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# *-DMP Elements in $*$-Semigroups and $*$-Rings

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**Abstract.** In this paper, we investigate *-DMP elements in $*$-semigroups and $*$-rings. The notion of *-DMP element was introduced by Patrício and Puystjens in 2004. An element $a$ is *-DMP if there exists a positive integer $m$ such that $a^m$ is EP. We first characterize *-DMP elements in terms of the {1,3}-inverse, Drazin inverse and pseudo core inverse, respectively. Then, we characterize the core-EP decomposition utilizing the pseudo core inverse, which extends the core-EP decomposition introduced by Wang for complex matrices to an arbitrary $*$-ring; and this decomposition turns to be a useful tool to characterize *-DMP elements. Further, we extend Wang's core-EP order from complex matrices to $*$-rings and use it to investigate *-DMP elements. Finally, we give necessary and sufficient conditions for two elements $a$, $b$ in $*$-rings to have $aa^{\textcircled{D}} = bb^{\textcircled{D}}$, which contribute to study *-DMP elements.

## 1. Introduction

Let $S$ and $R$ denote a semigroup and a ring with unit 1, respectively.

An element $a \in S$ is Drazin invertible [5] if there exists the unique element $a^D \in S$ such that

$$a^m a^D a = a^m \text{ for some positive integer } m, \quad a^D a a^D = a^D \text{ and } \quad a a^D = a^D a.$$

The smallest positive integer $m$ satisfying above equations is called the Drazin index of $a$, denoted by ind($a$). We denote by $a^{D_m}$ the Drazin inverse of $a$ with ind($a$) $= m$. If the Drazin index of $a$ equals one, then the Drazin inverse of $a$ is called the group inverse of $a$ and is denoted by $a^{\#}$.

$S$ is called a $*$-semigroup if $S$ is a semigroup with involution $*$. $R$ is called a $*$-ring if $R$ is a ring with involution $*$. In the following, unless otherwise indicated, $S$ and $R$ denote a $*$-semigroup and a $*$-ring, respectively.

An element $a \in S$ is Moore-Penrose invertible, if there exists $x \in S$ such that

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax \ \text{ and } \ (4)\ (xa)^* = xa.$$

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If such an $x$ exists, then it is unique, denoted by $a^{\dagger}$. $x$ satisfying equations (1) and (3) is called a $\{1,3\}$-inverse of $a$, denoted by $a^{(1,3)}$. Such a $\{1,3\}$-inverse of $a$ is not unique if it exists. We use $a\{1,3\}$, $S^{\{1,3\}}$ to denote the set of all the $\{1,3\}$-inverses of $a$ and the set of all the $\{1,3\}$-invertible elements in $S$, respectively.

An element $a \in S$ is symmetric if $a^* = a$. $a \in S$ is *-gMP if $a^{\#}$ and $a^{\dagger}$ exist with $a^{\#} = a^{\dagger}$ [19]. It should be pointed out that *-gMP element is also known as EP element (see [9–11, 16]). As a matter of convenience, we denote a *-gMP element as an EP element in this paper. $a \in S$ is *-DMP with index $m$ if $m$ is the smallest positive integer such that $(a^m)^{\#}$ and $(a^m)^{\dagger}$ exist with $(a^m)^{\#} = (a^m)^{\dagger}$ [19]. In other words, $a \in S$ is *-DMP with index $m$ if $m$ is the smallest positive integer such that $a^m$ is EP, which is equivalent to, $a^{D_m}$ exists and $a^m$ is EP. We call $a \in S$ a *-DMP element if there exists a positive integer $m$ such that $a^m$ is EP. The notion of *-DMP element is different from the notion of $m$-EP element [12, 26, 29], in some sense, they are parallel, are both generalizations of EP elements. Hence, it is of interest to investigate the notion of *-DMP element.

Baksalary and Trenkler [18] introduced the notion of core inverse for a complex matrix in 2010. This notion is also known as core-EP generalized inverse (see [13]). Then, Rakić, Dinčić and Djordjević [21] generalized the notion of core inverse to an arbitrary $*$-ring. Later, Xu, Chen and Zhang [28] characterized the core invertible elements in $*$-rings in terms of three equations. The core inverse of $a$, denoted by $a^{\circledast}$, is the unique solution to equations

$$xa^2 = a, \ \ ax^2 = x, \ \ (ax)^* = ax.$$

Recently, the notion of core inverse was extended to arbitrary index of elements in rings. The pseudo core inverse [7] of $a \in S$, denoted by $a^{\textcircled{D}}$, is the unique solution to equations

$$xa^{m+1} = a^m \text{ for some positive integer } m, \ ax^2 = x \text{ and } (ax)^* = ax.$$

Also, the pseudo core inverse extends core-EP inverse [13] from complex matrices to $*$-semigroups, in terms of equations. For consistency and convenience, we use the terminology pseudo core inverse throughout this paper. The smallest positive integer $m$ satisfying above equations is called the pseudo core index of $a$. If $a$ is pseudo core invertible, then it must be Drazin invertible, and the pseudo core index coincides with the Drazin index [7]. So here and subsequently, we denote the pseudo core index of $a$ by $\mathrm{ind}(a)$. The pseudo core inverse is a kind of outer inverse. If the pseudo core index equals one, then the pseudo core inverse of $a$ is the core inverse of $a$. Dually, the dual pseudo core inverse [7] of $a \in S$ is the unique element $a_{\textcircled{D}} \in S$ satisfying the following three equations

$$a^{m+1}a_{\textcircled{D}} = a^m \text{ for some positive integer } m, \ (a_{\textcircled{D}})^2 a = a_{\textcircled{D}} \text{ and } (a_{\textcircled{D}}a)^* = a_{\textcircled{D}}a.$$

The smallest positive integer $m$ satisfying above equations is called the dual pseudo core index of $a$. We denote by $a^{\textcircled{D}_m}$ and $a_{\textcircled{D}_m}$ the pseudo core inverse and dual pseudo core inverse of index $m$ of $a$, respectively. Note that $(a^*)^{\textcircled{D}_m}$ exists if and only if $a_{\textcircled{D}_m}$ exists with $(a^*)^{\textcircled{D}_m} = (a_{\textcircled{D}_m})^*$.

Lots of work have been done on EP elements in $*$-semigroups and $*$-rings in recent years, (see, for example, [3, 4, 15, 19, 21, 27]). In this paper, we use the setting of $*$-semigroups and $*$-rings, and our main goal is to characterize *-DMP elements. The paper is organized as follows: In Section 2, several characterizations of *-DMP elements are given in terms of generalized inverses: the {1,3}-inverse, Drazin inverse and pseudo core inverse respectively. Then, *-DMP elements are characterized in terms of equations and annihilators. After that, we consider conditions for the sum (resp. product) of two *-DMP elements to be *-DMP. It is known that Wang [23] introduced the core-EP decomposition and core-EP order for complex matrices. Core-EP decomposition was shown to be a useful tool in characterizing generalized inverses and partial orders (see [23, 24]). In Section 3, we extend the core-EP decomposition from complex matrices to an arbitrary $*$-ring, applying a purely algebraic technique. As applications, we use it to characterize *-DMP elements. Core partial order could be used to characterize EP elements (see [25]). Similarly, core-EP order can be used to investigate *-DMP elements. In Section 4, we obtain a characterization of *-DMP elements, in terms of this pre-order. In the final section, we aim to give equivalent conditions for $aa^{\textcircled{D}} = bb^{\textcircled{D}}$ in $*$-rings, which contribute to investigate *-DMP elements.

## 2. Characterizations of *-DMP Elements

In this section, several characterizations of *-DMP elements are given by conditions involving {1,3}-inverse, Drazin inverse, pseudo core inverse and dual pseudo core inverse. We begin with some auxiliary lemmas.

**Lemma 2.1.** [7] *Let $a \in S$. Then we have the following facts:*
(1) *$a^{\mathbb{D}_m}$ exists if and only if $a^{D_m}$ exists and $a^m \in S^{\{1,3\}}$. In this case $a^{\mathbb{D}_m} = a^{D_m}a^m(a^m)^{(1,3)}$.*
(2) *$a^{\mathbb{D}_m}$ and $a_{\mathbb{D}_m}$ exist if and only if $a^{D_m}$ and $(a^m)^\dagger$ exist. In this case, $a^{\mathbb{D}_m} = a^{D_m}a^m(a^m)^\dagger$ and $a_{\mathbb{D}_m} = (a^m)^\dagger a^m a^{D_m}$.*

**Lemma 2.2.** [11],[19] *Let $a \in S$. Then the following conditions are equivalent:*
(1) *$a$ is *-DMP with index $m$;*
(2) *$a^{D_m}$ exists and $aa^{D_m}$ is symmetric.*

**Lemma 2.3.** *Let $a \in S$. Then the following are equivalent:*
(1) *$a$ is *-DMP with index $m$;*
(2) *$a^{D_m}$ and $(a^m)^\dagger$ exist with $(a^{D_m})^m = (a^m)^\dagger$;*
(3) *$a^{\mathbb{D}_m}$ exists with $a^{\mathbb{D}_m} = a^{D_m}$;*
(4) *$a^{\mathbb{D}_m}$ and $(a^m)^\dagger$ exist with $(a^{\mathbb{D}_m})^m = (a^m)^\dagger$.*

