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Best Proximity Points for p–Cyclic Summing Iterated Contractions

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Abstract. We generalize the p – summing contractions maps. We found sufficient conditions for these new type of maps, that ensure the existence and uniqueness of best proximity points in uniformly convex Banach spaces. We apply the result for Kannan and Chatterjea type cyclic contractions and we obtain sufficient conditions for these maps, that ensure the existence and uniqueness of best proximity points in uniformly convex Banach spaces.

1. Introduction

A fundamental result in fixed point theory is the Banach Contraction Principle. One kind of a generalization of the Banach Contraction Principle is the notion of cyclic maps [17]. Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. Interesting application of cyclic maps to integro-differential equations is presented in [20]. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notion of a best proximity point is introduced in [11]. This definition is more general than the notion of cyclic maps [17], in sense that if the sets intersect then every best proximity point is a fixed point. A sufficient condition for the uniqueness of the best proximity points in uniformly convex Banach spaces is given in [11] for 2 sets and in [15] for p sets.

The notion of p–summing maps was introduced in [22] and sufficient conditions are found so that these maps to have fixed points and best proximity points. The p–summing maps are wider class of maps than the classical contraction maps and cyclic contraction maps [22]. A disadvantage of the classical results about best proximity points is that the conditions are so restrictive that the distances between the successive sets are equal. The p–summing maps overcome this disadvantage [22]. A cyclic orbital Meir–Keeler contraction was introduced in [14] and sufficient conditions are found for the existence of fixed points and best proximity points for these type of maps. The results in [14] were generalized in [25] for p–summing cyclic orbital Meir–Keeler contractions and later on in [16] another class of cyclic orbital Meir–Keeler contraction.

We generalize the notion of p – cyclic summing contraction maps in the sense of iterated contraction introduced in [23]. We show that a large class of cyclic maps are p – cyclic summing iterated contractions. As a consequence of the main result we obtain conditions sufficient that ensure the existence and uniqueness

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of best proximity points in uniformly convex Banach spaces for p – cyclic summing contraction, p – cyclic summing Kannan contractions and p – cyclic summing Chatterjea contractions.

2. Preliminary Results

In this section we give some basic definitions and concepts which are useful and related to the best proximity points. Let (X, ρ) be a metric space. Define a distance between two subset $A, B \subset X$ by $\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$.

Let $\{A_i\}_{i=1}^p$ be nonempty subsets of a metric space (X, ρ). We use the convention $A_{p+i} = A_i$ for every $i \in \mathbb{N}$. The map $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is called a cyclic map if $T(A_i) \subseteq T(A_{i+1})$ for every $i = 1, 2, \ldots p$. A point $\xi \in A_i$ is called a best proximity point of the cyclic map T in A_i if $\rho(\xi, T\xi) = \operatorname{dist}(A_i, A_{i+1})$.

When we investigate a Banach space $(X, \|\cdot\|)$ we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$.

Definition 2.1. *([9], p. 61) The norm $\|\cdot\|$ on a Banach space X is said to be uniformly convex if* $\lim_{n\to\infty} \|x_n - y_n\| = 0$ *whenever* $\|x_n\| = \|y_n\| = 1$, $n \in \mathbb{N}$ *are such that* $\lim_{n\to\infty} \|x_n + y_n\| = 2$.

We will use the following two lemmas, proved in [11].

Lemma 2.2. *([11]) Let A be a nonempty closed, convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in A and $\{y_n\}_{n=1}^\infty$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$;
2) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$, $\|x_m - y_n\| \leq \operatorname{dist}(A, B) + \varepsilon$,
then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, $\|x_m - z_n\| \leq \varepsilon$.

Lemma 2.3. *([11]) Let A be a nonempty closed, convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in A and $\{y_n\}_{n=1}^\infty$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A, B)$;
2) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$;
then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Definition 2.4. *([9], p. 42) We say that the Banach space $(X, \|\cdot\|)$ is strictly convex if $x = y$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$.*

Let us mention the well known fact, that any uniformly convex Banach space is strictly convex ([9], p.61).

Lemma 2.5. *([25]) Let A, B be closed subsets of a strictly convex Banach space $(X, \|\cdot\|)$, such that* $\operatorname{dist}(A, B) > 0$ *and let A be convex. If $x, z \in A$ and $y \in B$ be such that $\|x - y\| = \|z - y\| = \operatorname{dist}(A, B)$, then $x = z$.*

3. Main Results

Let $\{A_i\}_{i=1}^p$ be non empty subsets of the metric space (X, ρ) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$. We will use the notions $P = \sum_{i=1}^p \operatorname{dist}(A_i, A_{i+1})$ and

$$s_p(x_1, x_2, \ldots, x_p) = \sum_{j=1}^{p-1} \rho(x_j, x_{j+1}) + \rho(x_p, x_1), \tag{1}$$

where if $x_1 \in A_i$, then $x_{1+k} \in A_{i+k}$ for every $k = 1, 2, \ldots, p-1$. Just for simplicity of the notations we will denote

$$s_{p,n}(x) = s_p\left(T^n x, T^{n+1} x, T^{n+2} x, \ldots, T^{n+p-1} x\right)$$

for any $x \in \bigcup_{i=1}^p A_i$.

Definition 3.1. Let A_i, $i = 1, 2, \dots, p$ be subsets of a metric space (X, ρ). A map $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ will be called a **p – cyclic summing iterated contraction** if it satisfies the following conditions:
1) T is a cyclic map;
2) there exists $k \in (0, 1)$, such that for any $x \in \bigcup_{i=1}^{p} A_i$

$$s_{p,1}(x) \le k s_{p,0}(x) + (1-k)P. \tag{2}$$

We use in the sequel an equivalent form of (2)

$$s_{p,1}(x) - P \le k(s_{p,0}(x) - P). \tag{3}$$

Definition 3.2. *Let A_i, $i = 1, 2, \dots, p$ be nonempty subsets of a metric space (X, ρ), T be a cyclic map. We say that T satisfies the proximal property if whenever hold $\lim_{n\to\infty} x_n = x \in \bigcup_{i=1}^{p} A_i$ and $\lim_{n\to\infty} \rho(x_n, Tx_n) = \mathrm{dist}(A_i, A_{i+1})$ it follows that $\rho(x, Tx) = \mathrm{dist}(A_i, A_{i+1})$.*

The best proximity results need norm structure of the space. When we investigate a Banach space we will always consider the distance between the elements to be generated by the norm.

