the solution X is exponential.

Proposition 3.5. *Let the assumptions of Lemma 3.3 be satisfied. Then*

$$\sup_{t\in J} H_{L^2}(X_n(t), X(t)) \leqslant 2^{3/4}\left(M_1^2 \frac{(M_2\tilde{T})^n}{n!}\right)^{1/4} \exp\{2M_2\tilde{T}\} \textit{ for every } n \in \mathbb{N},$$

where the constants M_1 and M_2 are defined like in (3.7) *and* (3.8), *respectively.*

Proof. Similar calculations to those in the proof of Theorem 3.4 lead us to the inequality

$$\begin{aligned} H_{L^2}^2(X_n(t), X(t)) \leqslant\ & \Big[4t\Big(\int_{[0,t]\times\Omega} K_{\tilde{F}}^2(s)ds \times dP\Big)^{1/2} + 4\Big(\int_{[0,t]\times\Omega} K_{\tilde{G}}^2(s)ds \times dP\Big)^{1/2} \\ & + 4t\Big(\int_{[0,t]\times\Omega} K_F^2(s)ds \times dP\Big)^{1/2} + 4\Big(\int_{[0,t]\times\Omega} K_G^2(s)ds \times dP\Big)^{1/2}\Big] \\ & \times\Big(\int_{[0,t]\times\Omega} H_{L^2}^4(X_{n-1}(s), X(s))ds \times dP\Big)^{1/2} \end{aligned}$$

for $t \in J$. Thus

$$\begin{aligned} H_{L^2}^4(X_n(t), X(t)) &\leqslant M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X(s))ds \\ &\leqslant 8M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X_n(s))ds + 8M_2 \int_0^t H_{L^2}^4(X_n(s), X(s))ds \end{aligned}$$

and applying (3.9) we obtain

$$H_{L^2}^4(X_n(t), X(t)) \leqslant 8M_1^2 \frac{(M_2\tilde{T})^n}{n!} + 8M_2 \int_0^t H_{L^2}^4(X_n(s), X(s))ds,$$

where M_1, M_2 are like in (3.7) and (3.8). From Gronwall's inequality we get

$$H^4_{L^2}(X_n(t), X(t)) \leqslant 8M_1^2 \frac{(M_2\tilde{T})^n}{n!} \exp\{8M_2 t\} \text{ for every } t \in J.$$

Now the claim follows immediately. □

The next result gives an estimate confirming the boundednes of the solution to equation (3.1).

Proposition 3.6. *Under the assumptions of Lemma 3.3 for the solution X to* (3.1) *we have the estimate*

$$\sup_{t\in J} H_{L^2}(X(t), \{\Theta\}) \leqslant M_3 \exp\{M_4\tilde{T}\},$$

where

$$\begin{aligned} M_3 &= \Big(18H^4_{L^2}(X_0, \{\Theta\}) + 3\Big(12\tilde{T}\int_{J\times\Omega} C_{\tilde{F}}(s)ds \times dP + 12\int_{J\times\Omega} C_{\tilde{G}}(s)ds\times dP \\ &\quad + 8\tilde{T}\int_{J\times\Omega} C_F(s)ds\times dP + 8\int_{J\times\Omega} C_G(s)ds\times dP\Big)^2\Big)^{1/4} \end{aligned}$$

and

$$\begin{aligned} M_4 &= \frac{3}{4}\Big(12\tilde{T}\Big(\int_{J\times\Omega} K^2_{\tilde{F}}(s)ds\times dP\Big)^{1/2} + 12\Big(\int_{J\times\Omega} K^2_{\tilde{G}}(s)ds\times dP\Big)^{1/2} \\ &\quad + 8\tilde{T}\Big(\int_{J\times\Omega} K^2_F(s)ds\times dP\Big)^{1/2} + 8\Big(\int_{J\times\Omega} K^2_G(s)ds\times dP\Big)^{1/2}\Big)^2 \end{aligned}$$

Proof. Using (P3) and (P1) we have for $t \in J$ that

$$\begin{aligned} H^2_{L^2}(X(t), \{\Theta\}) &\leqslant 6H^2_{L^2}(X_0, \{\Theta\}) + 6H^2_{L^2}\left((S)\int_0^t \tilde{F}(s, X(s))ds, \{\Theta\}\right) \\ &\quad + 6H^2_{L^2}\left((S)\int_0^t \tilde{G}(s, X(s))dB(s), \{\Theta\}\right) \\ &\quad + 4H^2_{L^2}\left((S)\int_0^t F(s, X(s))ds, \{\Theta\}\right) + 4H^2_{L^2}\left((S)\int_0^t G(s, X(s))dB(s), \{\Theta\}\right). \end{aligned}$$

Applying Lemma 2.1 we get

$$\begin{aligned} H^2_{L^2}(X(t), \{\Theta\}) &\leqslant 6H^2_{L^2}(X_0, \{\Theta\}) + 12t\Big[\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{F}(s, X(s)), \tilde{F}(s, \{\Theta\}))ds\times dP \\ &\quad + \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{F}(s, \{\Theta\}), \{\theta\})ds\times dP\Big] + 12\Big[\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{G}(s, X(s)), \tilde{G}(s, \{\Theta\}))ds\times dP \\ &\quad + \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{G}(s, \{\Theta\}), \{\theta\})ds\times dP\Big] + 8t\Big[\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s, X(s)), F(s, \{\Theta\}))ds\times dP \\ &\quad + \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s, \{\Theta\}), \{\theta\})ds\times dP\Big] + 8\Big[\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s, X(s)), G(s, \{\Theta\}))ds\times dP \\ &\quad + \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s, \{\Theta\}), \{\theta\})ds\times dP\Big]. \end{aligned}$$