*Proof.* (1) ⇒ (2) is clear.
(2) ⇒ (3). Suppose $a^{D_m}$ and $(a^m)^\dagger$ exist with $(a^{D_m})^m = (a^m)^\dagger$. By Lemma 2.1, $a^{\mathbb{D}_m}$ exists with $a^{\mathbb{D}_m} = a^{D_m}a^m(a^m)^\dagger = a^{D_m}a^m(a^{D_m})^m = a^{D_m}$.
(3) ⇒ (4). Applying Lemma 2.1, $a^{\mathbb{D}_m}$ exists if and only if $a^{D_m}$ exists and $a^m \in S^{\{1,3\}}$, in which case, $a^{\mathbb{D}_m} = a^{D_m}a^m(a^m)^{(1,3)}$. From $a^{\mathbb{D}_m} = a^{D_m}$, it follows that $a^{D_m}a^m(a^m)^{(1,3)} = a^{D_m}$. Then, $aa^{D_m} = a^m(a^m)^{(1,3)}$. So, $(a^m)^\dagger$ exists with $(a^m)^\dagger = (a^{D_m})^m = (a^{\mathbb{D}_m})^m$.
(4) ⇒ (1). Since $(a^{D_m})^m a^m(a^m)^{(1,3)} = (a^{D_m}a^m(a^m)^{(1,3)})^m = (a^{\mathbb{D}_m})^m = (a^m)^\dagger$, then $aa^{D_m} = (a^m)^\dagger a^m$. Therefore $aa^{D_m}$ is symmetric. Hence $a$ is *-DMP with index $m$ by Lemma 2.2. □

The following result characterizes *-DMP elements in terms of $\{1,3\}$-inverses.

**Theorem 2.4.** *Let $a \in S$. Then $a$ is *-DMP with index $m$ if and only if $m$ is the smallest positive integer such that $a^m \in S^{\{1,3\}}$ and one of the following equivalent conditions holds:*
(1) *$a(a^m)^{(1,3)} = (a^m)^{(1,3)}a$ for some $(a^m)^{(1,3)} \in a^m\{1,3\}$;*
(2) *$a^m(a^m)^{(1,3)} = (a^m)^{(1,3)}a^m$ for some $(a^m)^{(1,3)} \in a^m\{1,3\}$.*

*Proof.* If $a$ is *-DMP with index $m$, then $m$ is the smallest positive integer such that $(a^m)^\dagger$ and $(a^m)^\#$ exist with $(a^m)^\dagger = (a^m)^\#$. So we may regard $(a^m)^\#$ as one of the $\{1,3\}$-inverses of $a^m$. Therefore (1) holds (see [5, Theorem 1]).

Conversely, we take $(a^m)^{(1,3)} \in a^m\{1,3\}$.
(1) ⇒ (2) is obvious.
(2). Equality $a^m(a^m)^{(1,3)} = (a^m)^{(1,3)}a^m$ yields that $(a^m)^\dagger = (a^m)^{(1,3)}a^m(a^m)^{(1,3)} = (a^m)^\#$. So $m$ is the smallest positive integer such that $(a^m)^\dagger = (a^m)^\#$. Hence $a$ is $*$-DMP with index $m$. □

**Corollary 2.5.** *Let $a \in S$. Then $a$ is EP if and only if $a \in S^{\{1,3\}}$ and $aa^{(1,3)} = a^{(1,3)}a$ for some $a^{(1,3)} \in a\{1,3\}$.*

In [11, Theorem 7.3], Koliha and Patrício characterized EP elements by using the group inverse. Similarly, we characterize *-DMP elements in terms of the Drazin inverse.

**Theorem 2.6.** *Let $a \in S$. Then $a$ is *-DMP with index $m$ if and only if $a^{D_m}$ exists and one of the following equivalent conditions holds:*
(1) *$a^{D_m} = a^{D_m}(aa^{D_m})^*$;*
(2) *$a^{D_m} = (a^{D_m}a)^*a^{D_m}$.*
*If $S$ is a $*$-ring, then* (1)-(2) *are equivalent to*
(3) *$a^{D_m}(1 - aa^{D_m})^* = (1 - aa^{D_m})(a^{D_m})^*$.*

*Proof.* If $a$ is *-DMP with index $m$, then $a^{D_m}$ exists and $aa^{D_m}$ is symmetric by Lemma 2.2. It is not difficult to verify that conditions (1)-(3) hold.

Conversely, we assume that $a^{D_m}$ exists.

(1) ⇒ (3). Since $a^{D_m} = a^{D_m}(aa^{D_m})^*$, we have

$$a^{D_m}(1 - aa^{D_m})^* = a^{D_m}(aa^{D_m})^*(1 - aa^{D_m})^* = a^{D_m}((1 - aa^{D_m})aa^{D_m})^* = 0.$$

Therefore $a^{D_m}(1 - aa^{D_m})^* = 0 = (1 - aa^{D_m})(a^{D_m})^*$.

(2) ⇒ (3) is analogous to (1) ⇒ (3).

Finally, we will prove $a$ is *-DMP with index $m$ under the assumption that $a^{D_m}$ exists with $a^{D_m}(1-aa^{D_m})^* = (1 - aa^{D_m})(a^{D_m})^*$. From $a^{D_m}(1 - a^*(a^{D_m})^*) = (1 - a^{D_m}a)(a^{D_m})^*$, we get $(a^{D_m})^* = a^{D_m}(1 - a^*(a^{D_m})^* + a(a^{D_m})^*)$. Post-multiply this equality by $(a^{D_m})^*(a^2)^*$, then we have $aa^{D_m} = aa^{D_m}(aa^{D_m})^*$. So $aa^{D_m}$ is symmetric. Applying Lemma 2.2, $a$ is *-DMP with index $m$. □

Let us recall that $a \in S$ is normal if $aa^* = a^*a$. It is known that an element $a \in S$ is EP may not imply it is normal (such as, take $S = \mathbb{R}^{2\times2}$ with transpose as involution, $a = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then $a$ is EP since $aa^\dagger = a^\dagger a = 1$, but $aa^* = \left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right) = a^*a$); $a$ is normal may not imply it is EP (such as, take $S = \mathbb{C}^{2\times2}$ with transpose as involution, $a = \left(\begin{smallmatrix}i&1\\-1&i\end{smallmatrix}\right)$. Then $aa^* = a^*a = 0$, i.e., $a$ is normal. But $a$ is not Moore-Penrose invertible and hence $a$ is not EP). So we may be of interest to know when $a$ is both EP and normal. Here we give a more extensive case.

**Theorem 2.7.** *Let $a \in S$. Then the following are equivalent:*
*(1) $a$ is *-DMP with index $m$ and $a(a^*)^m = (a^*)^m a$;*
*(2) $m$ is the smallest positive integer such that $(a^m)^\dagger$ exists and $a(a^*)^m = (a^*)^m a$;*
*(3) $a^{D_m}$ exists and $(a^m)^* = ua = au$ for some group invertible element $u \in S$.*

*Proof.* (1) ⇒ (2) is clear.

(2) ⇒ (1). The equality $a^m(a^m)^* = (a^m)^*a^m$ ensures that $a^m(a^m)^\dagger = (a^m)^\dagger a^m$ (see [8, Theorem 5]). So $a$ is *-DMP with index $m$ by Theorem 2.4.

(1) ⇒ (3). Since $a$ is *-DMP with index $m$, then $a^{D_m}$ exists and $aa^{D_m}$ is symmetric by Lemma 2.2. So,

$$(a^m)^* = (a^m a^{D_m} a)^* = aa^{D_m}(a^m)^*, \text{ and}$$
$$(a^m)^* = (aa^{D_m}a^m)^* = (a^m)^* aa^{D_m}.$$

Since $a^{D_m}$ exists and $(a^m)^*a = a(a^m)^*$, then we obtain $a^{D_m}(a^m)^* = (a^m)^*a^{D_m}$ (see [5, Theorem 1]). Take $u = a^{D_m}(a^m)^*$, then $au = ua = (a^m)^*$. In what follows, we show $u^\# = a((a^{D_m})^m)^*$. In fact,

$$\begin{aligned}\text{(i) } ua((a^{D_m})^m)^*u &= a^{D_m}(a^m)^*a((a^{D_m})^m)^*a^{D_m}(a^m)^* = (a^m)^*aa^{D_m}((a^{D_m})^m)^*a^{D_m}(a^m)^*\\ &= (a^m)^*((a^{D_m})^m)^*a^{D_m}(a^m)^* = (aa^{D_m})^*a^{D_m}(a^m)^* = a^{D_m}(a^m)^* = u;\end{aligned}$$

$$\begin{aligned}\text{(ii) } a((a^{D_m})^m)^*ua((a^{D_m})^m)^* &= a((a^{D_m})^m)^*a^{D_m}(a^m)^*a((a^{D_m})^m)^*\\ &= a((a^{D_m})^m)^*(a^m)^*a^{D_m}a((a^{D_m})^m)^*\\ &= a(aa^{D_m})^*a^{D_m}a((a^{D_m})^m)^* = a(aa^{D_m})^*((a^{D_m})^m)^*\\ &= a((a^{D_m})^m)^*;\end{aligned}$$

$$\begin{aligned}\text{(iii) } a((a^{D_m})^m)^*u &= a((a^{D_m})^m)^*a^{D_m}(a^m)^* = a((a^{D_m})^m)^*(a^m)^*a^{D_m} = a(aa^{D_m})^*a^{D_m}\\ &= aa^{D_m} \text{ and}\end{aligned}$$

$$ua((a^{D_m})^m)^* = a^{D_m}(a^m)^*a((a^{D_m})^m)^* = a^{D_m}a(a^m)^*((a^{D_m})^m)^* = aa^{D_m},$$

so, $a((a^{D_m})^m)^*u = ua((a^{D_m})^m)^*$.