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $A_i \subset X$, $i = 1, 2, \dots, p$ be closed, convex sets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction.*

Then for every $x \in A_1$ the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n\to\infty} T^{pn}x$ and T is continuous at z or T satisfies the proximal property, then $z \in A_1$ is a best proximity point of T in A_1, $T^i z \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 1, 2, \dots, p-1$ and $T^p z = z$.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and $A_i \subset X$, $i = 1, 2, \dots, p$ are weakly closed sets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. If T is weakly continuous on $\cup_{i=1}^{p} A_i$ or T satisfies the proximal property then for any $k = 1, 2, \dots p$ there exists $\xi_k \in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s \xi_k \in A_{k+s}$ is a best proximity point of T in A_{k+s}.*

4. Auxiliary Results

Lemma 4.1. *Let (X, ρ) be a metric space, $A_i \subset X$, $i = 1, 2, \dots, p$ be subsets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. Then $s_{p,n}(x) - P \le k^n (s_{p,0}(x) - P)$.*

Proof. By applying n–times (3) we get the inequality

$$s_{p,n}(x) - P \le k(s_{p,n-1}(x) - P) \le \cdots \le k^n (s_{p,0}(x) - P).$$

□

Lemma 4.2. *Let (X, ρ) be a metric space, $A_i \subset X$, $i = 1, 2, \dots p$ be subsets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. Then $\lim_{n\to\infty} s_{p,n}(x) = P$.*

Proof. By Lemma 4.1 we have the inequality

$$0 \le \lim_{n\to\infty} (s_{p,n}(x) - P) \le \lim_{n\to\infty} k^n (s_{p,0}(x) - P) = 0.$$

Hence we get $\lim_{n\to\infty} s_{p,n}(x) = P$. □

Lemma 4.3. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A_i \subset X$, $i = 1, 2, \dots p$ be closed, convex sets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. Then $\lim_{n\to\infty} \|T^{pn+k}x - T^{pn+p+k}x\| = 0$, $k = 0, 1, 2, \dots, p-1$.*

Proof. By Lemma 4.2 we have that $\lim_{n\to\infty} s_{p,pn}(x) = P$ and $\lim_{n\to\infty} s_{p,pn+1}(x) = P$. Consequently we get that $\lim_{n\to\infty} \|T^{pn}x - T^{pn+1}x\| = \operatorname{dist}(A_1, A_2)$ and $\lim_{n\to\infty} \|T^{pn+1}x - T^{pn+p}x\| = \operatorname{dist}(A_1, A_2)$. According to Lemma 2.3 it follows that $\lim_{n\to\infty} \|T^{pn}x - T^{pn+p}x\| = 0$.

The proof for $k = 1, 2, \dots, p-1$ is similar. □

The next Lemma is a generalization of a Lemma from [12].

Lemma 4.4. *Let (X, ρ) be a metric space, $A_i \subset X$, $i = 1, 2, \dots p$ be subsets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. Then*

$$Q_n = s_p(x, T^{pn+1}x, T^{pn+2}x, \dots, T^{pn+p-1}) - P \le 2\frac{1-k^{pn}}{1-k^p}\rho(x, T^p x) + k^{pn}(s_{p,0}(x) - P). \tag{4}$$

Proof. We will prove Lemma 4.4 by induction.

I) Let $n = 1$. Form Lemma 4.1 it follows that

$$\begin{aligned} Q_1 &= s_p(x, T^{p+1}x, T^{p+2}x, \dots, T^{2p-1}) - P \\ &= \rho(x, T^{p+1}x) + \rho(T^{p+1}x, T^{p+2}x) + \dots + \rho(T^{2p-2}x, T^{2p-1}x) + \rho(T^{2p-1}x, x) - P \\ &= 2\rho(x, T^p x) + s_{p,p}(x) - P \le 2\rho(x, T^p x) + k^p(s_{p,0}(x) - P). \end{aligned}$$

and therefore (4) holds true for $n = 1$.

II) Let suppose that (4) holds true for $n = m$.

III) We will prove that (4) holds true for $n = m + 1$. Indeed

$$\begin{aligned} Q_{m+1} &= s_p(x, T^{p(m+1)+1}x, T^{p(m+1)+2}x, \dots, T^{p(m+1)+p-1}x) - P \\ &= \rho(x, T^{p(m+1)+1}x) + \rho(T^{p(m+1)+1}x, T^{p(m+1)+2}x) + \cdots \\ &\quad + \rho(T^{p(m+1)+p-2}x, T^{p(m+1)+p-1}x) + \rho(T^{p(m+1)+p-1}x, x) - P \\ &\le 2\rho(x, T^p x) + s_p(T^p x, T^{p(m+1)+1}x, T^{p(m+1)+2}x, \dots, T^{p(m+1)+p-1}x) - P \\ &\le 2\rho(x, T^p x) + k^p(s_p(x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) - P) \\ &\le 2\rho(x, T^p x)\left(1 + k^p\frac{1-k^{pm}}{1-k^p}\right) + k^{p(m+1)}(s_{p,0}(x) - P) \\ &= 2\frac{1-k^{p(m+1)}}{1-k^p}\rho(x, T^p x) + k^{p(m+1)}(s_{p,0}(x) - P). \end{aligned}$$

□

Corollary 4.5. *Let (X, ρ) be a metric space, $A_i \subset X$, $i = 1, 2, \dots p$ be subsets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. Then*

$$\rho(x, T^{pn+1}x) - \operatorname{dist}(A_1, A_2) \le 2\frac{1-k^{pn}}{1-k^p}\rho(x, T^p x) + k^{pn}(s_{p,0}(x) - P). \tag{5}$$

Proof. From Lemma 4.4 we get

$$\begin{aligned} \rho(x, T^{pn+1}x) - \operatorname{dist}(A_1, A_2) &\le s_p(x, T^{pn+1}x, T^{pn+2}x, \dots, T^{pn+p-1}) - P \\ &\le 2\frac{1-k^{pn}}{1-k^p}\rho(x, T^p x) + k^{pn}(s_{p,0}(x) - P). \end{aligned}$$

□

Lemma 4.6. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A_i \subset X$, $i = 1, 2, \dots, p$ be closed, convex sets and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p – cyclic summing iterated contraction. Then*

a) For every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for every $m \ge n \ge N_0$, $\|T^{pn}x - T^{pm+1}x\| < \operatorname{dist}(A_1, A_2) + \varepsilon$.

b) For every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$, such that for every $m > n \ge M_0$, $\|T^{pm}x - T^{pn+1}x\| < \operatorname{dist}(A_1, A_2) + \varepsilon$.