From assumptions (H2) and (H3) we have

$$\begin{aligned}
H^2_{L^2}(X(t),\{\Theta\}) \leqslant\ & 6H^2_{L^2}(X_0,\{\Theta\}) \\
&+ 12t\int_{[0,t]\times\Omega} K_{\tilde F}(s)H^2_{L^2}(X(s),\{\Theta\})ds\times dP + 12t\int_{[0,t]\times\Omega} C_{\tilde F}(s)ds\times dP \\
&+ 12\int_{[0,t]\times\Omega} K_{\tilde G}(s)H^2_{L^2}(X(s),\{\Theta\})ds\times dP + 12\int_{[0,t]\times\Omega} C_{\tilde G}(s)ds\times dP \\
&+ 8t\int_{[0,t]\times\Omega} K_F(s)H^2_{L^2}(X(s),\{\Theta\})ds\times dP + 8t\int_{[0,t]\times\Omega} C_F(s)ds\times dP \\
&+ 8\int_{[0,t]\times\Omega} K_G(s)H^2_{L^2}(X(s),\{\Theta\})ds\times dP + 8\int_{[0,t]\times\Omega} C_G(s)ds\times dP \\
\leqslant\ & 6H^2_{L^2}(X_0,\{\Theta\}) \\
&+ 12t\int_{[0,t]\times\Omega} C_{\tilde F}(s)ds\times dP + 12\int_{[0,t]\times\Omega} C_{\tilde G}(s)ds\times dP \\
&+ 8t\int_{[0,t]\times\Omega} C_F(s)ds\times dP + 8\int_{[0,t]\times\Omega} C_G(s)ds\times dP \\
&+ \Big(12t\Big(\int_{[0,t]\times\Omega} K^2_{\tilde F}(s)ds\times dP\Big)^{1/2} + 12\Big(\int_{[0,t]\times\Omega} K^2_{\tilde G}(s)ds\times dP\Big)^{1/2} \\
&+ 8t\Big(\int_{[0,t]\times\Omega} K^2_F(s)ds\times dP\Big)^{1/2} + 8\Big(\int_{[0,t]\times\Omega} K^2_G(s)ds\times dP\Big)^{1/2}\Big) \\
&\times\Big(\int_{[0,t]\times\Omega} H^4_{L^2}(X(s),\{\Theta\})ds\times dP\Big)^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
H^4_{L^2}(X(t),\{\Theta\}) \leqslant\ & 18H^4_{L^2}(X_0,\{\Theta\}) \\
&+ 3\Big(12t\int_{[0,t]\times\Omega} C_{\tilde F}(s)ds\times dP + 12\int_{[0,t]\times\Omega} C_{\tilde G}(s)ds\times dP \\
&+ 8t\int_{[0,t]\times\Omega} C_F(s)ds\times dP + 8\int_{[0,t]\times\Omega} C_G(s)ds\times dP\Big)^2 \\
&+ 3\Big(12t\Big(\int_{[0,t]\times\Omega} K^2_{\tilde F}(s)ds\times dP\Big)^{1/2} + 12\Big(\int_{[0,t]\times\Omega} K^2_{\tilde G}(s)ds\times dP\Big)^{1/2} \\
&+ 8t\Big(\int_{[0,t]\times\Omega} K^2_F(s)ds\times dP\Big)^{1/2} + 8\Big(\int_{[0,t]\times\Omega} K^2_G(s)ds\times dP\Big)^{1/2}\Big)^2 \\
&\times\int_0^t H^4_{L^2}(X(s),\{\Theta\})ds\times dP \\
\leqslant\ & M_3^4 + 4M_2\int_0^t H^4_{L^2}(X(s),\{\Theta\})ds.
\end{aligned}$$

From Gronwall's inequality we get

$$H^4_{L^2}(X(t),\{\Theta\}) \leqslant M_3^4\exp\{4M_2t\} \text{ for every } t\in J,$$

and this inequality leads us to the assertion easily. □

Now we focus on the property of continuous dependence of the solution with respect to the initial value and the coefficients of the equation. This property is needed to ensure that the theory of bilateral set-valued stochastic integral equations is well-posed.

Consider equation (3.1) with initial value X_0 and the same equation with another initial value $\tilde{X}_0 \in \mathcal{K}_c^b(L_0^2)$, i.e.

$$\begin{aligned} &X(t) + (S)\int_0^t F(s,X(s))ds + (S)\int_0^t G(s,X(s))dB(s) \\ &= \tilde{X}_0 + (S)\int_0^t \tilde{F}(s,X(s))ds + (S)\int_0^t \tilde{G}(s,X(s))d\tilde{B}(s) \ \text{ for } \ t \in I. \end{aligned} \tag{3.10}$$

Let us denote by $X\colon J_1 \to \mathcal{K}_c^b(L^2)$ and $Y\colon J_2 \to \mathcal{K}_c^b(L^2)$ the unique solutions (if they exist) to equations (3.1) and (3.10), respectively, $J_1 = [0,\tilde{T}_1]$, $J_2 = [0,\tilde{T}_2]$ for some $\tilde{T}_1, \tilde{T}_2 \in (0,T]$. Let $J = J_1 \cap J_2$.