Hence $u^\# = a((a^{D_m})^m)^*$.

(3) ⇒ (1). Since $u^\#$ and $a^{D_m}$ exist with $au = ua$, then $au^\# = u^\# a$ and $(ua)^D = u^\# a^{D_m}$.

So, $(aa^{D_m})^* = (a^m(a^{D_m})^m)^* = ((a^m)^D a^m)^* = (a^m)^*((a^m)^*)^D = ua(ua)^D = uau^{\#}a^{D_m}$
$= uu^{\#}aa^{D_m}$.
Therefore $(aa^{D_m})^* aa^{D_m} = uu^{\#}aa^{D_m} = (aa^{D_m})^*$. That is, $aa^{D_m}$ is symmetric.
We thus have $a$ is *-DMP with index $m$. □

**Corollary 2.8.** *Let $a \in S$. Then the following are equivalent:*
*(1) $a$ is EP and normal;*
*(2) $a^\dagger$ exists and $a$ is normal;*
*(3) $a^{\#}$ exists and $a^* = ua = au$ for some group invertible element $u \in S$.*

In what follows, *-DMP elements are characterized in terms of the pseudo core inverse and dual pseudo core inverse.

**Theorem 2.9.** *Let $a \in S$. Then the following are equivalent:*
*(1) $a$ is *-DMP with index $m$;*
*(2) $a^{\mathbb{D}_m}$ and $a_{\mathbb{D}_m}$ exist with $a^{\mathbb{D}_m} = a_{\mathbb{D}_m}$;*
*(3) $a^{\mathbb{D}_m}$ and $a_{\mathbb{D}_m}$ exist with $aa^{\mathbb{D}_m} = a_{\mathbb{D}_m}a$.*

*Proof.* (1) ⇒ (2), (3). If $a$ is *-DMP with index $m$, then by Lemma 2.3, $a^{D_m}$ and $(a^m)^\dagger$ exist with $(a^m)^\dagger = (a^{D_m})^m$. Hence $a^{\mathbb{D}_m}$ and $a_{\mathbb{D}_m}$ exist by Lemma 2.1 (2). It is not difficult to verify that $a_{\mathbb{D}_m} = a^{\mathbb{D}_m}$ and $aa^{\mathbb{D}_m} = a_{\mathbb{D}_m}a$.

(2) ⇒ (1). If $a^{\mathbb{D}_m}$ and $a_{\mathbb{D}_m}$ exist, then $a^{D_m}$ and $(a^m)^\dagger$ exist with $a^{\mathbb{D}_m} = a^{D_m}a^m(a^m)^\dagger$, $a_{\mathbb{D}_m} = (a^m)^\dagger a^m a^{D_m}$. Equality $a_{\mathbb{D}_m} = a^{\mathbb{D}_m}$ would imply that $a^{D_m}a^m(a^m)^\dagger = (a^m)^\dagger a^m a^{D_m}$. Post-multiply this equality by $a^{m+1}(a^{D_m})^m$, then we obtain $aa^{D_m} = (a^m)^\dagger a^m$. So $aa^{D_m}$ is symmetric. According to Lemma 2.2, $a$ is *-DMP with index $m$.

(3) ⇒ (1). By the hypothesis, we have $aa^{D_m}a^m(a^m)^\dagger = (a^m)^\dagger a^m a^{D_m}a$. That is, $a^m(a^m)^\dagger = (a^m)^\dagger a^m$. So $aa^{D_m} = a^m(a^{D_m})^m = a^m(a^m)^\dagger a^m(a^{D_m})^m = (a^m)^\dagger a^m a^m(a^{D_m})^m = (a^m)^\dagger a^m$. Therefore $aa^{D_m}$ is symmetric. Hence $a$ is *-DMP with index $m$. □

The following result characterizes *-DMP elements merely in terms of the pseudo core inverse.

**Theorem 2.10.** *Let $a \in S$. Then $a$ is *-DMP with index $m$ if and only if $a^{\mathbb{D}_m}$ exists and one of the following equivalent conditions holds:*
*(1) $aa^{\mathbb{D}_m} = a^{\mathbb{D}_m}a$;*
*(2) $a^{D_m}a^{\mathbb{D}_m} = a^{\mathbb{D}_m}a^{D_m}$;*
*(3) $a^{\mathbb{D}_m} = (a^m)^{(1,3)}a^m a^{D_m}$ for some $(a^m)^{(1,3)} \in a^m\{1,3\}$;*
*(4) $a^{m+1}a^{\mathbb{D}_m} = a^m$;*
*(5) $(a^{\mathbb{D}_m})^2 a = a^{\mathbb{D}_m}$;*
*(6) $a^{\mathbb{D}_m}a$ is symmetric;*
*(7) $aa^{\mathbb{D}_m}$ commutes with $a^{\mathbb{D}_m}a$.*

*Proof.* If $a$ is *-DMP with index $m$, then $(a^{D_m})^m = (a^m)^\dagger$, $a^{\mathbb{D}_m} = a^{D_m}$ by Lemma 2.3 and $aa^{D_m}$ is symmetric by Lemma 2.2. So (1)-(7) hold.

Conversely, we assume that $a^{\mathbb{D}_m}$ exists.

(1). By the definition of the pseudo core inverse, we have $a^{\mathbb{D}_m}a^{m+1} = a^m$, and we also have $a^{\mathbb{D}_m}aa^{\mathbb{D}_m} = a^{\mathbb{D}_m}$ by calculation. The equalities $aa^{\mathbb{D}_m} = a^{\mathbb{D}_m}a$, $a^{\mathbb{D}_m}aa^{\mathbb{D}_m} = a^{\mathbb{D}_m}$ and $a^{\mathbb{D}_m}a^{m+1} = a^m$ yield that $a^{D_m} = a^{\mathbb{D}_m}$. Therefore $a$ is *-DMP with index $m$ by Lemma 2.3.

(2). Since $a^{D_m}a^{\mathbb{D}_m} = a^{\mathbb{D}_m}a^{D_m}$, then $(a^{D_m})^{\#}a^{\mathbb{D}_m} = a^{\mathbb{D}_m}(a^{D_m})^{\#}$ (see [5, Theorem 1]). Namely,

$$a^2a^{D_m}a^{\mathbb{D}_m} = a^{\mathbb{D}_m}a^2a^{D_m}.$$

So $aa^{\mathbb{D}_m} = a^m(a^{\mathbb{D}_m})^m = aa^{D_m}a^m(a^{\mathbb{D}_m})^m = aa^{D_m}aa^{\mathbb{D}_m} = a^2a^{D_m}a^{\mathbb{D}_m} = a^{\mathbb{D}_m}a^2a^{D_m}$
$= a^{\mathbb{D}_m}a^{m+1}(a^{D_m})^m = a^m(a^{D_m})^m = aa^{D_m}$.
Therefore $aa^{D_m}$ is symmetric. Hence $a$ is *-DMP with index $m$ by Lemma 2.2.

(3). Since $a^{\mathbb{D}_m}$ exists, then by Lemma 2.1 (1), $a^{D_m}$ and $(a^m)^{(1,3)}$ exist. From equality (3) and $a^{\mathbb{D}_m} = a^{D_m}a^m(a^m)^{(1,3)}$, it follows that $a^{D_m}a^m(a^m)^{(1,3)} = (a^m)^{(1,3)}a^m a^{D_m}$. Pre-multiply this equality by $(a^{D_m})^{m-1}a^m$, then we get

$$a^m(a^m)^{(1,3)} = aa^{D_m}.$$

So $aa^{D_m}$ is symmetric. Hence $a$ is *-DMP with index $m$ by Lemma 2.2.
(4). The equalities $a^{m+1}a^{\textcircled{D}_m} = a^m$ and $a^{\textcircled{D}_m}a^{m+1} = a^m$ yield that $a$ is strongly $\pi$-regular and $a^{D_m} = a^m(a^{\textcircled{D}_m})^{m+1} = a^{\textcircled{D}_m}$ (see [5, Theorem 4]). So $a$ is *-DMP with index $m$ by Lemma 2.3.
(5) ⇒ (1). Pre-multiply (5) by $a$, then we get $a(a^{\textcircled{D}_m})^2a = aa^{\textcircled{D}_m}$. That is, $a^{\textcircled{D}_m}a = aa^{\textcircled{D}_m}$.
(6) ⇒ (1). Pre-multiply $(a^{\textcircled{D}_m}a)^* = a^{\textcircled{D}_m}a$ by $aa^{\textcircled{D}_m}$, then we obtain

$$aa^{\textcircled{D}_m}(a^{\textcircled{D}_m}a)^* = aa^{\textcircled{D}_m}a^{\textcircled{D}_m}a = a^{\textcircled{D}_m}a.$$

So,

$$\begin{aligned} aa^{\textcircled{D}_m} &= a^m(a^{\textcircled{D}_m})^m = (a^m(a^{\textcircled{D}_m})^m)^* = (a^{\textcircled{D}_m}a^{m+1}(a^{\textcircled{D}_m})^m)^* = (a^{\textcircled{D}_m}aaa^{\textcircled{D}_m})^* \\ &= (aa^{\textcircled{D}_m})^*(a^{\textcircled{D}_m}a)^* = aa^{\textcircled{D}_m}(a^{\textcircled{D}_m}a)^* = a^{\textcircled{D}_m}a. \end{aligned}$$

(7) ⇒ (1). From $aa^{\textcircled{D}_m}(a^{\textcircled{D}_m}a) = (a^{\textcircled{D}_m}a)aa^{\textcircled{D}_m}$, $aa^{\textcircled{D}_m}(a^{\textcircled{D}_m}a) = a^{\textcircled{D}_m}a$ and $(a^{\textcircled{D}_m}a)aa^{\textcircled{D}_m} = a^{\textcircled{D}_m}a^{m+1}(a^{\textcircled{D}_m})^m = aa^{\textcircled{D}_m}$, it follows that $aa^{\textcircled{D}_m} = a^{\textcircled{D}_m}a$. □

In [27], Xu and Chen characterized EP elements in terms of equations. Similarly, we utilize equations to characterize *-DMP elements.

