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T_0-Reflection and Some Separation Axioms in PRETOP

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Abstract. We give the T_0-reflection in the category of pretopological spaces with p-continuous maps as arrows. After that we will study some separation axioms in this category.

1. Introduction

The notion of pretopological spaces represent a generalization of topological spaces. Those spaces give larger variety of properties for sets. Today, pretopological spaces are applied in different areas like complex modelling, image analysis, graph theory and economical modelling (for more information see [3, 4, 11]).

The construction of the T_0-reflection in the category **TOP** is given by Herrlich and Strecker in [7]. After that, some authors have been interested in the T_0-reflection in other categories as a generalization. Künzi and Richmond considered the category **PREORDTOP** whose objects are preorder-topological spaces $(X, \tau, \leq)$ and continuous increasing maps as arrows [8]. Mirhosseinkhani in [11] gave the T_0-reflection in the category **GenTOP** with objects generalized topological spaces and arrows g-continuous maps. As a continuation of this work, in this paper we consider the construction of the T_0-reflection in the category **PreTOP**.

In the second section we introduce some preliminary results in order to define the category **PreTOP** with p-continuous maps as arrows.

In the third section, we give the construction of the T_0-reflection in **PreTOP**. Morphisms rendered invertible by this reflector are given. Finally, in the fourth section we investigate some new separation axioms in the category **PreTOP**. Some interesting results in [2] are deduced.

2. Preliminary Results

Definition 2.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A pseudo-closure on X is a map a from $\mathcal{P}(X)$ onto itself such that:

$a(\emptyset) = \emptyset,$

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$A \subseteq a(A)$ for all $A \in \mathcal{P}(X)$.

The couple (X, a) is called a pretopological space.

Examples 2.2. (1) Let X be a topological space and a the operator from $\mathcal{P}(X)$ to itself defined by $a(A) = \overline{A}$ (Kuratowski operator). (X, a) is a pretopological space. Note that a pretopological space (X, a) is a topological space if, additionally, a preserves binary unions and is idempotent.

(2) Consider $X = \mathbb{R}$ and a the operator from $\mathcal{P}(\mathbb{R})$ to itself defined by

$a(\emptyset) = \emptyset$,

$a([x, y]) = [x, y + 1]$ for all $x \leq y \in \mathbb{R}$,

$a(A) = \mathbb{R}$ if $A \neq [x, y]$ for any $x \leq y$ in $\mathbb{R}$ and $A \neq \emptyset$.

(X, a) is a pretopological space which is not a topological space since a is not idempotent.

Now, we will present the category of pretopological spaces but, first, let us start with some definitions and notations.

Definition 2.3. Let (X, a) be a pretopological space and $A \in \mathcal{P}(X)$.

A is called *p-closed* if there exists $B \in \mathcal{P}(X)$ such that $A = a(B)$,

A is called *p-open* if A^c is *p-closed.*

We denote by $\mathcal{PC}((X, a))$ (respectively, $\mathcal{PO}((X, a))$) the set of all p-closed (respectively, p-open) subsets of X.

Notation 2.4. Let (X, a) be a pretopological space.
We denote by i the pseudo-interior in (X, a) defined by:

$$i(A) = (a(A^c))^c \text{ for all } A \in \mathcal{P}(X).$$

Proposition 2.5. *Let (X, a) be a pretopological space and $A \in \mathcal{P}(X)$.*
A is p-open if and only if there exists $B \in \mathcal{P}(X)$ such that $A = i(B)$.

Proof. If A is p-open then there exists $B \in \mathcal{P}(X)$ such that $A^c = a(B)$ and then $A = i(B^c)$. Conversely, if $A = i(B)$ for some $B \in \mathcal{P}(X)$ then $A = (a(B^c))^c$ and $A^c = a(B^c)$ so that A^c is p-closed which implies that A is p-open. □

Definition 2.6. Let (X, a) and (Y, b) be two pretopological spaces and f a map from X to Y. f is called *p-continuous* if $f^{-1}(A) \in \mathcal{PO}((X, a))$ for all $A \in \mathcal{PO}((Y, b))$.

The following result is immediate.