Theorem 3.7. *Let $F,\tilde{F},G,\tilde{G}$ satisfy the conditions (H1)-(H3). Assume also that $F,\tilde{F},G,\tilde{G}$ satisfy (H4) with X_0, and that $F,\tilde{F},G,\tilde{G}$ satisfy (H4) with $\tilde{X}_0$, too. Then*

$$\sup_{t\in J} H_{L^2}(X(t),Y(t)) \leqslant 12^{1/4} H_{L^2}(X_0,\tilde{X}_0)\exp\{\tilde{M}\min\{T_1,T_2\}\},$$

where

$$\begin{aligned} \tilde{M} &= \frac{1}{2}\Big(6\min\{T_1,T_2\}\Big(\int_{J\times\Omega} K_{\tilde{F}}^2(s)ds\times dP\Big)^{1/2} + 6\Big(\int_{J\times\Omega} K_{\tilde{G}}^2(s)ds\times dP\Big)^{1/2} \\ &\quad + 4\min\{T_1,T_2\}\Big(\int_{J\times\Omega} K_F^2(s)ds\times dP\Big)^{1/2} + 4\Big(\int_{J\times\Omega} K_G^2(s)ds\times dP\Big)^{1/2}\Big)^2. \end{aligned}$$

Proof. Notice that

$$\begin{aligned} H_{L^2}^2(X(t),Y(t)) &\leqslant 6H_{L^2}^2(X_0,\tilde{X}_0) \\ &\quad + 6H_{L^2}^2\left(\int_0^t \tilde{F}(s,X(s))ds, \int_0^t \tilde{F}(s,Y(s))ds\right) \\ &\quad + 6H_{L^2}^2\left(\int_0^t \tilde{G}(s,X(s))d\tilde{B}(s), \int_0^t \tilde{G}(s,Y(s))d\tilde{B}(s)\right) \\ &\quad + 4H_{L^2}^2\left(\int_0^t F(s,X(s))ds, \int_0^t F(s,Y(s))ds\right) \\ &\quad + 4H_{L^2}^2\left(\int_0^t G(s,X(s))dB(s), \int_0^t G(s,Y(s))dB(s)\right). \end{aligned}$$

Using Lemma 2.1 and assumptions (H2) and (H3) we get

$$\begin{aligned} H_{L^2}^2(X(t),Y(t)) &\leqslant 6H_{L^2}^2(X_0,\tilde{X}_0) \\ &\quad + 4t\int_{[0,t]\times\Omega} K_{\tilde{F}}(s)H_{L^2}^2(X(s),Y(s))ds\times dP \\ &\quad + 4\int_{[0,t]\times\Omega} K_{\tilde{G}}(s)H_{L^2}^2(X(s),Y(s))ds\times dP \\ &\quad + 4t\int_{[0,t]\times\Omega} K_F(s)H_{L^2}^2(X(s),Y(s))ds\times dP \\ &\quad + 4\int_{[0,t]\times\Omega} K_G(s)H_{L^2}^2(X(s),Y(s))ds\times dP. \end{aligned}$$

Hence

$$\begin{aligned} H^4_{L^2}(X(t),Y(t)) \leqslant\ & 12H^4_{L^2}(X_0,\tilde{X}_0)+2\Big(6t\Big(\int_{[0,t]\times\Omega}K^2_{\tilde{F}}(s)ds\times dP\Big)^{1/2}+6\Big(\int_{[0,t]\times\Omega}K^2_{\tilde{G}}(s)ds\times dP\Big)^{1/2} \\ & +4t\Big(\int_{[0,t]\times\Omega}K^2_{F}(s)ds\times dP\Big)^{1/2}+4\Big(\int_{[0,t]\times\Omega}K^2_{G}(s)ds\times dP\Big)^{1/2}\Big)^2\int_0^t H^4_{L^2}\big(X(s),Y(s)\big)ds. \end{aligned}$$

Gronwall's inequality allows us to infer that

$$H^4_{L^2}(X(t),Y(t))\leqslant \tilde{M}_5\exp\{\tilde{M}_6 t\}\text{ for every } t\in J,$$

where $\tilde{M}_5=12H^4_{L^2}(X_0,\tilde{X}_0)$ and

$$\begin{aligned} \tilde{M}_6 =\ & 2\Big(6\min\{T_1,T_2\}\Big(\int_{J\times\Omega}K^2_{\tilde{F}}(s)ds\times dP\Big)^{1/2}+6\Big(\int_{J\times\Omega}K^2_{\tilde{G}}(s)ds\times dP\Big)^{1/2} \\ & +4\min\{T_1,T_2\}\Big(\int_{J\times\Omega}K^2_{F}(s)ds\times dP\Big)^{1/2}+4\Big(\int_{J\times\Omega}K^2_{G}(s)ds\times dP\Big)^{1/2}\Big)^2. \end{aligned}$$