Proof. a) Put $v = T^{pn}x$. Then $T^{pm+1}x = T^{p(m-n)+1}v$. From Corollary 4.5 we have that

$$\begin{aligned} Q_{n,m} &= \|T^{pn}x - T^{pm+1}x\| - \operatorname{dist}(A_1, A_2) = \|v - T^{p(m-n)+1}v\| - \operatorname{dist}(A_1, A_2) \\ &\leq 2\frac{1-k^{p(m-n)}}{1-k^p}\|v - T^p v\| + k^{p(m-n)}(s_{p,0}(v) - P) \\ &\leq \frac{2}{1-k^p}\|v - T^p v\| + k^{p(m-n)}(s_{p,0}(v) - P) \\ &= \frac{2}{1-k^p}\|T^{pn}x - T^{pn+p}x\| + k^{p(m-n)}(s_{p,pn}(x) - P). \end{aligned}$$

From Lemma 4.3 and Lemma 4.2 it follows that there exists $N_0 \in \mathbb{N}$, such that for every $n \geq N_0$ hold $\|T^{pn}x - T^{pn+p}x\| < \frac{(1-k^p)\varepsilon}{4}$ and $s_{p,pn}(x) - P < \frac{\varepsilon}{2}$. Consequently for every $m \geq n \geq N_0$, $\|T^{pn}x - T^{pm+1}x\| - \operatorname{dist}(A_1, A_2) < \varepsilon$.

b) Put $v = T^{pn+1}x$. Then $T^{pm}x = T^{p(m-n)-1}v$. From Corollary 4.5 we have that

$$\begin{aligned} Q_{m,n} &= \|T^{pn+1}x - T^{pm}x\| - \operatorname{dist}(A_1, A_2) = \|v - T^{p(m-n)-1}v\| - \operatorname{dist}(A_1, A_2) \\ &\leq 2\frac{1-k^{p(m-n-1)}}{1-k^p}\|v - T^p v\| + k^{p(m-n-1)}(s_{p,0}(v) - P) \\ &\leq \frac{2}{1-k^p}\|v - T^p v\| + k^{p(m-n-1)}(s_{p,0}(v) - P) \\ &= \frac{2}{1-k^p}\|T^{pn+1}x - T^{pn+p+1}x\| + k^{p(m-n-1)}(s_{p,pn+1}(x) - P). \end{aligned}$$

From Lemma 4.3 and Lemma 4.2 it follows that there exists $M_0 \in \mathbb{N}$, such that for any $n \geq M_0$, $s_{p,pn+1}(x) - P < \frac{\varepsilon}{2}$ and $\|T^{pn+1}x - T^{pn+p+1}x\| < \frac{(1-k^p)\varepsilon}{4}$. Consequently for every $m \geq n \geq M_0$, $\|T^{pn+1}x - T^{pm}x\| - \operatorname{dist}(A_1, A_2) < \varepsilon$. □

Lemma 4.7. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A_i \subset X$ be closed, convex sets and $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p – cyclic summing iterated contraction. Then for every $x \in A_1$ the sequences $\{T^{pn}x\}_{n=1}^\infty$ and $\{T^{pn+1}x\}_{n=1}^\infty$ are Cauchy sequences.*

Proof. By Lemma 4.2 we have that $\lim_{n\to\infty} \|T^{pn}x - T^{pn+1}x\| = \operatorname{dist}(A_1, A_2)$. From Lemma 4.6 b) we have that for every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$, so that for every $m \geq n \geq M_0$,

$$\|T^{pm}x - T^{pn+1}x\| < \operatorname{dist}(A_1, A_2) + \varepsilon.$$

According to Lemma 2.2 there exists $N_1 \in \mathbb{N}$, such that $\|T^{pm}x - T^{pn}x\| < \varepsilon$ holds for every $m > n \geq N_1$.

The proof that the sequence $\{T^{pn+1}x\}_{n=1}^\infty$ is a Cauchy sequence is similar. □

Lemma 4.8. *Let (X, ρ) be a metric space, $A_i \subset X$ be subsets and $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p – cyclic summing iterated contraction. Then for every $x \in A_1$ the sequences $\{T^{pn+k}x\}_{n=1}^\infty$, $k = 1, 2, \ldots, p$ are bounded sequences.*

Proof. From Corollary 4.5 we have the inequality

$$\rho(x, T^{pn+1}x) \leq \frac{2}{1-k^p}\rho(x, T^p x) + (s_{p,0}(x) - P) + \operatorname{dist}(A_1, A_2)$$

and consequently the sequence $\{T^{pn+1}x\}_{n=1}^\infty$ is bounded. By Lemma 4.2 we get $\lim_{n\to\infty} \rho(T^{pn+1}x, T^{pn+2}x) = \operatorname{dist}(A_2, A_3)$ and thus from the fact that the sequence $\{T^{pn+1}x\}_{n=1}^\infty$ is bounded it follows that the sequence $\{T^{pn+2}x\}_{n=1}^\infty$ is bounded.

The proof that the sequences $\{T^{pn+k}x\}_{n=1}^\infty$, $k = 3, 4, \ldots, p$ are bounded sequences can be done in a similar fashion. □

Lemma 4.9. *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $A_i \subset X$ be nonempty weakly closed subsets and $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p – cyclic summing iterated contraction. Then there exist $\xi_i \in A_i$ such that $\|\xi_i - \xi_{i+1}\| = \operatorname{dist}(A_i, A_{i+1})$.*