**Theorem 2.11.** *Let $a \in S$. Then the following are equivalent:*
(1) *$a$ is *-DMP with index $m$;*
(2) *$m$ is the smallest positive integer such that $xa^{m+1} = a^m$, $ax^2 = x$ and $(x^m a^m)^* = x^m a^m$ for some $x \in S$;*
(3) *$m$ is the smallest positive integer such that $xa^{m+1} = a^m$, $ax = xa$ and $(x^m a^m)^* = x^m a^m$ for some $x \in S$.*

*Proof.* (1) ⇒ (2), (3). Suppose $a$ is *-DMP with index $m$, then $a^{D_m}$ exists and $a^{D_m}a$ is symmetric by Lemma 2.2. Take $x = a^{D_m}$, then (2) and (3) hold.
(2) ⇒ (1). From $xa^{m+1} = a^m$ and $a^m = xa^{m+1} = (ax^2)a^{m+1} = (a^{m+1}x^{m+2})a^{m+1} = a^{m+1}(x^{m+2}a^{m+1}) = a^{m+1}(x^{m+1}a^m) = a^{m+1}x^{m+1}a^m$, it follows that $a$ is strongly $\pi$-regular and $a^{D_m} = x^{m+1}a^m$. So $aa^{D_m} = ax^{m+1}a^m = x^m a^m$. Therefore $a^{D_m}$ exists and $aa^{D_m}$ is symmetric. Hence $a$ is *-DMP with index $m$ by Lemma 2.2.
(3) ⇒ (1). Equalities $xa^{m+1} = a^m$ and $a^m = a^{m+1}x$ yield that $a^{D_m} = x^{m+1}a^m$. So $a^{D_m}a = x^{m+1}a^{m+1} = x^m a^m$. Therefore $a^{D_m}$ exists and $aa^{D_m}$ is symmetric. Hence $a$ is *-DMP with index $m$. □

Let $S^0$ denote a $*$-semigroup with zero element 0. The left annihilator of $a \in S^0$ is denoted by $^\circ a$ and is defined by $^\circ a = \{x \in S^0 : xa = 0\}$. The following result characterizes *-DMP elements in $S^0$ in terms of left annihilators. We begin with an auxiliary lemma.

**Lemma 2.12.** [7] *Let $a,\ x \in S^0$. Then $a^{\textcircled{D}_m} = x$ if and only if $m$ is the smallest positive integer such that one of the following equivalent conditions holds:*
(1) *$xax = x$ and $xS^0 = x^*S^0 = a^m S^0$;*
(2) *$xax = x$, $^\circ x = {}^\circ(a^m)$ and $^\circ(x^*) \subseteq {}^\circ(a^m)$.*

**Theorem 2.13.** *Let $a \in S^0$. Then $a$ is *-DMP with index $m$ if and only if $m$ is the smallest positive integer such that one of the following equivalent conditions holds:*
(1) *$xax = x$, $xS^0 = x^*S^0 = a^m S^0$ and $x^m S^0 = (a^m)^* S^0$ for some $x \in S^0$;*
(2) *$xax = x$, $^\circ x = {}^\circ(a^m)$, $^\circ(x^*) \subseteq {}^\circ(a^m)$ and $^\circ(a^m)^* \subseteq {}^\circ(x^m)$ for some $x \in S^0$.*

*Proof.* Suppose $a$ is *-DMP with index $m$. Then $a^{\textcircled{D}_m}$, $(a^m)^\dagger$ exist with $(a^{\textcircled{D}_m})^m = (a^m)^\dagger$ by Lemma 2.3. Take $x = a^{\textcircled{D}_m}$, then $xax = x$, $xS^0 = x^*S^0 = a^m S^0$ by Lemma 2.12. Further, from $x^m = (a^m)^\dagger$, it follows that $x^m = (x^m a^m)^* x^m = (a^m)^*(x^m)^* x^m \in (a^m)^* S^0$ and $(a^m)^* = (a^m x^m a^m)^* = x^m a^m (a^m)^* \in x^m S^0$. Hence (1) holds.
(1) ⇒ (2) is clear.
(2). From $xax = x$, $^\circ x = {}^\circ(a^m)$ and $^\circ(x^*) \subseteq {}^\circ(a^m)$, it follows that $a^{\textcircled{D}_m} = x$ by Lemma 2.12. Then $1 - (x^m a^m)^* \in {}^\circ(a^m)^* \subseteq {}^\circ(x^m)$ implies $x^m = (x^m a^m)^* x^m$. So $x^m a^m = (x^m a^m)^* x^m a^m$. Therefore $(x^m a^m)^* = x^m a^m$, together with $xa^{m+1} = a^m$, $ax^2 = x$, implies $a$ is *-DMP with index $m$ by Theorem 2.11. □

It is known that $a^D$ exists if and only if $(a^k)^D$ exists for any positive integer $k$ if and only if $(a^k)^D$ exists for some positive integer $k$ [5]. We find this property is inherited by *-DMP.

**Theorem 2.14.** *Let $a \in S$ and $k$ a positive integer, then $a$ is *-DMP if and only if $a^k$ is *-DMP.*

*Proof.* Observe that $a^D$ exists and $aa^D$ is symmetric if and only if $(a^k)^D$ exists and $a^k(a^k)^D$ is symmetric. So $a$ is *-DMP if and only if $a^k$ is *-DMP by Lemma 2.2. □

Given two *-DMP elements $a$ and $b$, we may be of interest to consider conditions for the product $ab$ (resp. sum $a + b$) to be *-DMP.

**Theorem 2.15.** *Let $a, b \in S$ with $ab = ba$, $ab^* = b^*a$. If both $a$ and $b$ are *-DMP, then $ab$ is *-DMP.*

*Proof.* Suppose that both $a$ and $b$ are *-DMP, then $a^{\textcircled{D}}$, $a^D$ and $b^{\textcircled{D}}$, $b^D$ exist with $a^{\textcircled{D}} = a^D$, $b^{\textcircled{D}} = b^D$ by Lemma 2.3. Since $a^{\textcircled{D}}$ and $b^{\textcircled{D}}$ exist with $ab = ba$, $ab^* = b^*a$, then $(ab)^{\textcircled{D}}$ exists with $(ab)^{\textcircled{D}} = a^{\textcircled{D}}b^{\textcircled{D}}$ (see [7, Theorem 4.3]). Also, $(ab)^D$ exists with $(ab)^D = a^Db^D$. So,

$$(ab)^{\textcircled{D}} = a^{\textcircled{D}}b^{\textcircled{D}} = a^Db^D = (ab)^D.$$

Hence $ab$ is *-DMP by Lemma 2.3. □

**Theorem 2.16.** *Let $a, b \in R$ with $ab = ba = 0$, $a^*b = 0$. If both $a$ and $b$ are *-DMP, then $a + b$ is *-DMP.*

*Proof.* If both $a$ and $b$ are *-DMP, then $a^{\textcircled{D}}$, $a^D$ and $b^{\textcircled{D}}$, $b^D$ exist with $a^{\textcircled{D}} = a^D$, $b^{\textcircled{D}} = b^D$ by Lemma 2.3. Since $a^{\textcircled{D}}$ and $b^{\textcircled{D}}$ exist with $ab = ba = 0$, $a^*b = 0$, then $(a+b)^{\textcircled{D}}$ exists with $(a+b)^{\textcircled{D}} = a^{\textcircled{D}} + b^{\textcircled{D}}$ (see [7, Theorem 4.4]). Also, $(a+b)^D$ exists with $(a+b)^D = a^D + b^D$ (see [5, Corollary 1]). So we have

$$(a+b)^{\textcircled{D}} = a^{\textcircled{D}} + b^{\textcircled{D}} = a^D + b^D = (a+b)^D.$$

Hence $a + b$ is *-DMP by Lemma 2.3. □

**Example 2.17.** *The condition $ab = 0$, $a^*b = 0$ (without $ba = 0$) is not sufficient to show that $a + b$ is *-DMP, although both $a$ and $b$ are *-DMP.*

*Let $R = \mathbb{C}^{2\times 2}$ with transpose as involution, $a = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, then $ab = a^*b = 0$, but $ba \neq 0$. Since $a^{\textcircled{D}} = a^{\textcircled{\#}} = a^{\#}aa^{(1,3)} = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix} = a^{\#} = a^D$, $a$ is *-DMP. It is clear that $b$ is *-DMP. Observe that $a + b = \begin{pmatrix} i & 0 \\ -1 & 0 \end{pmatrix}$, by calculation, we find that neither $a+b$ nor $(a+b)^2$ has any {1,3}-inverse. Since $(a+b)^m = \begin{cases} (-1)^{\frac{m-1}{2}}(a+b) & m \text{ is odd} \\ (-1)^{\frac{m}{2}+1}(a+b)^2 & m \text{ is even} \end{cases}$, we conclude that $(a+b)^m$ has no {1,3}-inverse for arbitrary positive integer $m$. Hence $a + b$ is not *-DMP.*

## 3. Core-EP Decomposition

Core-nilpotent decomposition was introduced in [2] for complex matrices. Later, Patrício and Puystjens [19] generalized this decomposition from complex matrices to rings. Let $a \in R$ with $a^{D_m}$ exists. The sum $a = c_a + n_a$ is called the core-nilpotent decomposition of $a$, where $c_a = aa^{D_m}a$ is the core part of $a$, $n_a = (1 - aa^{D_m})a$ is the nilpotent part of $a$. This decomposition is unique and it brings $n_a^m = 0$, $c_an_a = n_ac_a = 0$, $c_a^{\#}$ exists with $c_a^{\#} = a^{D_m}$.