Therefore we get

$$\sup_{t\in J} H_{L^2}(X(t),Y(t))\leqslant(\tilde{M}_5)^{1/4}\exp\{\tilde{M}_6\min\{T_1,T_2\}/4\},$$

which ends the proof. □

Now consider equation (3.1) and equations (for $n\in\mathbb{N}$)

$$\begin{aligned} & X(t)+(S)\int_0^t F_n(s,X(s))ds+(S)\int_0^t G_n(s,X(s))dB(s) \\ & = X_0+(S)\int_0^t \tilde{F}_n(s,X(s))ds+(S)\int_0^t \tilde{G}_n(s,X(s))d\tilde{B}(s)\quad\text{for}\quad t\in I \end{aligned}\tag{3.11}$$

with other coefficients $F_n,\tilde{F}_n,G_n,\tilde{G}_n$. Let us denote by X and Y_n unique solutions (if they exist) to equations (3.1) and (3.11), respectively. Suppose that they all are defined on a common interval $J=[0,\tilde{T}]$ with $\tilde{T}\in(0,T]$.

Theorem 3.8. *Let $X_0,F,\tilde{F},G,\tilde{G}$ satisfy the conditions (H1)-(H4). Assume also that $X_0,F_n,\tilde{F}_n,G_n,\tilde{G}_n$ satisfy (H1)-(H4), in particular the conditions (H2) and (H3) are satisfied with the processes $K_{F_n},K_{\tilde{F}_n}K_{G_n},K_{\tilde{G}_n}$ and $C_{F_n},C_{\tilde{F}_n},C_{G_n},C_{\tilde{G}_n}$, respectively. Assume that there exist constants $S_F,S_{\tilde{F}},S_G,S_{\tilde{G}}>0$ such that for every $n\in\mathbb{N}$*

$$\int_{I\times\Omega}K^2_{F_n}(s)ds\times dP\leqslant S_F,\ \int_{I\times\Omega}K^2_{G_n}(s)ds\times dP\leqslant S_G,$$

$$\int_{I\times\Omega}K^2_{\tilde{F}_n}(s)ds\times dP\leqslant S_{\tilde{F}}\ \textit{and}\ \int_{I\times\Omega}K^2_{\tilde{G}_n}(s)ds\times dP\leqslant S_{\tilde{G}}.$$

Suppose that for every $A\in\mathcal{K}^b_c(L^2)$

$$\begin{aligned} & \int_{I\times\Omega}H^2_{\mathbb{R}^d}(F_n(s,A),F(s,A))ds\times dP\to 0 \textit{ as } n\to\infty, \\ & \int_{I\times\Omega}H^2_{\mathbb{R}^d}(G_n(s,A),G(s,A))ds\times dP\to 0 \textit{ as } n\to\infty, \\ & \int_{I\times\Omega}H^2_{\mathbb{R}^d}(\tilde{F}_n(s,A),\tilde{F}(s,A))ds\times dP\to 0 \textit{ as } n\to\infty \textit{ and} \\ & \int_{I\times\Omega}H^2_{\mathbb{R}^d}(\tilde{G}_n(s,A),\tilde{G}(s,A))ds\times dP\to 0 \textit{ as } n\to\infty. \end{aligned}$$

Then for the local solution X to (3.1) *and solutions X_n to* (3.11) *we have*

$$\sup_{t \in J} H_{L^2}(Y_n(t), X(t)) \to 0 \text{ as } n \to \infty.$$

Proof. Using the form (3.6) of the solutions X and Y_n and properties (P3), (P2) and (P1) we have for $t \in J$ that

$$\begin{aligned}
H^2_{L^2}(Y_n(t), X(t)) \quad \leqslant \quad & 4H^2_{L^2}\left(\int_0^t \tilde{F}_n(s, Y_n(s))ds, \int_0^t \tilde{F}(s, X(s))ds\right) \\
& + 4H^2_{L^2}\left(\int_0^t \tilde{G}_n(s, Y_n(s))d\tilde{B}(s), \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s)\right) \\
& + 4H^2_{L^2}\left(\int_0^t F_n(s, Y_n(s))ds, \int_0^t F(s, X(s))ds\right) \\
& + 4H^2_{L^2}\left(\int_0^t G_n(s, Y_n(s))dB(s), \int_0^t G(s, X(s))dB(s)\right) \\
\leqslant \quad & 8H^2_{L^2}\left(\int_0^t \tilde{F}_n(s, Y_n(s))ds, \int_0^t \tilde{F}_n(s, X(s))ds\right) \\
& + 8H^2_{L^2}\left(\int_0^t \tilde{F}_n(s, X(s))ds, \int_0^t \tilde{F}(s, X(s))ds\right) \\
& + 8H^2_{L^2}\left(\int_0^t \tilde{G}_n(s, Y_n(s))d\tilde{B}(s), \int_0^t \tilde{G}_n(s, X(s))d\tilde{B}(s)\right) \\
& + 8H^2_{L^2}\left(\int_0^t \tilde{G}_n(s, X(s))d\tilde{B}(s), \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s)\right) \\
& + 8H^2_{L^2}\left(\int_0^t F_n(s, Y_n(s))ds, \int_0^t F_n(s, X(s))ds\right) \\
& + 8H^2_{L^2}\left(\int_0^t F_n(s, X(s))ds, \int_0^t F(s, X(s))ds\right) \\
& + 8H^2_{L^2}\left(\int_0^t G_n(s, Y_n(s))dB(s), \int_0^t G_n(s, X(s))dB(s)\right) \\
& + 8H^2_{L^2}\left(\int_0^t G_n(s, X(s))dB(s), \int_0^t G(s, X(s))dB(s)\right).
\end{aligned}$$