Proof. Let $\text{dist}(A_1, A_2) > 0$. For an arbitrary $x \in A_1$ we define the sequence $x_n = T^{n-1}x$. By Lemma 4.8 the sequences $\{x_{pn+k}\}_{n=1}^{\infty}$, $k = 1, 2, \dots, p$ are bounded sequences. From the assumption that the sets A_i are weakly closed it follows that we can choose a subsequence of naturals $\{n_j\}_{j=1}^{\infty}$, such that the sequences $\{x_{pn_j+k}\}_{j=1}^{\infty}$ are weakly convergent for every $k = 1, 2, \dots, p$. Let us denote $w\lim_{j\to\infty} x_{pn_j+k} = \xi_k$. Then $w\lim_{j\to\infty} x_{pn_j+k} - x_{pn_j+k+1} = \xi_k - \xi_{k+1} \neq 0$. There exist bounded linear functionals $f_k \in S_X$, such that $f_k(\xi_k - \xi_{k+1}) = \|\xi_k - \xi_{k+1}\|$. From the inequalities

$$\left|f_k(\xi_k - \xi_{k+1})\right| \leq \|f_k\| \, \|\xi_k - \xi_{k+1}\| = \|\xi_k - \xi_{k+1}\|$$

and $\lim_{j\to\infty} f_k(x_{pn_j+k} - x_{pn_j+k+1}) = f_k(\xi_k - \xi_{k+1}) = \|\xi_k - \xi_{k+1}\|$ we obtain the inequality

$$\begin{array}{rcl} \|\xi_k - \xi_{k+1}\| & = & \lim_{j\to\infty} \left|f_k(x_{pn_j+k} - x_{pn_j+k+1})\right| \leq \lim_{j\to\infty} \|x_{pn_j+k} - x_{pn_j+k+1}\| \\ & = & \text{dist}(A_k, A_{k+1}) \end{array}$$

and therefore $\|\xi_k - \xi_{k+1}\| = \text{dist}(A_k, A_{k+1})$. □

5. Proof of Main Result

Proof of Theorem 3.3. Let $x \in A_1$ be arbitrary chosen. From Lemma 4.7 it follows that $\{T^{pn}\}_{n=1}^{\infty}$ is a Cauchy sequence and therefore it is convergent. Let us denote $z = \lim_{n\to\infty} T^{pn}x$.

We will consider the two cases I) the map T is continuous at z and II) the map T satisfies the proximal property separately.

I) By the continuity of the norm and the assumption that T is continuous at z we can write the equality

$$s_{p,0}(z) - P = \lim_{n\to\infty} \left(s_{p,0}(T^{pn}x) - P\right) = 0. \tag{6}$$

Thus $\|z - Tz\| - \text{dist}(A_1, A_2) \leq s_{p,0}(z) - P = 0$ and consequently $z \in A_1$ is a best proximity point of T in A_1. From (6) we get that $\|T^iz - T^{i+1}z\| - \text{dist}(A_{i+1}, A_{i+2}) \leq s_{p,0}(z) - P = 0$ for $i = 1, 2, \dots, p-2$ and therefore $T^iz \in A_{i+1}$ is a best proximity point of T in A_{i+1}.

II) If T satisfies the proximal property then from the equality

$$\lim_{n\to\infty} \|x_{pn} - Tx_{pn}\| = \lim_{n\to\infty} \|T^{pn}x - T^{pn+1}x\| = \text{dist}(A_1, A_2)$$

it follows the equality

$$\|z - Tz\| = \text{dist}(A_1, A_2) \tag{7}$$

and consequently $z \in A_1$ is a best proximity point of T in A_1. By the continuity of the norm we get

$$\lim_{n\to\infty} \|z - x_{pn+1}\| = \lim_{n\to\infty} \|T^{pn}x - T^{pn+1}x\| = \text{dist}(A_1, A_2). \tag{8}$$

From (7), (8) and the uniform convexity we get

$$Tz = \lim_{n\to\infty} x_{pn+1} = \lim_{n\to\infty} T^{pn+1}x.$$

As T satisfies the proximal property then from the equality

$$\lim_{n\to\infty} \|x_{pn+1} - Tx_{pn+1}\| = \lim_{n\to\infty} \|T^{pn+1}x - T^{pn+2}x\| = \text{dist}(A_2, A_3). \tag{9}$$

it follows that $\|Tz - T^2z\| = \text{dist}(A_2, A_3)$ and consequently $Tz \in A_1$ is a best proximity point of T in A_2.

Proceeding in a similar fashion we get that $T^iz \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 2, 3, \dots, p-1$.

To finish the proof, let z be arbitrary best proximity point of T in A_1, which is obtained as a limit of a sequence $\{T^{pn}x\}_{n=1}^{\infty}$. From the inequality

$$\|T^p z - Tz\| - \operatorname{dist}(A_1, A_2) \le s_{p,1}(z) - P \le k\left(s_{p,0}(z) - P\right) = 0$$

we obtain that $\|T^p z - Tz\| = \operatorname{dist}(A_1, A_2)$. By Lemma 2.5 we get that $T^p z = z$.

Proof of Theorem 3.4. Without loss of generality we can assume that $\operatorname{dist}(A_1, A_2) > 0$. If not by $P > 0$ there exists i, such that $\operatorname{dist}(A_i, A_{i+1}) > 0$ and we can enumerate the sets so that $\operatorname{dist}(A_1, A_2) > 0$.

Let $\operatorname{dist}(A_1, A_2) > 0$. For an arbitrary $x_1 \in A_1$ we define the sequence $x_n = T^{n-1}x_1$. By Lemma 4.8 the sequences $\{x_{pn+k}\}_{n=1}^{\infty}$, $k = 0, 1, 2, \dots, p-1$ are bounded sequences. From the assumption that the all the sets A_k, $k = 1, 2, \dots, p$ are weakly closed it follows that we can choose a subsequence of naturals $\{n_j\}_{j=1}^{\infty}$, such that the sequences $\{x_{pn_j+k}\}_{j=1}^{\infty}$, $k = 0, 1, \dots, p-1$ are weakly convergent for every $k = 0, 1, \dots, p-1$. Let us denote $w\lim_{j\to\infty} x_{pn_j+k} = \xi_{k+1}$.