Wang [23] introduced the core-EP decomposition for a complex matrix, and proved its uniqueness by using the rank of a matrix and matrix decomposition. Let $A$ be a square complex matrix with index $m$, then $A = A_1 + A_2$, where $A_1^{\#}$ exists, $A_2^m = 0$ and $A_1^*A_2 = A_2A_1 = 0$. In the following, we show that neither the rank nor the matrix decomposition are necessary for the characterization of core-EP decomposition in rings.

**Theorem 3.1.** *Let $a \in R$ with $a^{\textcircled{D}_m}$ exists. Then $a = a_1 + a_2$, where*
(1) *$a_1^{\#}$ exists;*
(2) *$a_2^m = 0$;*
(3) *$a_1^* a_2 = a_2 a_1 = 0$.*

*Proof.* Since $a^{\textcircled{D}_m}$ exists. Take $a_1 = aa^{\textcircled{D}_m}a$ and $a_2 = a - aa^{\textcircled{D}_m}a$, then $a_2^m = 0$ and $a_1^* a_2 = a_2 a_1 = 0$. Next, we will prove that $a_1^{\#}$ exists. In fact,

$$a_1 = aa^{\textcircled{D}_m}a = (aa^{\textcircled{D}_m}a)^2(a^{\textcircled{D}_m})^2 a \in a_1^2 R \text{ and } a_1 = aa^{\textcircled{D}_m}a = a^{\textcircled{D}_m}(aa^{\textcircled{D}_m}a)^2 \in Ra_1^2.$$

Hence $a_1^{\#}$ exists with $a_1^{\#} = (a^{\textcircled{D}_m})^2 a$ (see [9, Proposition 7]). □

**Theorem 3.2.** *The core-EP decomposition of an element in $R$ is unique.*

*Proof.* The proof is similar to [23, Theorem 2.4], the matrices case. We give the proof for completeness.

Let $a = a_1 + a_2$ be the core-EP decomposition of $a \in R$, where $a_1 = aa^{\textcircled{D}_m}a$, $a_2 = a - aa^{\textcircled{D}_m}a$. Let $a = b_1 + b_2$ be another core-EP decomposition of $a$. Then $a^m = \sum\limits_{i=0}^{m} b_1^i b_2^{m-i}$. Since $b_1^* b_2 = 0$ and $b_2^m = 0$, then $(a^m)^* b_2 = 0$. Since $b_2 b_1 = 0$, then $a^m b_1 (b_1^m)^{\#} = b_1$. Therefore,

$$\begin{aligned} b_1 - a_1 &= b_1 - aa^{\textcircled{D}_m}a = b_1 - aa^{\textcircled{D}_m}b_1 - aa^{\textcircled{D}_m}b_2 = b_1 - a^m(a^{\textcircled{D}_m})^m b_1 - [a^m(a^{\textcircled{D}_m})^m]^* b_2 \\ &= b_1 - a^m(a^{\textcircled{D}_m})^m a^m b_1 (b_1^m)^{\#} = b_1 - a^m b_1 (b_1^m)^{\#} = 0. \end{aligned}$$

Thus, $b_1 = a_1$. Hence the core-EP decomposition of $a$ is unique. □

Next, we exhibit two applications of the core-EP decomposition. On one hand, we give a characterization of the pseudo core inverse by using the core-EP decomposition.

**Theorem 3.3.** *Let $a \in R$ with $a^{\textcircled{D}_m}$ exists and let the core-EP decomposition of $a$ be as in Theorem 3.1. Then $a_1^{\textcircled{\#}} = a^{\textcircled{D}_m}$.*

*Proof.* Suppose $a^{\textcircled{D}_m}$ exists, then $a^{D_m}$ and $(a^m)^{(1,3)}$ exist by Lemma 2.1, as well as
$a^{\textcircled{D}_m}(a_1)^2 = a^{\textcircled{D}_m}(aa^{\textcircled{D}_m}a)^2 = aa^{\textcircled{D}_m}a = a_1$; $a_1(a^{\textcircled{D}_m})^2 = aa^{\textcircled{D}_m}a(a^{\textcircled{D}_m})^2 = a^{\textcircled{D}_m}$;
$a_1 a^{\textcircled{D}_m} = aa^{\textcircled{D}_m}aa^{\textcircled{D}_m} = aa^{\textcircled{D}_m}$, which implies $(a_1 a^{\textcircled{D}_m})^* = a_1 a^{\textcircled{D}_m}$.
We thus get $a_1^{\textcircled{\#}} = a^{\textcircled{D}_m}$. □

On the other hand, we use core-EP decomposition to characterize *-DMP elements.

**Theorem 3.4.** *Let $a \in R$ with $a^{\textcircled{D}_m}$ exists and let the core-EP decomposition of $a$ be as in Theorem 3.1. Then the following are equivalent:*
(1) *$a$ is *-DMP with index $m$;*
(2) *$a_1$ is EP.*

*Proof.* (1) ⇔ (2). $a$ is *-DMP with index $m$ if and only if $a^{\textcircled{D}_m}$ exists with $aa^{\textcircled{D}_m} = a^{\textcircled{D}_m}a$ by Theorem 2.10 (1). According to Theorem 3.3, $a_1^{\textcircled{\#}} = a^{\textcircled{D}_m}$. By a simple calculation, $a_1 a_1^{\textcircled{\#}} = aa_1^{\textcircled{\#}} = aa^{\textcircled{D}_m}$, and $a_1^{\textcircled{\#}} a_1 = a_1^{\textcircled{\#}} a = a^{\textcircled{D}_m}a$. So $aa^{\textcircled{D}_m} = a^{\textcircled{D}_m}a$ is equivalent to $a_1 a_1^{\textcircled{\#}} = a_1^{\textcircled{\#}} a_1$, which is equivalent to, $a_1$ is EP (see [21, Theorem 3.1]). □

**Remark 3.5.** *If $a$ is *-DMP with index $m$. Then the core-EP decomposition of $a$ coincides with its core-nilpotent decomposition. In fact, if $a$ is *-DMP with index $m$, then $a^{\textcircled{D}_m} = a^{D_m}$ by Lemma 2.3. Hence the core-EP decomposition and core-nilpotent decomposition coincide.*

## 4. Core-EP Order

In the following, $R^{\textcircled{\#}}$ and $R^{\textcircled{\scriptsize D}}$ denote the sets of all core invertible and pseudo core invertible elements in $R$, respectively. $R^{\textcircled{\scriptsize D}_m}$ and $R_{\textcircled{\scriptsize D}_m}$ denote the sets of all pseudo core invertible and dual pseudo core invertible elements of index $m$, respectively.

Baksalary and Trenkler [1] introduced the core partial order for complex matrices of index one. Then, Rakić and Djordjević [22] generalized the core partial order from complex matrices to $*$-rings. Let $a, b \in R^{\textcircled{\#}}$, the core partial order $a \overset{\textcircled{\#}}{\leq} b$ was defined as

$$a \overset{\textcircled{\#}}{\leq} b: \ a^{\textcircled{\#}}a = a^{\textcircled{\#}}b \text{ and } aa^{\textcircled{\#}} = ba^{\textcircled{\#}}.$$

In [23], Wang introduced the core-EP order for complex matrices. Let $A, B \in \mathbb{C}^{n\times n}$, the core-EP order $A \overset{\textcircled{\dagger}}{\leq} B$ was defined as

$$A \overset{\textcircled{\dagger}}{\leq} B: \ A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B \text{ and } AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}},$$

where $A^{\textcircled{\dagger}}$ denotes the core-EP inverse [13] of $A$.

One can see [6], [14] for a deep study of the partial order.

In what follows, we generalize the core-EP order from complex matrices to $*$-rings and give some properties.

**Definition 4.1.** *Let $a, b \in R^{\textcircled{\scriptsize D}}$. The core-EP order $a \overset{\textcircled{\scriptsize D}}{\leq} b$ is defined as*

$$a \overset{\textcircled{\scriptsize D}}{\leq} b: \ a^{\textcircled{\scriptsize D}}a = a^{\textcircled{\scriptsize D}}b \text{ and } aa^{\textcircled{\scriptsize D}} = ba^{\textcircled{\scriptsize D}}. \tag{4.1}$$

We extend some results of the core-EP order [23] from matrices to an arbitrary $*$-ring, using a different method. First, we have the following result.