Proposition 2.7. *Let (X, a) and (Y, b) be two pretopological spaces and f a map from X to Y. f is p-continuous, if and only if, $f^{-1}(A) \in \mathcal{PC}((X, a))$ for all $A \in \mathcal{PC}((Y, b))$.*

Let (X, a) be a pretopological space. For all $x \in X$ we denote by $pc(x)$ the intersection of all p-closed subsets of X containing x and we call it the *preclosure* of x. More generally, given a subset A of X, we define the preclosure of A, denoted by $pc(A)$, to be the intersection of all p-closed subsets of X containing A.

Now, we introduce and characterize some separation axioms in the category **PreTOP**.

Proposition 2.8. *Let (X, a) be a pretopological space. The following statements are equivalent:*

1. $pc(x) = pc(y) \implies x = y$.
2. *If $x \neq y$ then there exists a p-closed subset of X containing one of the points x, y and not the other.*
3. *If $x \neq y$ then there exists a p-open subset of X containing one of the points x, y and not the other.*

Definition 2.9. A pretopological space satisfying one of the previous equivalent statements is called T_0-*pretopological*.

Definition 2.10. A pretopological space (X, a) is called a T_D-*pretopological* space if for all $x \in X$ we have $pc(x) \setminus \{x\}$ is p-closed.

Definition 2.11. A pretopological space (X, a) is said to be T_1-*pretopological* if for all distinct points x and y, there exists a p-open set containing x which does not contain y.

The following proposition characterizes T_1-pretopological spaces.

Proposition 2.12. *A pretopological space (X, a) is T_1-pretopological if and only if $pc(x) = \{x\}$ for all $x \in X$.*

Remark 2.13. In the category **TOP**, a topological space is T_1 if and only if every point is closed. In **PreTOP**, it is clear that if $\{x\}$ is p-closed for any $x \in X$, then (X, a) is a T_1-pretopological space. The following example shows that the converse is not true.

Let $X = \{1, 2, 3\}$ and a the precolsure on A defined by:
$a(\emptyset) = \emptyset$, $a(\{1\}) = \{1, 2\}$, $a(\{2\}) = \{2, 3\}$, $a(\{3\}) = \{1, 3\}$ else $a(A) = X$.
Then (X, a) is a T_1 pretopological space but $\{1\}, \{2\}$ and $\{3\}$ are not p-closed subsets.

Regarding the previous remark, we define a new pretopological space as follow.

Definition 2.14. A pretopological space (X, a) is called a T_1^k-*pretopological* space if for all $x \in X$, $\{x\}$ is p-closed.

Finally, we close this section by giving the definition of T_2-*pretopological* spaces.

Definition 2.15. A pretopological space (X, a) is said to be a T_2-pretopological space if for all distinct points x and y, there exist two disjoint p-open subsets A and B such that $x \in A$ and $y \in B$.

Remark 2.16. Let (X, a) be a pretopological space.
1. (X, a) is a T_1^k-pretopological space if and only if for any $x \in X$, $a(\{x\}) = \{x\}$.
2. If a is a Kuratowski closure operator (that is, furthermore, $a(A \cup B) = a(A) \cap a(B)$ and $a(a(A)) = a(A)$ for all $A, B \subseteq X$), then $T_1^k = T_1$.
3. It is clear that in the category **PreTOP**, we have the following implications:

$$T_2 \Longrightarrow T_1^k \Longrightarrow T_1 \Longrightarrow T_D \Longrightarrow T_0.$$

4. A T_1^k-pretopological space need not be a T_2-pretopological space. For this consider a set X with cardinality greater than or equal to 3 and define a on X by:
$a(\{x\}) = \{x\}$ for any $x \in X$, $a(\emptyset) = \emptyset$ and $a(A) = X$ if not. Then by definition X is a T_1^k-pretopological space which is not T_2-pretopological.

3. T_0-Reflection

First, let us denote by **PreTop**$_0$, the full subcategory of **PreTop** whose objects are T_0-pretopological spaces.