1) Let T be weakly continuous on $\bigcup_{k=1}^{p} A_k$. Then

$$w\lim_{j\to\infty} x_{pn_j+k} = w\lim_{j\to\infty} Tx_{pn_j+k-1} = T\xi_{k+1}.$$

Consequently $w\lim_{j\to\infty}(x_{pn_j+k-1} - x_{pn_j+k}) = \xi_k - T\xi_k$. There exists a bounded linear functional $f_k \in S_{X^*}$, such that $f_k(\xi_k - T\xi_k) = \|\xi_k - T\xi_k\|$. From the inequalities

$$\left|f_k(x_{pn_j+k-1} - Tx_{pn_j+k-1})\right| \le \|f_k\|.\|x_{pn_j+k-1} - Tx_{pn_j+k-1}\| = \|x_{pn_j+k-1} - Tx_{pn_j+k-1}\|$$

and $\lim_{j\to\infty} f_k(x_{pn_j+k-1} - Tx_{pn_j+k-1}) = f_k(\xi_k - T\xi_k) = \|\xi_k - T\xi_k\|$ we obtain the inequality

$$\begin{array}{rcl} \|\xi_k - T\xi_k\| & = & \lim_{j\to\infty}\left|f_k(x_{pn_j+k-1} - Tx_{pn_j+k-1})\right| \\ & \le & \lim_{j\to\infty}\|x_{pn_j+k-1} - Tx_{pn_j+k-1}\| = \operatorname{dist}(A_k, A_{k+1}) \end{array}$$

and therefore $\|\xi_k - T\xi_k\| = \operatorname{dist}(A_k, A_{k+1})$. Thus ξ_k is a best proximity point of T in A_k.

We will prove that $T\xi_k$ is a best proximity point of T in A_{k+1}. There exists bounded linear functional $g_k \in S_{X^*}$, such that $g_k(T\xi_k - T^2\xi_k) = \|T\xi_k - T^2\xi_k\|$. From the inequalities

$$\begin{array}{rcl} \left|g_k(Tx_{pn_j+k-1} - T^2x_{pn_j+k-1})\right| & \le & \|g_k\|\,\|Tx_{pn_j+k-1} - T^2x_{pn_j+k-1}\| \\ & = & \|Tx_{pn_j+k-1} - T^2x_{pn_j+k-1}\| \end{array}$$

and $\lim_{j\to\infty} g_k(Tx_{pn_j+k-1} - T^2x_{pn_j+k-1}) = g_k(T\xi_k - T^2\xi_k) = \|T\xi_k - T^2\xi_k)\|$ we obtain the inequality

$$\begin{array}{rcl} \|T\xi_k - T^2\xi_k\| & = & \lim_{j\to\infty}\left|g_k(Tx_{pn_j+k-1} - T^2x_{pn_j+k-1})\right| \\ & \le & \lim_{j\to\infty}\|Tx_{pn_j+k-1} - T^2x_{pn_j+k-1}\| = \operatorname{dist}(A_{k+1}, A_{k+2}) \end{array}$$

and therefore $\|T\xi_k - T^2\xi_k\| = \operatorname{dist}(A_k, A_{k+1})$, i.e. $T\xi_k$ is a best proximity point of T in A_{k+1}.

It can be proved in a similar fashion that $T^s(\xi_{k+s})$ is a best proximity point of T in A_{k+s}.

2) Let T satisfy the proximal property. From Lemma 4.4 we get

$$\lim_{j\to\infty}\|x_{pn_j+k-1} - T(x_{pn_j+k-1})\| = \lim_{j\to\infty}\|x_{pn_j+k-1} - x_{pn_j+k}\| = \operatorname{dist}(A_k, A_{k+1}).$$

By the assumption that T satisfies the proximal property it follows that

$$\|\xi_k - T\xi_k\| = \operatorname{dist}(A_k, A_{k+1})$$

and thus ξ_k is a best proximity point of T in A_k.

We will prove that $T\xi_k$ is a best proximity point of T in A_{k+1}. From Lemma 4.4 we get

$$\lim_{j\to\infty}\|Tx_{pn_j+k-1} - T^2(x_{pn_j+k-1})\| = \lim_{j\to\infty}\|x_{pn_j+k} - x_{pn_j+k+1}\| = \operatorname{dist}(A_{k+1}, A_{k+2}).$$

By the assumption that T satisfies the proximal property it follows that

$$\|\xi_{k+1} - T\xi_{k+1}\| = \operatorname{dist}(A_{k+1}, A_{k+2})$$

It can be proved in a similar fashion that $T^s(\xi_{k+s})$ is a best proximity point of T in A_{k+s}.

6. Applications of Theorem 3.3

We would like to point out that the maps, which were investigated in [15, 21, 22] satisfy the proximal property. Thus the original result in [15, 21, 22] are more general. Let us point out that the conditions of Theorem 3.3 are not sufficient for uniqueness of the best proximity point, therefore the results in [15, 21, 22] are stronger.

Theorem 6.1. *(p–summing cyclic contraction [22]) Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and* $A_i \subset X$, $i = 1, 2, \ldots, p$ *be closed, convex sets,* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *be a cyclic map and there exists* $k \in (0, 1)$, *such that*

$$s_p(Tx_1, Tx_2, \ldots, Tx_p) \le k s_p(x_1, x_2, \ldots, x_p) + (1 - k)P \tag{10}$$

for every $x_i \in A_i$, $i = 1, 2, \ldots, p$.

Then for every $x \in A_1$ *the sequence* $\{T^{pn}x\}_{n=1}^{\infty}$ *is convergent. If* $z = \lim_{n\to\infty} T^{pn}x$ *and* T *is continuous at* z *or* T *satisfies the proximal property, then* $z \in A_1$ *is a best proximity point of* T *in* A_1, $T^i z \in A_{i+1}$ *is a best proximity point of* T *in* A_{i+1} *for* $i = 1, 2, \ldots, p - 1$ *and* $T^p z = z$.

Proof. If we put $x_1 = x \in A_1$, $x_i = T^{i-1}x$, $i = 2, 3, \ldots, p$ in (10) then T satisfies (2) and the proof follows from Theorem 3.3 □

Corollary 6.2. *(Cyclic type contraction, [15]) Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and* $A_i \subset X$, $i = 1, 2, \ldots, p$ *be closed, convex sets,* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *be a cyclic map and there exists* $k \in (0, 1)$, *such that*

$$\|Tx_i - Tx_{i+1}\| \le k\|x_i - x_{i+1}\| + (1 - k)\mathrm{dist}(A_i, A_{i+1}), \tag{11}$$

for every $x_i \in A_i$, $i = 1, 2, \ldots, p$.

Then for every $x \in A_1$ *the sequence* $\{T^{pn}x\}_{n=1}^{\infty}$ *is convergent. If* $z = \lim_{n\to\infty} T^{pn}x$ *and* T *is continuous at* z *or* T *satisfies the proximal property, then* $z \in A_1$ *is a best proximity point of* T *in* A_1, $T^i z \in A_{i+1}$ *is a best proximity point of* T *in* A_{i+1} *for* $i = 1, 2, \ldots, p - 1$ *and* $T^p z = z$.