Applying Lemma 2.1 we obtain the estimate

$$\begin{aligned}
H^2_{L^2}(Y_n(t), X(t)) \quad \leqslant \quad & 8t \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{F}_n(s, Y_n(s)), \tilde{F}_n(s, X(s)))ds \times dP \\
& + 8t \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{F}_n(s, X(s)), \tilde{F}(s, X(s)))ds \times dP \\
& + 8 \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{G}_n(s, X_n(s)), \tilde{G}_n(s, X(s)))ds \times dP \\
& + 8 \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(\tilde{G}_n(s, X(s)), \tilde{G}(s, X(s)))ds \times dP \\
& + 8t \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F_n(s, Y_n(s)), F_n(s, X(s)))ds \times dP
\end{aligned}$$

$$+ 8t \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds \times dP$$
$$+ 8 \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X_n(s)),G_n(s,X(s)))ds \times dP$$
$$+ 8 \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds \times dP.$$

From assumption (H2) which is supposed to be satisfied by $F_n, \tilde{F}_n, G_n, \tilde{G}_n$ with the processes K_{F_n}, $K_{\tilde{F}_n}$, K_{G_n} and $K_{\tilde{G}_n}$, respectively, we notice that

$$\begin{aligned}
H^2_{L^2}(Y_n(t),X(t)) \quad \leqslant \quad & 8\tilde{T} \int_{[0,t]\times\Omega} K_{\tilde{F}_n}(s) H^2_{L^2}(Y_n(s),X(s))ds \times dP \\
& + 8\tilde{T} \int_{J\times\Omega} H^2_{\mathbb{R}^d}(\tilde{F}_n(s,X(s)),\tilde{F}(s,X(s)))ds \times dP \\
& + 8 \int_{[0,t]\times\Omega} K_{\tilde{G}_n}(s) H^2_{L^2}(Y_n(s),X(s))ds \times dP \\
& + 8 \int_{J\times\Omega} H^2_{\mathbb{R}^d}(\tilde{G}_n(s,X(s)),\tilde{G}(s,X(s)))ds \times dP \\
& + 8\tilde{T} \int_{[0,t]\times\Omega} K_{F_n}(s) H^2_{L^2}(Y_n(s),X(s))ds \times dP \\
& + 8\tilde{T} \int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds \times dP \\
& + 8 \int_{[0,t]\times\Omega} K_{G_n}(s) H^2_{L^2}(Y_n(s),X(s))ds \times dP \\
& + 8 \int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds \times dP.
\end{aligned}$$

Therefore

$$H^2_{L^2}(Y_n(t),X(t)) \quad \leqslant \quad M_7(n) + M_8(n)\Big(\int_0^t H^4_{L^2}(Y_n(s),X(s))ds\Big)^{1/2},$$

where

$$\begin{aligned}
M_7(n) \quad = \quad & 8\tilde{T} \int_{J\times\Omega} H^2_{\mathbb{R}^d}(\tilde{F}_n(s,X(s)),\tilde{F}(s,X(s)))ds \times dP \\
& + 8 \int_{J\times\Omega} H^2_{\mathbb{R}^d}(\tilde{G}_n(s,X(s)),\tilde{G}(s,X(s)))ds \times dP \\
& + 8\tilde{T} \int_{J\times\Omega} H^2_{\mathbb{R}^d}(F_n(s,X(s)),F(s,X(s)))ds \times dP \\
& + 8 \int_{J\times\Omega} H^2_{\mathbb{R}^d}(G_n(s,X(s)),G(s,X(s)))ds \times dP
\end{aligned}$$

and

$$\begin{aligned}
M_8(n) \quad = \quad & 8\tilde{T}\Big(\int_{J\times\Omega} K^2_{\tilde{F}_n}(s)ds \times dP\Big)^{1/2} + 8\Big(\int_{J\times\Omega} K^2_{\tilde{G}_n}(s)ds \times dP\Big)^{1/2} \\
& + 8\tilde{T}\Big(\int_{J\times\Omega} K^2_{F_n}(s) \times dP\Big)^{1/2} + 8\Big(\int_{J\times\Omega} K^2_{G_n}(s) \times dP\Big)^{1/2}.
\end{aligned}$$

Thus

$$H^4_{L^2}(Y_n(t), X(t)) \leqslant 2M_7^2(n) + 2M_8^2(n)\int_0^t H^4_{L^2}(Y_n(s), X(s))ds$$

Applying the Gronwall inequality we get for every $t \in J$ that

$$H^4_{L^2}(Y_n(t), X(t)) \leqslant 2M_7^2(n)\exp\{2M_8^2(n)t\}.$$

Hence

$$\sup_{t\in J} H^4_{L^2}(Y_n(t), X(t)) \leqslant 2M_7^2(n)\exp\{2M_8^2(n)\tilde{T}\}.$$