The main goal of this paper is to construct the T_0 reflection of a pretopological space (X, a). For this reason consider the equivalence relation on X defined by:

$$x \sim y \text{ if and only if } pc(x) = pc(y).$$

Let us denote by μ_X the canonical surjection from X to $X/\sim$. We define the map $\tilde{a}$ from $\mathcal{P}(X/\sim)$ to itself by:
$\tilde{a}(A) = A$ if $\mu_X^{-1}(A)$ is a p-closed subset in X and $\tilde{a}(A) = X/\sim$ if not.

One can see easily that with this construction $(X/\sim, \tilde{a})$ is a pretopological space and μ_X is a p-continuous map from (X, a) to $(X/\sim, \tilde{a})$.

Theorem 3.1. *$({}^{X}/_{\sim}, \tilde{a})$ is a T_0-pretopological space.*

Proof. We start by showing that : $\mu_X^{-1}(\mu_X(a(A))) = a(A)$.

It is clear that $a(A) \subseteq \mu_X^{-1}(\mu_X(a(A)))$. Conversely, let $x \in \mu_X^{-1}(\mu_X(a(A)))$ then there exists $y \in a(A)$ such that $\mu_X(x) = \mu_X(y)$ so that $pc(x) = pc(y)$ and thus $x \in pc(y) \subset a(A)$. Therefore, $x \in a(A)$.

If $\mu_X(x) \neq \mu_X(y)$, then there exists $A \subseteq X$ such that, after possible relabeling of x and y, $x \in a(A)$ and $y \notin a(A)$. Using the previous result we can see that

$\mu_X(x) \in \mu_X(a(A)) = \tilde{a}(\mu_X(a(A)))$ but $\mu_X(y) \notin \mu_X(a(A)) = \tilde{a}(\mu_X(a(A)))$. □

Theorem 3.2. ***PreTop**$_0$ is reflective in **PreTop**.*

Proof. It is sufficient to prove that for any pretopological space (X, a), $({}^{X}/_{\sim}, \tilde{a})$ is the T_0 reflection of (X, a).

For this, using the characterization given by MacLane in [10, page 89], we must prove that for every T_0-pretopological space and every p-continuous map from (X, a) to (Y, b) there exists a unique p-continuous map $\tilde{f}$ rending the following diagram commutative.

$$\begin{array}{ccc} (X,a) & \xrightarrow{\mu_X} & ({}^{X}/_{\sim}, \tilde{a}) \\ & {\scriptstyle f}\searrow \quad \triangledown \quad \swarrow{\scriptstyle \tilde{f}} & \\ & (Y,b) & \end{array}$$

Uniqueness:

Clearly, if $\tilde{f}$ exist then it is unique and naturally defined by $\tilde{f}(\mu_X(x)) = f(x)$.

$\tilde{f}$ is well defined:

Suppose $x, y \in X$ and $\mu_X(x) = \mu_X(y)$. If $f(x) \neq f(y)$ then there exists a p-closed subset F containing for example $f(x)$ and not containing $f(y)$ so that $f^{-1}(F)$ is a p-closed subset containing x and not containing y which is impossible, so that $f(x) = f(y)$.

$\tilde{f}$ is p-continuous :

Let F be a p-closed subset in (Y, b). Then $\mu_X^{-1}\left(\tilde{f}^{-1}(F)\right) = f^{-1}(F)$ is a p-closed subset in (X, a). Since f is p-continuous, then by the construction of $\tilde{a}$, $\tilde{a}\left(\tilde{f}^{-1}(F)\right) = \tilde{f}^{-1}(F)$ which is p-closed. □

Now, using the definition of the T_0-reflection of a pretopological space, for any p-continuous map f from (X, a) to (Y, b) there exists a unique p-continuous map $T_0(f)$ from $T_0((X, a)) = ({}^{X}/_{\sim}, \tilde{a})$ to $T_0((Y, b)) = \left({}^{Y}/_{\sim}, \tilde{b}\right)$ making commutative the following diagram.

$$\begin{array}{ccc} (X,a) & \xrightarrow{f} & (Y,b) \\ {\scriptstyle \mu_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_Y} \\ T_0((X,a)) & \xrightarrow[T_0(f)]{} & T_0((Y,b)) \end{array}$$

The notion of quasihomeomorphisms between topological spaces was introduced for the first time by Grothendieck and Dieudonné in [5] to answer some problems in algebraic topology.