Theorem 6.3. *(p – cyclic Kannan summing iterated contraction [21]) Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and* $A_i \subset X$, $i = 1, 2, \ldots, p$ *be closed, convex sets,* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *be a cyclic map and there exists* $k \in (0, 1/4)$, *such that for every* $x_i \in A_i$, $i = 1, 2, \ldots, p$

$$s_p(Tx_1, Tx_2, \ldots, Tx_p) \le 2k \sum_{i=1}^{p} \|Tx_i - x_i\| + (1 - 2k)P. \tag{12}$$

Then for every $x \in A_1$ *the sequence* $\{T^{pn}x\}_{n=1}^{\infty}$ *is convergent. If* $z = \lim_{n\to\infty} T^{pn}x$ *and* T *is continuous at* z *or* T *satisfies the proximal property, then* $z \in A_1$ *is a best proximity point of* T *in* A_1, $T^i z \in A_{i+1}$ *is a best proximity point of* T *in* A_{i+1} *for* $i = 1, 2, \ldots, p - 1$ *and* $T^p z = z$.

Proof. Let us choose an arbitrary $x \in A_1$ and let us put $x_i = T^{i-1}x$, $i \in \mathbb{N}$. From (12) we get the inequality

$$\begin{aligned} s_{p,1}(x) &= \|Tx - T^2x\| + \|T^2x - T^3x\| + \cdots \|T^{p-1}x - T^px\| + \|T^px - Tx\| \\ &\le 2k \sum_{k=0}^{p-1} \|T^kx - T^{k+1}x\| + (1 - 2k)P. \end{aligned} \tag{13}$$

From (13) and the inequality

$$P \le \|Tx - T^2x\| + \|T^2x - T^3x\| + \cdots + \|T^{p-1}x - x\| + \|T^px - Tx\|$$

we obtain the inequality

$$\begin{aligned} (1 - 2k)s_{p,1}(x) &\le 2k\,(\|Tx - x\| - \|T^px - Tx\|) + (1 - 2k)P \\ &\le 2k\,(\|Tx - x\| - \|T^px - Tx\| + P) + (1 - 4k)P \\ &\le 2k s_{p,0}(x) + (1 - 4k)P. \end{aligned} \tag{14}$$

From (14) we get

$$s_{p,1}(x) \le \frac{2k}{1-2k} s_{p,0}(x) + \frac{1-4k}{1-2k} P. \tag{15}$$

From $k \in (0,1/4)$ it follows that $\frac{2k}{1-2k} \in (0,1)$ and if we put $\alpha = \frac{2k}{1-2k}$ then (15) can be written as $s_{p,1}(x) \le \alpha s_{p,0}(x) + (1-\alpha)P$. Therefore T satisfies (2) and we can apply Theorem 3.3. □

We can weaken the assumption that $k \in (0,1/4)$ in the next result.

Theorem 6.4. *(p – cyclic Kannan summing iterated contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $A_i \subset X$, $i = 1,2,\dots,p$ be closed, convex sets, $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a cyclic map and there exists $k \in (0,1/2)$, such that for every $x_i \in A_i$*

$$s_p(Tx_1, Tx_2, \dots, Tx_p) \le 2k \sum_{i=1}^{p} \|Tx_i - x_i\| + (1-2k)P \tag{16}$$

and for every $x \in \cup_{i=1}^{p} A_i$

$$\|x - Tx\| + \|T^{p-1}x - T^p x\| \le \|T^{p-1}x - x\| + \|T^p x - Tx\|. \tag{17}$$

Then for every $x \in A_1$ the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n\to\infty} T^{pn}x$ and T is continuous at z or T satisfies the proximal property, then $z \in A_1$ is a best proximity point of T in A_1, $T^i z \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 1,2,\dots,p-1$ and $T^p z = z$.

Proof. Let us choose an arbitrary $x \in A_1$ and let us put $x_i = T^{i-1}x$, $i \in \mathbb{N}$. From (16) we get the inequality

$$s_{p,1}(x) \le 2k \sum_{k=0}^{p-1} \|T^k x - T^{k+1}x\| + (1-2k)P. \tag{18}$$

From (18) and (17) we obtain the inequality

$$\begin{aligned} (1-k)s_{p,1}(x) &\le k \sum_{i=0}^{p-2} \|T^i x - T^{i+1}x\| + k\left(\|x - Tx\| + \|T^{p-1}x - T^p x\| - \|T^p x - Tx\|\right) + (1-2k)P \\ &\le k\left(\sum_{i=0}^{p-2} \|T^i x - T^{i+1}x\| + \|T^{p-1}x - x\|\right) + (1-2k)P = k s_{p,0}(x) + (1-2k)P. \end{aligned} \tag{19}$$

From (19) we get

$$s_{p,1}(x) \le \frac{k}{1-k} s_{p,0}(x) + \frac{1-2k}{1-k} P. \tag{20}$$

From $k \in (0,1/2)$ it follows that $\frac{k}{1-k} \in (0,1)$ and if we put $\alpha = \frac{k}{1-k}$ then (20) can be written as $s_{p,1}(x) \le \alpha s_{p,0}(x) + (1-\alpha)P$. Therefore T satisfies (2) and we can apply Theorem 3.3. □

Corollary 6.5. *(Kannan type p–cyclic contraction [21]) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $A_i \subset X$, $i = 1,2,\dots,p$ be closed, convex sets, $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a cyclic map and there exists $k \in (0,1/4)$, such that for every $x_i \in A_i$, $i = 1,2,\dots,p$*

$$\|Tx_i - Tx_{i+1}\| \le k(\|Tx_i - x_i\| + \|Tx_{i+1} - x_{i+1}\|) + (1-2k)\mathrm{dist}(A_i, A_{i+1}). \tag{21}$$

Then for every $x \in A_1$ the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n\to\infty} T^{pn}x$ and T is continuous at z or T satisfies the proximal property, then $z \in A_1$ is a best proximity point of T in A_1, $T^i z \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 1,2,\dots,p-1$ and $T^p z = z$.