**Theorem 4.2.** *The core-EP order is not a partial order but merely a pre-order.*

*Proof.* It is clear that the core-EP order (4.1) is reflexive. Let $a, b, c \in R^{\textcircled{\scriptsize D}}$, $a \overset{\textcircled{\scriptsize D}}{\leq} b$ and $b \overset{\textcircled{\scriptsize D}}{\leq} c$. Next, we prove $a \overset{\textcircled{\scriptsize D}}{\leq} c$.

Suppose $k = max\{\text{ind}(a), \text{ind}(b)\}$. From $aa^{\textcircled{\scriptsize D}} = ba^{\textcircled{\scriptsize D}}$ and $bb^{\textcircled{\scriptsize D}} = cb^{\textcircled{\scriptsize D}}$, it follows that

$$\begin{aligned} aa^{\textcircled{\scriptsize D}} &= ba^{\textcircled{\scriptsize D}} = ba(a^{\textcircled{\scriptsize D}})^2 = b^2(a^{\textcircled{\scriptsize D}})^2 = b^{k+1}(a^{\textcircled{\scriptsize D}})^{k+1} = bb^{\textcircled{\scriptsize D}}b^{k+1}(a^{\textcircled{\scriptsize D}})^{k+1} = cb^{\textcircled{\scriptsize D}}b^{k+1}(a^{\textcircled{\scriptsize D}})^{k+1} \\ &= cb^k(a^{\textcircled{\scriptsize D}})^{k+1} = cb(a^{\textcircled{\scriptsize D}})^2 = ca^{\textcircled{\scriptsize D}}. \end{aligned}$$

Since $aa^{\textcircled{\scriptsize D}} = ba^{\textcircled{\scriptsize D}}$, then $a^{\textcircled{\scriptsize D}} = a^{\textcircled{\scriptsize D}}(aa^{\textcircled{\scriptsize D}})^* = a^{\textcircled{\scriptsize D}}(ba^{\textcircled{\scriptsize D}})^* = a^{\textcircled{\scriptsize D}}[b^k(a^{\textcircled{\scriptsize D}})^k]^* = a^{\textcircled{\scriptsize D}}[bb^{\textcircled{\scriptsize D}}b^k(a^{\textcircled{\scriptsize D}})^k]^* = a^{\textcircled{\scriptsize D}}[b^k(a^{\textcircled{\scriptsize D}})^k]^*bb^{\textcircled{\scriptsize D}}$. Equalities $a^{\textcircled{\scriptsize D}}a = a^{\textcircled{\scriptsize D}}b$, $b^{\textcircled{\scriptsize D}}b = b^{\textcircled{\scriptsize D}}c$ and $a^{\textcircled{\scriptsize D}} = a^{\textcircled{\scriptsize D}}[b^k(a^{\textcircled{\scriptsize D}})^k]^*bb^{\textcircled{\scriptsize D}}$ yield that $a^{\textcircled{\scriptsize D}}a = a^{\textcircled{\scriptsize D}}b = a^{\textcircled{\scriptsize D}}[b^k(a^{\textcircled{\scriptsize D}})^k]^*bb^{\textcircled{\scriptsize D}}b = a^{\textcircled{\scriptsize D}}[b^k(a^{\textcircled{\scriptsize D}})^k]^*bb^{\textcircled{\scriptsize D}}c = a^{\textcircled{\scriptsize D}}c$.

We thus have $a \overset{\textcircled{\scriptsize D}}{\leq} c$.

However, the core-EP order is not anti-symmetric (see [23, Example 4.1]). □

The following result give some characterizations of the core-EP order, generalizing [23, Theorem 4.2] from matrices to an arbitrary $*$-ring without using matrix decomposition.

**Theorem 4.3.** *Let $a, b \in R^{\textcircled{\scriptsize D}}$ with $k = max\{ind(a), ind(b)\}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions. Then the following are equivalent:*

*(1) $a \overset{\textcircled{\scriptsize D}}{\leq} b$;*
*(2) $a^{k+1} = ba^k$ and $a^*a^k = b^*a^k$;*
*(3) $a_1 \overset{\textcircled{\#}}{\leq} b_1$.*

*Proof.* (1) ⇒ (2). Post-multiply $aa^{\textcircled{D}} = ba^{\textcircled{D}}$ by $a^{k+1}$, then we derive $a^{k+1} = ba^k$. From $a^{\textcircled{D}}a = a^{\textcircled{D}}b$, it follows that $a^*(a^{\textcircled{D}})^* = b^*(a^{\textcircled{D}})^*$. Post-multiply this equality by $a^*a^k$, then $a^*a^k = b^*a^k$.
(2) ⇒ (1). Equality $a^*a^k = b^*a^k$ yields that $(a^k)^*a = (a^k)^*b$. Pre-multiply this equality by $a^{\textcircled{D}}((a^{\textcircled{D}})^k)^*$, then $a^{\textcircled{D}}a = a^{\textcircled{D}}b$. Post-multiply $a^{k+1} = ba^k$ by $(a^{\textcircled{D}})^{k+1}$, then $aa^{\textcircled{D}} = ba^{\textcircled{D}}$.
(1) ⇒ (3). From Theorem 3.3 and $aa^{\textcircled{D}} = ba^{\textcircled{D}}$, it follows that

$$\begin{aligned} a_1a_1^{\textcircled{\#}} &= aa_1^{\textcircled{\#}} = aa^{\textcircled{D}} = ba^{\textcircled{D}} = ba(a^{\textcircled{D}})^2 = b^2(a^{\textcircled{D}})^2 = \cdots = b^k(a^{\textcircled{D}})^k = bb^{\textcircled{D}}b^k(a^{\textcircled{D}})^k \\ &= bb^{\textcircled{D}}ba^{\textcircled{D}} = b_1a_1^{\textcircled{\#}}. \end{aligned}$$

Meanwhile, we have $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}}$ by taking an involution on $aa^{\textcircled{D}} = bb^{\textcircled{D}}ba^{\textcircled{D}} = bb^{\textcircled{D}}aa^{\textcircled{D}}$. So $a^{\textcircled{D}} = a^{\textcircled{D}}bb^{\textcircled{D}}$. Therefore $a_1^{\textcircled{\#}}a_1 = a_1^{\textcircled{\#}}a = a^{\textcircled{D}}a = a^{\textcircled{D}}b = a^{\textcircled{D}}bb^{\textcircled{D}}b = a_1^{\textcircled{\#}}b_1$.
(3) ⇒ (1). Since $aa^{\textcircled{D}} = a_1a_1^{\textcircled{\#}} = b_1a_1^{\textcircled{\#}} = bb^{\textcircled{D}}ba^{\textcircled{D}}$, then

$$\begin{aligned} aa^{\textcircled{D}} &= bb^{\textcircled{D}}baa^{\textcircled{D}}a^{\textcircled{D}} = (bb^{\textcircled{D}}b)^2(a^{\textcircled{D}})^2 = bb^{\textcircled{D}}bb^k(b^{\textcircled{D}})^kb(a^{\textcircled{D}})^2 = b(bb^{\textcircled{D}}ba^{\textcircled{D}})a^{\textcircled{D}} = ba(a^{\textcircled{D}})^2 \\ &= ba^{\textcircled{D}}. \end{aligned}$$

Equalities $aa^{\textcircled{D}} = bb^{\textcircled{D}}ba^{\textcircled{D}}$ and $aa^{\textcircled{D}} = ba^{\textcircled{D}}$ yield that $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}}$. Therefore $a^{\textcircled{D}} = a^{\textcircled{D}}bb^{\textcircled{D}}$. Hence $a^{\textcircled{D}}b = a^{\textcircled{D}}bb^{\textcircled{D}}b = a_1^{\textcircled{\#}}b_1 = a_1^{\textcircled{\#}}a_1 = a^{\textcircled{D}}a$. □

Wang and Chen [25] gave some equivalences to $a \overset{\textcircled{\#}}{\leq} b$ under the assumption that $a$ is EP. Similarly, we give a characterization of $a \overset{\textcircled{D}}{\leq} b$ whenever $a$ is *-DMP. In the following result, $c_a$ and $c_b$ are the core parts of the core-nilpotent decompositions of $a$, $b$ respectively.

**Theorem 4.4.** *Let $a,\ b \in R^{\textcircled{D}}$. If $a$ is *-DMP, then the following are equivalent:*
(1) $a \overset{\textcircled{D}}{\leq} b$;
(2) $c_a \overset{\textcircled{\#}}{\leq} c_b$;
(3) $a^{\textcircled{D}}b^{\textcircled{D}} = b^{\textcircled{D}}a^{\textcircled{D}}$ *and* $a^{\textcircled{D}}b = a^{\textcircled{D}}a$;
(4) $a^{\textcircled{D}} \overset{\textcircled{D}}{\leq} b^{\textcircled{D}}$ *and* $a^{\textcircled{D}}b = a^{\textcircled{D}}a$.