From our assumptions we obtain that $M_7(n) \to 0$ as $n \to \infty$ and $M_8(n) \leqslant 8\tilde{T}\sqrt{S_{\tilde{F}}} + 8\sqrt{S_{\tilde{G}}} + 8\tilde{T}\sqrt{S_F} + 8\sqrt{S_G}$. Thus

$$\sup_{t\in J} H^4_{L^2}(Y_n(t), X(t)) \longrightarrow 0$$

which completes the proof. □

Although condition (H4), which requires the existence of some Hukuhara differences, seems to be restrictive, as we mentioned before it is important and indispensable in the study of bilateral set-valued integral equations. It is needed, since in the representation (3.6) of equation (3.1) some Hukuhara differences are involved. In general, having in the background subsets of infinitely dimensional space L^2, some assumption concerning the existence of Hukuhara's differences must appear. An assumption of this kind, in the authors' opinion, is also needed in considering subsets of at least a two-dimensional space. Our reasoning is from the fact that in such spaces there is no convenient way to check the existence of Hukuhara's differences. Some slightly different conditions that lead to (H4) are presented below.

Remark 3.9. *In each of the conditions mentioned below one can replace (H4) in the study of the bilateral set-valued stochastic integral equation* (3.1).

(H41) There exists $\tilde{T} \in (0, T]$ such that for every $t \in [0, \tilde{T}]$ and for every H_{L^2}-continuous mapping $X\colon [0, \tilde{T}] \to \mathcal{K}^b_c(L^2)$ satisfying $X(t) \in \mathcal{K}^b_c(L^2_t)$, $t \in [0, \tilde{T}]$, there exist Hukuhara's differences

$$\left[X_0 + (S)\int_0^t \tilde{F}(s, X(s))ds + (S)\int_0^t \tilde{G}(s, X(s))d\tilde{B}(s)\right] \ominus \left[(S)\int_0^t F(s, X(s))ds + (S)\int_0^t G(s, X(s))dB(s)\right].$$

(H42) There exists $\tilde{T} \in (0, T]$ such that for every $t \in [0, \tilde{T}]$ and for every $A \in \mathcal{K}^b_c(L^2)$, there exist Hukuhara's differences

$$\left[X_0 + (S)\int_0^t \tilde{F}(s, A)ds + (S)\int_0^t \tilde{G}(s, A)d\tilde{B}(s)\right] \ominus \left[(S)\int_0^t F(s, A)ds + (S)\int_0^t G(s, A)dB(s)\right].$$

Unfortunately, checking the existence of differences in the case of subsets of the abstract space L^2 is still a very difficult task.

In the next section we show that (H4) can be replaced with another condition that guarantees the existence of the desired Hukuhara differences. However, we limit ourselves to considering subsets of a one-dimensional space. For subsets of such a space, there is a convenient criterion for checking the existence of the Hukuhara difference.

4. Implications for Deterministic Bilateral Set-Valued Integral Equations

Since $\mathbb{R}$ can be embedded into $L^2(\Omega, \mathcal{A}, P; \mathbb{R})$, the following deterministic, bilateral set-valued integral equation

$$X(t) + (S)\int_0^t \Psi(s, X(s))ds = X_0 + (S)\int_0^t \Phi(s, X(s))ds \quad \text{for } t \in I, \tag{4.1}$$

where $\Psi, \Phi \colon I \times \mathcal{K}_c^b(\mathbb{R}) \to \mathcal{K}_c^b(\mathbb{R})$, $X_0 \in \mathcal{K}_c^b(\mathbb{R})$ and the integral is the set-valued Aumann integral, is a particular case of equation (3.1). If $\Psi \equiv \{0\}$ then (4.1) constitutes an integral form of set-valued differential equations studied in [24]. We show that assuming a kind of boundedness of the coefficient Ψ we get the existence of the Hukuhara differences written in (H4).

We assume that $\Psi, \Phi \colon I \times \mathcal{K}_c^b(\mathbb{R}) \to \mathcal{K}_c^b(\mathbb{R})$ satisfy

(S1) for every $A \in \mathcal{K}(\mathbb{R})$ the set-valued mappings $\Psi(\cdot, A), \Phi(\cdot, A) \colon I \to \mathcal{K}_c^b(\mathbb{R})$ are β_I-measurable,

(S2) there exist $K_\Psi, K_\Phi \in L^2(I, \beta_I, \lambda; \mathbb{R})$ such that λ-a.e. for every $A, B \in \mathcal{K}_c^b(\mathbb{R})$

$$H_{\mathbb{R}}\Big(\Psi(t, A), \Psi(t, B)\Big) \leqslant K_\Psi(t) H_{\mathbb{R}}(A, B),$$

$$H_{\mathbb{R}}\Big(\Phi(t, A), \Phi(t, B)\Big) \leqslant K_\Phi(t) H_{\mathbb{R}}(A, B),$$

(S3) there exist $C_\Psi, C_\Phi \in L^1(I, \beta_I, \lambda; \mathbb{R})$ such that λ-a.e.