Theorem 6.6. *(3 – cyclic Chatterjea summing iterated contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $A_1, A_2, A_3 \subset X$ be closed, convex sets, $T : \cup_{i=1}^{3} A_i \to \cup_{i=1}^{3} A_i$ be a cyclic map, there exists $k \in (0, 1/2)$, such that for every $x_i \in A_i$, $i = 1, 2, 3$*

$$\begin{array}{rcl} s_3(Tx_1, Tx_2, Tx_3) & \leq & k(\|Tx_1 - x_2\| + \|Tx_1 - x_3\|) + k(\|Tx_2 - x_1\| + \|Tx_2 - x_3\|) \\ & & +k(\|Tx_3 - x_1\| + \|Tx_3 - x_2\|) + (1 - 2k)P. \end{array} \tag{22}$$

Then for every $x \in A_1$ the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n\to\infty} T^{pn}x$ and T is continuous at z or T satisfies the proximal property, then $z \in A_1$ is a best proximity point of T in A_1, $T^i z \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 1, 2, \ldots, p - 1$ and $T^p z = z$.

Proof. Let us choose an arbitrary $x \in A_1$ and let us put $x_2 = Tx$ and $x_3 = T^2x$. From (22) we get the inequality

$$\begin{array}{rcl} s_3(Tx, T^2x, T^3x) & = & \|Tx - T^2x\| + \|T^2x - T^3x\| + \|T^3x - Tx\| \\ & \leq & k(\|Tx - T^2x\| + \|T^2x - x\|) + \|T^3x - x\| + \|T^3x - Tx\|) + (1 - 2k)P \end{array} \tag{23}$$

From (23) we obtain the inequality

$$\begin{array}{rcl} Q_3 & = & (1-k)\Big(\|Tx - T^2x\| + \|T^2x - T^3x\| + \|T^3x - Tx\|\Big) \\ & \leq & k\Big(\|x - T^2x\| + \|T^3x - x\| - \|T^2x - T^3x\|\Big) + (1 - 2k)P \\ & \leq & k\Big(\|x - T^2x\| + \|x - Tx\| + \|Tx - T^2x\|\Big) + (1 - 2k)P. \end{array} \tag{24}$$

From (24) we get

$$s_3(Tx, T^2x, T^3x) \leq \frac{2k}{1-2k} s_3(x, Tx, T^2x) + \frac{1-4k}{1-2k} P. \tag{25}$$

From $k \in (0, 1/2)$ it follows that $\frac{k}{1-k} \in (0, 1)$ and if we put $\alpha = \frac{k}{1-k}$ then (25) can be written as $s_3(Tx, T^2x, T^3x) \leq \alpha s_3(x, Tx, T^2x) + (1 - \alpha)P$. Therefore T satisfies (2) and we can apply Theorem 3.3. □

Corollary 6.7. *(3 – cyclic Chatterjea contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $A_1, A_2, A_3 \subset X$ be closed, convex sets, $T : \cup_{i=1}^{3} A_i \to \cup_{i=1}^{3} A_i$ be a cyclic map, there exists $k \in (0, 1/2)$, such that for every $x_i \in A_i$, $i = 1, 2, 3$*

$$\|Tx_i - Tx_{i+1}\| \leq k(\|Tx_i - x_{i+1}\| + \|Tx_{i+1} - x_i\|) + (1 - 2k)\mathrm{dist}(A_i, A_{i+1}). \tag{26}$$

Then for every $x \in A_1$ the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n\to\infty} T^{pn}x$ and T is continuous at z or T satisfies the proximal property, then $z \in A_1$ is a best proximity point of T in A_1, $T^i z \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 1, 2, \ldots, p - 1$ and $T^p z = z$.

Theorem 6.8. *(Chatterjea Type cyclic contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $A_i \subset X$, $i = 1, 2, \ldots, p$ be closed, convex sets, $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a cyclic map and there exists $k \in (0, 1/2)$, such that*

$$s_p(Tx_1, Tx_2, \ldots, Tx_p) \leq k \sum_{i=1}^{p} (\|Tx_i - x_{i-1}\| + \|Tx_i - x_{i+1}\|) - (1 - 2k)P \tag{27}$$

for every $x_i \in A_i$, $i = 1, 2, 3$, where we use the convention $x_0 = x_p$ and $x_{p+1} = x_1$, and for every $x \in A_i$, $i = 1, 2, 3$

$$\|T^p x - x\| + \|T^{p-1}x - Tx\| \leq \|T^{p-1}x - x\| + \|T^p x - Tx\|. \tag{28}$$

Then for every $x \in A_1$ the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n\to\infty} T^{pn}x$ and T is continuous at z or T satisfies the proximal property, then $z \in A_1$ is a best proximity point of T in A_1, $T^i z \in A_{i+1}$ is a best proximity point of T in A_{i+1} for $i = 1, 2, \ldots, p - 1$ and $T^p z = z$.

Proof. Let us choose an arbitrary $x \in A_1$ and let us put $x_i = T^{i-1}x$, $i \in \mathbb{N}$. From (27) we get the inequality

$$\begin{aligned} s_{p,1}(x) &= \sum_{i=2}^{p} \|T^{i-1}x - T^i x\| + \|T^p x - Tx\| \\ &\le k\left(\sum_{i=2}^{p} \|T^i x - T^{i-2}x\| + \|T^p x - x\| + \|T^{p-1}x - Tx\|\right) + (1-2k)P \\ &\le k\left(\sum_{i=2}^{p} \|T^{i-1}x - T^i x\| + \|T^p x - Tx\|\right) \\ &\quad + k\left(\sum_{i=2}^{p} \|T^{i-1}x - T^{i-2}x\| + \|T^p x - x\| + \|T^{p-1}x - T^p x\|\right) + (1-2k)P \end{aligned} \tag{29}$$

From (29) and (28) we obtain the inequality

$$(1-k)s_{p,1}(x) \le k s_{p,0}(x) + (1-2k)P. \tag{30}$$

From (30) we get

$$s_{p,1}(x) \le \frac{k}{1-k} s_{p,0}(x) + \frac{1-2k}{1-k} P. \tag{31}$$

From $k \in (0, 1/2)$ it follows that $\frac{k}{1-k} \in (0,1)$ and if we put $\alpha = \frac{k}{1-k}$ then (31) can be written as $s_{p,1}(x) \le \alpha s_{p,0}(x) + (1-\alpha)P$. Therefore T satisfies (2) and we can apply Theorem 3.3. □

7. Applications of Theorem 3.4

The next results are proved in a similar fashion as like as it is done in the previous section.