*Proof.* Let $k$ =max{ind($a$), ind($b$)}. If $a$ is *-DMP, then $a^{\textcircled{D}} = a^D$ by Lemma 2.3 and $aa^{\textcircled{D}} = a^{\textcircled{D}}a$ by Theorem 2.10.
(1) ⇒ (2). $a^{\textcircled{D}} = c_a^{\textcircled{\#}}$ (see [7, Theorem 2.9]) and $a^{\textcircled{D}}a = a^{\textcircled{D}}b$ imply $c_a^{\textcircled{\#}}a = c_a^{\textcircled{\#}}b$. From $a^{\textcircled{D}}b = a^{\textcircled{D}}a = aa^{\textcircled{D}} = ba^{\textcircled{D}}$, we have $a^{\textcircled{D}}b^D = b^Da^{\textcircled{D}}$. So, $a^{\textcircled{D}}bb^Db = bb^Dba^{\textcircled{D}} = bb^Db^k(a^{\textcircled{D}})^k = b^k(a^{\textcircled{D}})^k = aa^{\textcircled{D}}$. Therefore $c_a^{\textcircled{\#}}c_b = c_bc_a^{\textcircled{\#}} = c_ac_a^{\textcircled{\#}} = c_a^{\textcircled{\#}}c_a$.
(2) ⇒ (1). $aa^{\textcircled{D}} = c_ac_a^{\textcircled{\#}} = c_bc_a^{\textcircled{\#}} = bb^Dba^{\textcircled{D}} = (bb^Db)^2(a^{\textcircled{D}})^2 = b^2b^Db(a^{\textcircled{D}})^2 = b(bb^Dba^{\textcircled{D}})a^{\textcircled{D}} = baa^{\textcircled{D}}a^{\textcircled{D}} = ba^{\textcircled{D}}$, and $a^{\textcircled{D}}a = c_a^{\textcircled{\#}}c_a = c_a^{\textcircled{\#}}c_b = a^{\textcircled{D}}bb^Db = a^{\textcircled{D}}a^{\textcircled{D}}(bb^Db)^2 = a^{\textcircled{D}}a^{\textcircled{D}}ab = a^{\textcircled{D}}b$.
(1) ⇒ (3). From $a^{\textcircled{D}}a = a^{\textcircled{D}}b$ and $aa^{\textcircled{D}} = ba^{\textcircled{D}}$, it follows that

$$aa^{\textcircled{D}}b = aa^{\textcircled{D}}a = ba^{\textcircled{D}}a = baa^{\textcircled{D}},$$

which forces, by [7, Proposition 4.2], $aa^{\textcircled{D}}b^{\textcircled{D}} = b^{\textcircled{D}}aa^{\textcircled{D}} = b^{\textcircled{D}}b^{k+1}(a^{\textcircled{D}})^{k+1} = b^k(a^{\textcircled{D}})^{k+1} = a^{\textcircled{D}}$. So $a^{\textcircled{D}}b^{\textcircled{D}} = (a^{\textcircled{D}})^2 = b^{\textcircled{D}}a^{\textcircled{D}}$.
(3) ⇒ (1). $ba^{\textcircled{D}} = b(a^{\textcircled{D}})^2a = b(a^{\textcircled{D}})^2b = b(a^{\textcircled{D}})^{k+1}b^k = b(a^{\textcircled{D}})^{k+1}b^{\textcircled{D}}b^{k+1} = bb^{\textcircled{D}}(a^{\textcircled{D}})^{k+1}b^{k+1} = bb^{\textcircled{D}}aa^{\textcircled{D}}$, together with $aa^{\textcircled{D}} = a^{\textcircled{D}}a = a^{\textcircled{D}}b = (a^{\textcircled{D}})^kb^k = (a^{\textcircled{D}})^kb^{\textcircled{D}}b^{k+1} = b^{\textcircled{D}}(a^{\textcircled{D}})^kb^{k+1} = bb^{\textcircled{D}}aa^{\textcircled{D}}$, implies $aa^{\textcircled{D}} = ba^{\textcircled{D}}$.
(3) ⇒ (4). From $a^{\textcircled{D}}b^{\textcircled{D}} = b^{\textcircled{D}}a^{\textcircled{D}}$, it follows that (1) holds and

$$\begin{aligned} (a^{\textcircled{D}})^{\textcircled{D}}a^{\textcircled{D}} &= a^2(a^{\textcircled{D}})^2 = a^2b^k(a^{\textcircled{D}})^{k+2} = a^2b^{\textcircled{D}}b^{k+1}(a^{\textcircled{D}})^{k+2} = a^2b^{\textcircled{D}}a(a^{\textcircled{D}})^2 \\ &= a^2b^{\textcircled{D}}a^{\textcircled{D}} = a^2a^{\textcircled{D}}b^{\textcircled{D}} = (a^{\textcircled{D}})^{\textcircled{D}}b^{\textcircled{D}}, \\ a^{\textcircled{D}}(a^{\textcircled{D}})^{\textcircled{D}} &= a^{\textcircled{D}}a^2a^{\textcircled{D}} = aa^{\textcircled{D}} = b^{\textcircled{D}}a^2a^{\textcircled{D}} = b^{\textcircled{D}}(a^{\textcircled{D}})^{\textcircled{D}}. \end{aligned}$$

(4) ⇒ (3). Since $(a^{\textcircled{D}})^{\textcircled{D}}a^{\textcircled{D}} = (a^{\textcircled{D}})^{\textcircled{D}}b^{\textcircled{D}}$ and $a^{\textcircled{D}}(a^{\textcircled{D}})^{\textcircled{D}} = b^{\textcircled{D}}(a^{\textcircled{D}})^{\textcircled{D}}$, then we obtain $aa^{\textcircled{D}} = a^2a^{\textcircled{D}}b^{\textcircled{D}}$ and $aa^{\textcircled{D}} = b^{\textcircled{D}}a^2a^{\textcircled{D}}$. So $b^{\textcircled{D}}a^{\textcircled{D}} = (a^{\textcircled{D}})^2 = a^{\textcircled{D}}b^{\textcircled{D}}$. □

Wang and Chen [25] proved that if $a \overset{*}{\leq} b$, $a^\dagger$ exists, then $b^\dagger$ exists if and only if $[b(1-aa^\dagger)]^\dagger$ exists. Similarly, we have the following result.

**Theorem 4.5.** *Let $a,\ b \in R^{\textcircled{\tiny D}}$ with $a \overset{\textcircled{\tiny D}}{\leq} b$. Suppose that $a$ is *-DMP. Then $b$ is *-DMP if and only if $b(1-aa^{\textcircled{\tiny D}})$ is *-DMP.*

*Proof.* From $a^{\textcircled{\tiny D}}a = a^{\textcircled{\tiny D}}b$ and $aa^{\textcircled{\tiny D}} = ba^{\textcircled{\tiny D}}$, it follows that

$$aa^{\textcircled{\tiny D}}b = aa^{\textcircled{\tiny D}}a = ba^{\textcircled{\tiny D}}a = baa^{\textcircled{\tiny D}}.$$

Suppose that $b$ is *-DMP, then $bb^{\textcircled{\tiny D}} = b^{\textcircled{\tiny D}}b$. Next, we prove $[b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}} = b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}}$. In fact, suppose $\mathrm{ind}(b) = k$, then

$$\begin{aligned}(b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}})[b(1-aa^{\textcircled{\tiny D}})]^{k+1} &= (b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}})b^{k+1}(1-aa^{\textcircled{\tiny D}}) = b^k(1-aa^{\textcircled{\tiny D}}) - a^{\textcircled{\tiny D}}b^{k+1}(1-aa^{\textcircled{\tiny D}})\\ &= b^k(1-aa^{\textcircled{\tiny D}}) = [b(1-aa^{\textcircled{\tiny D}})]^k;\end{aligned}$$

$b(1-aa^{\textcircled{\tiny D}})(b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}}) = bb^{\textcircled{\tiny D}} - aa^{\textcircled{\tiny D}}$;
$b(1-aa^{\textcircled{\tiny D}})(b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}})^2 = (bb^{\textcircled{\tiny D}} - aa^{\textcircled{\tiny D}})(b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}}) = b^{\textcircled{\tiny D}} - b^{\textcircled{\tiny D}}aa^{\textcircled{\tiny D}} = b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}}$.
We thus have $[b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}} = b^{\textcircled{\tiny D}} - a^{\textcircled{\tiny D}}$.
So, $b(1-aa^{\textcircled{\tiny D}})[b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}} = bb^{\textcircled{\tiny D}} - aa^{\textcircled{\tiny D}}$ and $[b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}}b(1-aa^{\textcircled{\tiny D}}) = b^{\textcircled{\tiny D}}b - b^{\textcircled{\tiny D}}baa^{\textcircled{\tiny D}} = bb^{\textcircled{\tiny D}} - aa^{\textcircled{\tiny D}}$.
Therefore, $b(1-aa^{\textcircled{\tiny D}})[b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}} = [b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}}b(1-aa^{\textcircled{\tiny D}})$. Hence $b(1-aa^{\textcircled{\tiny D}})$ is *-DMP.