$$H_{\mathbb{R}}\Big(\Psi(t, \{0\}), \{0\}\Big) \leqslant C_\Psi(t),$$

$$H_{\mathbb{R}}\Big(\Phi(t, \{0\}), \{0\}\Big) \leqslant C_\Phi(t).$$

Additionally suppose that Ψ satisfies the following condition

(S4) for every $N \in \mathbb{N}$ there exists a positive constant M_N such that for every $t \in I$ and for every $A \in \mathcal{K}_c^b(\mathbb{R})$ we have

$$H_{\mathbb{R}}(A, \{0\}) \leqslant N \implies H_{\mathbb{R}}(\Psi(t, A), \{0\}) \leqslant M_N. \tag{4.2}$$

We claim in this case that the sequence $\{X_n\}$ described in (H4) by $X_0(t) = X_0$ and

$$X_n(t) = \Big(X_0 + (S)\int_0^t \Phi(s, X_{n-1}(s))ds\Big) \ominus (S)\int_0^t \Psi(s, X_{n-1}(s))ds$$

(for $n \in \mathbb{N}$) is well defined on an interval $J \subset I$. Indeed, there exists $N_0 \in \mathbb{N}$ such that $H_{\mathbb{R}}(X_0, \{0\}) \leqslant N_0$ and this means, in view of (4.2), that there exists M_{N_0} such that $H_{\mathbb{R}}(\Psi(t, X_0), \{0\}) \leqslant M_{N_0}$ for every $t \in I$. Hence for $t \in [0, \mathrm{diam} X_0/(2M_{N_0})]$ we have

$$\begin{aligned}
\mathrm{diam}\left((S)\int_0^t \Psi(s, X_0)ds\right) &\leqslant 2\int_0^t H_{\mathbb{R}}(\Psi(s, X_0), \{0\})ds \\
&\leqslant 2M_{N_0} t \\
&\leqslant \mathrm{diam} X_0 \\
&\leqslant \mathrm{diam} X_0 + \mathrm{diam}\left((S)\int_0^t \Phi(s, X_0)ds\right).
\end{aligned}$$

In the hyperspace $\mathcal{K}_c^b(\mathbb{R})$ we have: if $\mathrm{diam} A \geqslant \mathrm{diam} B$ then $A \ominus B$ exists, $A, B \in \mathcal{K}_c^b(\mathbb{R})$. Hence the Hukuhara difference $\Big(X_0 + (S)\int_0^t \Phi(s, X_0)ds\Big) \ominus (S)\int_0^t \Psi(s, X_0)ds$ exists for each $t \in [0, \mathrm{diam} X_0/(2M_{N_0})]$ which means that

$X_1(t)$ is well defined for $t \in [0, \mathrm{diam} X_0/(2M_{N_0})]$. Now it can be verified that there exist a positive constant a_1 such that $\sup_t H_{\mathbb{R}}(X_1(t), X_0) \leqslant a_1$.

Observe also that this reasoning can be repeated starting from the estimate $H_{\mathbb{R}}(X_0, \{0\}) \leqslant \tilde{N}_0$, where $\tilde{N}_0 = N_0 + \Big[a_1 \sum_{n=1}^{\infty} \Big(\frac{(aT)^{n-1}}{(n-1)!}\Big)^{1/2}\Big] + 1$ with $[x]$ denoting the greatest integer not greater than x and $a = \int_I (K_\Phi(s) + K_\Psi(s))^2 ds$. Thus $X_1(t)$ is well defined for every $t \in J = [0, \tilde{T}]$, where $\tilde{T} = \mathrm{diam} X_0/2M_{\tilde{N}_0}$.

Further one can observe that

$$H_{\mathbb{R}}(X_1(t), \{0\}) \leqslant H_{\mathbb{R}}(X_1(t), X_0) + H_{\mathbb{R}}(X_0, \{0\}) \leqslant a_1 + N_0 \leqslant \tilde{N}_0$$

for every $t \in J$. Hence, by (4.2), $H_{\mathbb{R}}(\Psi(t, X_1(t)), \{0\}) \leqslant M_{\tilde{N}_0}$ for every $t \in J$. Thus for $t \in J$

$$\begin{aligned} \mathrm{diam}\left((S)\int_0^t \Psi(s, X_1(s))ds\right) &\leqslant 2\int_0^t H_{\mathbb{R}}(\Psi(s, X_1(s)), \{0\})ds \\ &\leqslant 2M_{\tilde{N}_0} t \\ &\leqslant \mathrm{diam} X_0 \\ &\leqslant \mathrm{diam} X_0 + \mathrm{diam}\left((S)\int_0^t \Phi(s, X_1(s))ds\right). \end{aligned}$$

Hence $X_2(t) = \Big(X_0 + (S)\int_0^t \Phi(s, X_1(s))ds\Big) \ominus (S)\int_0^t \Psi(s, X_1(s))ds$ is well defined for $t \in J$, and

$$\begin{aligned} H_{\mathbb{R}}(X_2(t), \{0\}) &\leqslant H_{\mathbb{R}}(X_2(t), X_1(t)) + H_{\mathbb{R}}(X_1(t), X_0) + H_{\mathbb{R}}(X_0, \{0\}) \\ &\leqslant a_1(a\tilde{T})^{1/2} + a_1 + N_0 \leqslant \tilde{N}_0. \end{aligned}$$

Repeating this procedure we obtain that for every $n \in \mathbb{N}$

$$X_n(t) = \Big(X_0 + (S)\int_0^t \Phi(s, X_{n-1}(s))ds\Big) \ominus (S)\int_0^t \Psi(s, X_{n-1}(s))ds \quad \text{is well defined for} \quad t \in J,$$

and $\sup_t H_{\mathbb{R}}(X_n(t), \{0\}) \leqslant \tilde{N}_0$. Hence, in the study of the deterministic set-valued equation (4.1) we can use the more convenient condition (S4) which does not involve any Hukuhara difference. Now we are able to state some results for solutions to equation (4.1) using condition (S4). The proofs are similar to those established for solutions to equation (3.1), so we omit them. We begin with an existence and uniqueness theorem.