Theorem 7.1. *(p–summing cyclic contraction [22]) Let $(X, \|\cdot\|)$ be a reflexive Banach space and $A_i \subset X$, $i = 1, 2, \ldots, p$ are weakly closed sets, $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k \in (0, 1)$, such that*

$$s_p(Tx_1, Tx_2, \ldots, Tx_p) \le k s_p(x_1, x_2, \ldots, x_p) + (1-k)P \tag{32}$$

for every $x_i \in A_i$, $i = 1, 2, \ldots, p$. Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k \in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k \in A_{k+s}$ is a best proximity point of T in A_{k+s}.

Corollary 7.2. *(Cyclic type contraction) Let $(X, \|\cdot\|)$ be a reflexive Banach space and $A_i \subset X$, $i = 1, 2, \ldots, p$ are weakly closed sets, $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k \in (0, 1)$, such that for every $x_i \in A_i$, $i = 1, 2, \ldots, p$*

$$\|Tx_i - Tx_{i+1}\| \le k\|x_i - x_{i+1}\| + (1-k)\mathrm{dist}(A_i, A_{i+1}). \tag{33}$$

Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k \in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k \in A_{k+s}$ is a best proximity point of T in A_{k+s}.

For $p = 1$ in Corollary 7.2 we get the results from [1].

Theorem 7.3. *(p – cyclic Kannan summing iterated contraction) Let $(X, \|\cdot\|)$ be a reflexive Banach space and $A_i \subset X$, $i = 1, 2, \ldots, p$ are weakly closed sets, $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k \in (0, 1/4)$, such that*

$$s_p(Tx_1, Tx_2, \ldots, Tx_p) \le 2k \sum_{i=1}^{p} \|Tx_i - x_i\| + (1-2k)P \tag{34}$$

for every $x_i \in A_i$, $i = 1, 2, \ldots, p$. Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k \in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k \in A_{k+s}$ is a best proximity point of T in A_{k+s}.

Theorem 7.4. *($p-$cyclic Kannan summing iterated contraction) Let $(X,\|\cdot\|)$ be a reflexive Banach space and $A_i \subset X$, $i=1,2,\dots,p$ are weakly closed sets, $T:\cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k\in(0,1/2)$, such that*

$$s_p(Tx_1,Tx_2,\dots,Tx_p) \le 2k\sum_{i=1}^p \|Tx_i - x_i\| + (1-2k)P \tag{35}$$

for every $x_i\in A_i$, $i=1,2,\dots,p$ and

$$\|x-Tx\| + \|T^{p-1}x - T^p x\| \le \|T^{p-1}x - x\| + \|T^p x - Tx\| \tag{36}$$

for every $x\in\cup_{i=1}^p A_i$. Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k\in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k\in A_{k+s}$ is a best proximity point of T in A_{k+s}.

Corollary 7.5. *(Kannan type $p-$cyclic contraction [21]) Let $(X,\|\cdot\|)$ be a reflexive Banach space and $A_i\subset X$, $i=1,2,\dots,p$ are weakly closed sets, $T:\cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k\in(0,1/4)$, such that*

$$\|Tx_i - Tx_{i+1}\| \le k(\|Tx_i - x_i\| + \|Tx_{i+1} - x_{i+1}\|) + (1-2k)\mathrm{dist}(A_i,A_{i+1}), \tag{37}$$

for every $x_i\in A_i$, $i=1,2,\dots,p$. Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k\in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k\in A_{k+s}$ is a best proximity point of T in A_{k+s}.

Theorem 7.6. *($3-$cyclic Chatterjea summing iterated contraction) Let $(X,\|\cdot\|)$ be a reflexive Banach space and $A_i\subset X$, $i=1,2,\dots,p$ are weakly closed sets, $T:\cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k\in(0,1)$, such that for every $x_i\in A_i$, $i=1,2,3$*

$$\begin{aligned} s_3(Tx_1,Tx_2,Tx_3) \le\ & k(\|Tx_1-x_2\|+\|Tx_1-x_3\|)+k(\|Tx_2-x_1\|+\|Tx_2-x_3\|)\\ &+k(\|Tx_3-x_1\|+\|Tx_3-x_2\|)-(1-2k)P. \end{aligned} \tag{38}$$

Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k\in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k\in A_{k+s}$ is a best proximity point of T in A_{k+s}.

Corollary 7.7. *($3-$cyclic Chatterjea contraction) Let $(X,\|\cdot\|)$ be a reflexive Banach space and $A_i\subset X$, $i=1,2,\dots,p$ are weakly closed sets, $T:\cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k\in(0,1)$, such that*

$$\|Tx_i - Tx_{i+1}\| \le k(\|Tx_i - x_{i+1}\| + \|Tx_{i+1} - x_i\|) + (1-2k)\mathrm{dist}(A_i,A_{i+1}), \tag{39}$$

for every $x_i\in A_i$, $i=1,2,3$. Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k\in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k\in A_{k+s}$ is a best proximity point of T in A_{k+s}.

Theorem 7.8. *(Chatterjea Type cyclic contraction) Let $(X,\|\cdot\|)$ be a reflexive Banach space and $A_i\subset X$, $i=1,2,\dots,p$ are weakly closed sets, $T:\cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclic map and there exists $k\in(0,1)$, such that*

$$s_p(Tx_1,Tx_2,\dots,Tx_p) \le k\sum_{i=1}^p (\|Tx_i - x_{i-1}\| + \|Tx_i - x_{i+1}\|) - (1-2k)P, \tag{40}$$

for every $x_i\in A_i$, $i=1,2,3$, where we use the convention $x_0=x_p$ and $x_{p+1}=x_1$, and for every $x\in A_i$, $i=1,2,3$

$$\|T^p x - x\| + \|T^{p-1}x - Tx\| \le \|T^{p-1}x - x\| + \|T^p x - Tx\|. \tag{41}$$

Let there holds one of the following T is weakly continuous on $\cup_{i=1}^p A_i$ or T satisfies the proximal property. Then there exists $\xi_k\in A_k$, which is a best proximity point of T in A_k. In the case when T is weakly continuous the point $T^s\xi_k\in A_{k+s}$ is a best proximity point of T in A_{k+s}.

We would like to finish with an open question. Is it possible to generalize the results about best proximity points in reflexive Banach spaces for other type of maps such as the investigated in [2–5, 18]?

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