Conversely, suppose that $b(1-aa^{\textcircled{\tiny D}})$ is *-DMP. Then, $[b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}} = [b(1-aa^{\textcircled{\tiny D}})]^D$. We can easily check that

$$(baa^{\textcircled{\tiny D}})^{\textcircled{\tiny D}} = (baa^{\textcircled{\tiny D}})^{\textcircled{\tiny \#}} = (baa^{\textcircled{\tiny D}})^{\#} = a^{\textcircled{\tiny D}}.$$

Since $b = b(1-aa^{\textcircled{\tiny D}}) + baa^{\textcircled{\tiny D}}$, $[b(1-aa^{\textcircled{\tiny D}})]baa^{\textcircled{\tiny D}} = b(1-aa^{\textcircled{\tiny D}})aa^{\textcircled{\tiny D}}b = 0$, $baa^{\textcircled{\tiny D}}[b(1-aa^{\textcircled{\tiny D}})] = baa^{\textcircled{\tiny D}}(1-aa^{\textcircled{\tiny D}})b = 0$, and $(baa^{\textcircled{\tiny D}})^*b(1-aa^{\textcircled{\tiny D}}) = b^*aa^{\textcircled{\tiny D}}(1-aa^{\textcircled{\tiny D}})b = 0$, then $b^{\textcircled{\tiny D}} = [b(1-aa^{\textcircled{\tiny D}})]^{\textcircled{\tiny D}} + a^{\textcircled{\tiny D}}$ (see [7, Theorem 4.4]) and $b^D = [b(1-aa^{\textcircled{\tiny D}})]^D + (baa^{\textcircled{\tiny D}})^{\#} = [b(1-aa^{\textcircled{\tiny D}})]^D + a^{\textcircled{\tiny D}}$. Thus, $b$ is *-DMP. □

## 5. Characterizations for $aa^{\textcircled{\tiny D}} = bb^{\textcircled{\tiny D}}$

Let $a,\ b \in R$. If $a^{\odot}$ and $b^{\odot}$ are some kind of generalized inverses of $a$ and $b$. It is very interesting to discuss when $aa^{\odot} = bb^{\odot}$. Koliha et al. [11, Theorem 6.1], Mosić et al. [17, Theorem 3.7] and Patrício et al. [18, Theorem 2.3] gave some equivalences for generalized Drazin inverses, image-kernel $(p,q)$-inverses and Moore-Penrose inverses, respectively. Here we give a characterization for $aa^{\textcircled{\tiny D}} = bb^{\textcircled{\tiny D}}$.

**Proposition 5.1.** *Let $a,\ b \in R^{\textcircled{\tiny D}}$. Then the following are equivalent:*
(1) $aa^{\textcircled{\tiny D}} = bb^{\textcircled{\tiny D}}aa^{\textcircled{\tiny D}}$;
(2) $aa^{\textcircled{\tiny D}} = aa^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$;
(3) $a^{\textcircled{\tiny D}} = a^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$;
(4) $Ra^{\textcircled{\tiny D}} \subseteq Ra^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$.

*Proof.* (1) ⇔ (2) by taking an involution.
(2) ⇒ (3). Pre-multiply $aa^{\textcircled{\tiny D}} = aa^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$ by $a^{\textcircled{\tiny D}}$, then we get $a^{\textcircled{\tiny D}} = a^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$.
(3) ⇒ (4) is clear.
(4) ⇒ (2). From $Ra^{\textcircled{\tiny D}} \subseteq Ra^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$, it follows that $a^{\textcircled{\tiny D}} = xa^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$ for some $x \in R$. Then, $aa^{\textcircled{\tiny D}} = axa^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}} = (axa^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}})bb^{\textcircled{\tiny D}} = aa^{\textcircled{\tiny D}}bb^{\textcircled{\tiny D}}$. □

The above proposition gives some equivalences to $aa^{\textcircled{\tiny D}} = bb^{\textcircled{\tiny D}}aa^{\textcircled{\tiny D}}$, which enrich the following result. $R^{-1}$ denotes the set of all invertible elements in $R$.

**Theorem 5.2.** *Let $a,\ b \in R^{\textcircled{D}}$ with $ind(a) = m$. Then the following are equivalent:*
(1) $aa^{\textcircled{D}} = bb^{\textcircled{D}}$;
(2) $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}}$ *and* $u = aa^{\textcircled{D}} + 1 - bb^{\textcircled{D}} \in R^{-1}$;
(3) $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}}$ *and* $v = a^m + 1 - bb^{\textcircled{D}} \in R^{-1}$;
(4) $aa^{\textcircled{D}}$ *commutes with* $bb^{\textcircled{D}}$, $u = aa^{\textcircled{D}} + 1 - bb^{\textcircled{D}} \in R^{-1}$ *and* $s = bb^{\textcircled{D}} + 1 - aa^{\textcircled{D}} \in R^{-1}$;
(5) $aa^{\textcircled{D}}$ *commutes with* $bb^{\textcircled{D}}$ *and* $w = 1 - (aa^{\textcircled{D}} - bb^{\textcircled{D}})^2 \in R^{-1}$;
(6) $aa^{\textcircled{D}}$ *commutes with* $bb^{\textcircled{D}}$ *and* $b^{\textcircled{D}}aa^{\textcircled{D}} - a^{\textcircled{D}}bb^{\textcircled{D}} = b^{\textcircled{D}} - a^{\textcircled{D}}$.

*Proof.* (1) $\Rightarrow$ (2)-(6) is clear.
(2)$\Leftrightarrow$(3). Since $a^{\textcircled{D}_m}$ exists, then $a^{D_m}$ exists by Lemma 2.1. So $(a^m)^\#$ exists. Therefore $a^m + 1 - aa^{\textcircled{D}_m} \in R^{-1}$ (see [20, Theorem 1]). From $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}}$, it follows that $aa^{\textcircled{D}}bb^{\textcircled{D}} = bb^{\textcircled{D}}aa^{\textcircled{D}} = aa^{\textcircled{D}}$ by Proposition 5.1. Observe that $(aa^{\textcircled{D}} + 1 - bb^{\textcircled{D}})(a^m + 1 - aa^{\textcircled{D}}) = a^m + 1 - bb^{\textcircled{D}}$, and hence $u \in R^{-1}$ if and only if $v \in R^{-1}$.
(3)$\Rightarrow$(1). Notice that $aa^{\textcircled{D}}v = a^m + aa^{\textcircled{D}} - aa^{\textcircled{D}}bb^{\textcircled{D}} = a^m$ and $bb^{\textcircled{D}}v = bb^{\textcircled{D}}a^m = bb^{\textcircled{D}}aa^{\textcircled{D}}a^m = aa^{\textcircled{D}}a^m = a^m$. Therefore $aa^{\textcircled{D}} = bb^{\textcircled{D}}$.
(4)$\Rightarrow$(1). Since $ubb^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}} = uaa^{\textcircled{D}}bb^{\textcircled{D}}$, $saa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}} = saa^{\textcircled{D}}bb^{\textcircled{D}}$. Hence $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}} = bb^{\textcircled{D}}$.
(5)$\Rightarrow$(4). Note that $1 - (aa^{\textcircled{D}} - bb^{\textcircled{D}})^2 = (bb^{\textcircled{D}} + 1 - aa^{\textcircled{D}})(aa^{\textcircled{D}} + 1 - bb^{\textcircled{D}}) = (aa^{\textcircled{D}} + 1 - bb^{\textcircled{D}})(bb^{\textcircled{D}} + 1 - aa^{\textcircled{D}})$. Hence $w \in R^{-1}$ implies $u,\ s \in R^{-1}$.
(6)$\Rightarrow$(1). Post-multiply $b^{\textcircled{D}}aa^{\textcircled{D}} - a^{\textcircled{D}}bb^{\textcircled{D}} = b^{\textcircled{D}} - a^{\textcircled{D}}$ by $aa^{\textcircled{D}}$, then $b^{\textcircled{D}}aa^{\textcircled{D}} - a^{\textcircled{D}}bb^{\textcircled{D}}aa^{\textcircled{D}} = b^{\textcircled{D}}aa^{\textcircled{D}} - a^{\textcircled{D}}$. So, $a^{\textcircled{D}} = a^{\textcircled{D}}bb^{\textcircled{D}}aa^{\textcircled{D}} = a^{\textcircled{D}}bb^{\textcircled{D}}$. Therefore, $b^{\textcircled{D}} = b^{\textcircled{D}}aa^{\textcircled{D}}$. Hence $aa^{\textcircled{D}} = aa^{\textcircled{D}}bb^{\textcircled{D}} = bb^{\textcircled{D}}aa^{\textcircled{D}} = bb^{\textcircled{D}}$. $\square$

Take $b = a^*$ in Theorem 5.2, then we obtain a characterization of *-DMP elements by applying Theorem 2.9.

**Corollary 5.3.** *Let $a \in R^{\textcircled{D}_m} \cap R_{\textcircled{D}_m}$. Then the following are equivalent:*
(1) *$a$ is *-DMP with index $m$;*
(2) $aa^{\textcircled{D}_m} = a_{\textcircled{D}_m}a$;
(3) $aa^{\textcircled{D}_m} = aa^{\textcircled{D}_m}a_{\textcircled{D}_m}a$ *and* $u = aa^{\textcircled{D}_m} + 1 - a_{\textcircled{D}_m}a \in R^{-1}$;
(4) $aa^{\textcircled{D}_m} = aa^{\textcircled{D}_m}a_{\textcircled{D}_m}a$ *and* $v = a^m + 1 - a_{\textcircled{D}_m}a \in R^{-1}$;
(5) $aa^{\textcircled{D}_m}$ *commutes with* $a_{\textcircled{D}_m}a$, $u = aa^{\textcircled{D}_m} + 1 - a_{\textcircled{D}_m}a \in R^{-1}$ *and* $s = a_{\textcircled{D}_m}a + 1 - aa^{\textcircled{D}_m} \in R^{-1}$;
(6) $aa^{\textcircled{D}_m}$ *commutes with* $a_{\textcircled{D}_m}a$ *and* $w = 1 - (aa^{\textcircled{D}_m} - a_{\textcircled{D}_m}a)^2 \in R^{-1}$;
(7) $aa^{\textcircled{D}_m}$ *commutes with* $a_{\textcircled{D}_m}a$ *and* $a^*_{\textcircled{D}_m}aa^{\textcircled{D}_m} - a^{\textcircled{D}_m}a_{\textcircled{D}_m}a = a^*_{\textcircled{D}_m} - a^{\textcircled{D}_m}$.

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