Proposition 4.1. *Let $X_0 \in \mathcal{K}_c^b(\mathbb{R})$, and $\Psi, \Phi \colon I \times \mathcal{K}_c^b(\mathbb{R}) \to \mathcal{K}_c^b(\mathbb{R})$ satisfy conditions (S1)-(S4). Then equation* (4.1) *has a unique (possibly local) solution.*

Next we state an estimate on the error between the nth approximation X_n and the exact solution X to equation (4.1).

Proposition 4.2. *Let the assumptions of Proposition 4.1 be satisfied. Then for every $n \in \mathbb{N}$ we have*

$$\sup_{t \in J} H_{L^2}(X_n(t), X(t)) \leqslant \left(k_1 \frac{(k_2\tilde{T})^n}{n!}\right)^{1/2} \exp\{k_3\tilde{T}\},$$

where k_1, k_2, k_3 are some positive constants.

The boundedness of the solution can also be obtained.

Proposition 4.3. *Under the assumptions of Proposition 4.1 for the local solution X to* (4.1) *we have*

$$\sup_{t \in J} H_{L^2}(X(t), \{0\}) \leqslant k_4,$$

where k_4 is a positive constant.

To consider the sensitivity of solutions to small changes of initial value, we study equation (4.1) and the same equation with another initial value $\tilde{X}_0$

$$X(t) + (S)\int_0^t \Psi(s, X(s))ds \quad = \quad \tilde{X}_0 + (S)\int_0^t \Phi(s, X(s))ds \ \text{ for } \ t \in I. \tag{4.3}$$

Let $X\colon J_1 \to \mathcal{K}_c^b(\mathbb{R})$ and $Y\colon J_2 \to \mathcal{K}_c^b(\mathbb{R})$ denote the unique solutions (if they exist) to these equations, respectively, $J_1 = [0, \tilde{T}_1]$, $J_2 = [0, \tilde{T}_2]$ for some $\tilde{T}_1, \tilde{T}_2 \in (0, T]$. Let $J = J_1 \cap J_2$.

Proposition 4.4. *Let $X_0, \tilde{X}_0 \in \mathcal{K}_c^b(\mathbb{R})$. Assume that Ψ, Φ satisfy the conditions (S1)-(S4). Then*

$$\sup_{t\in J} H_{\mathbb{R}}(X(t), Y(t)) \leqslant k_5 H_{\mathbb{R}}(X_0, \tilde{X}_0)\exp\{k_6 \min\{T_1, T_2\}\},$$

where k_5, k_6 are some positive constants.

To consider stability of solutions to (4.1) with respect to small changes of coefficients Ψ and Φ, we consider equation (4.1) and equations (for $n \in \mathbb{N}$)

$$X(t) + (S)\int_0^t \Psi_n(s, X(s))ds \quad = \quad X_0 + (S)\int_0^t \Phi_n(s, X(s))ds \ \text{ for } \ t \in I \tag{4.4}$$

with other coefficients Ψ_n and Φ_n than Ψ and Φ in (4.1). Let X, Y_n denote the unique solutions (if they exist) to these equations, respectively. Assume that they all are defined on a common interval $J = [0, \tilde{T}]$ with $\tilde{T} \in (0, T]$.

Proposition 4.5. *Let $X_0 \in \mathcal{K}_c^b(\mathbb{R})$. Suppose that Ψ, Φ satisfy the conditions (S1)-(S4). Assume also that Ψ_n, Φ_n satisfy (S1)-(S4), in particular the conditions (S2) and (S3) are satisfied with the functions K_{Ψ_n}, K_{Φ_n} and C_{Ψ_n}, C_{Φ_n}, respectively. Assume that there exist constants $S_\Psi, S_\Phi > 0$ such that for every $n \in \mathbb{N}$*

$$\int_I K^2_{\Psi_n}(s)ds \leqslant S_\Psi, \quad \int_I K^2_{\Phi_n}(s)ds \leqslant S_\Phi.$$

Suppose that for every $A \in \mathcal{K}_c^b(\mathbb{R})$

$$\int_I H_{\mathbb{R}^d}(\Psi_n(s, A), \Psi(s, A))ds \to 0 \text{ as } n \to \infty,$$
$$\int_I H_{\mathbb{R}^d}(\Phi_n(s, A), \Phi(s, A))ds \to 0 \text{ as } n \to \infty.$$

Then for the solution X to (4.1) *and the solutions X_n to* (4.4) *we have*

$$\sup_{t\in J} H_{\mathbb{R}}(Y_n(t), X(t)) \to 0 \text{ as } n \to \infty.$$

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