Now, let us introduce the notion of a quasihomeomorphism between two pretopological spaces.

Definition 3.3. Let $f : (X, a) \longrightarrow (Y, b)$ be a p-continuous map between two pretopological spaces. f is said to be a *quasihomeomorphism* if the correspondence $b(A) \longmapsto f^{-1}(b(A))$ defines a bijection from $\mathcal{PC}((Y, b))$ (respectively, $\mathcal{PO}((Y, b))$) to $\mathcal{PC}((X, a))$ (respectively, $\mathcal{PO}((X, a))$.

Clearly every homeomorphism between pretopological spaces is a quasihomeomorphism. The converse is not true as shown by the following example.

Example 3.4. Let $X = \{1, 2, 3, 4\}$ and a, b the two pseudo-closures defined by:

$$a(\emptyset) = \emptyset,$$

$$a(\{1,2\}) = \{1,2\},$$

$$a(\{1,2,3\}) = \{1,2,3\},$$

$$a(A) = X \ \forall A \in \mathcal{P}(X) \setminus \left\{\emptyset, \{1,2\}, \{1,2,3\}\right\}$$

and

$$b(\emptyset) = \emptyset,$$

$$b(\{2,3\}) = \{2,3\},$$

$$b(\{1,2,3\}) = \{1,2,3\},$$

$$b(A) = X \ \forall A \in \mathcal{P}(X) \setminus \left\{\emptyset, \{2,3\}, \{1,2,3\}\right\}.$$

Let $f : (X, a) \longrightarrow (X, b)$ be defined by $f(1) = f(2) = 2$, $f(3) = 1$ and $f(4) = 4$.
f is a quasi-homeomorphism but is not a homeomorphism.

Remark 3.5. (Never two without three) Let $f : (X, a) \longrightarrow (Y, b)$ and $g : (Y, b) \longrightarrow (Z, d)$ be two p-continuous maps. If two of the three maps f, g, $g \circ f$ are quasihomeomorphisms then so is the third one.

Proposition 3.6. *μ_X is a quasihomemorphism.*

Proof. It is sufficient to use $\mu_X^{-1}(\mu_X(a(A))) = a(A)$. □

Proposition 3.7. *Let $f : (X, a) \longrightarrow (Y, b)$ be a quasihomemorphism.*

1. *If (X, a) is a T_0-pretopological space then f is one to one.*
2. *If (Y, b) is a T_1^k-pretopological space then f is onto.*
3. *If (X, a) is a T_0-pretopological space and (Y, b) is a T_1^k-pretopological space then f is a homeomorphism.*

Proof. 1. Let $x, y \in X$ such that $f(x) = f(y)$. Suppose that $x \neq y$. Since (X, a) is a T_0 preordered topological space then there exists a subset $A \subseteq X$ such that $x \in a(A)$ and $y \notin a(A)$. Let $B \subseteq Y$ such that $f^{-1}(b(B)) = a(A)$. Now, $f(x) \in b(B)$ and thus $f(y) \in b(B)$ which implies that $y \in a(A)$ which leads to a contradiction.

2. Let $y \in Y$. Since (Y, b) is a T_1^k-pretopological space, $\{y\} = a(\{y\})$ is a non empty p-closed subset of Y. Now, since f is a quasihomeomorphism, then $f^{-1}(\{y\})$ is a non empty p-closed subset of X. Therefore, f is onto.

3. An immediate consequence of (1) and (2). □

Remark 3.8. In [2, Lemma 3.7], the authors showed that given a quasihomeomorphism $q : X \longrightarrow Y$ between two topological spaces, if Y is a T_D-space, then q is onto. The following example shows that this result is not true in the category **PreTOP** even if Y is a T_1 pretopological space.

Indeed, let $X = \{1,2,3\}$ and $Y = \{1,2,3,4\}$.

Define a on X by $a(\emptyset) = \emptyset$, $a(\{1\}) = \{1\}$, $a(\{2\}) = \{2\}$, $a(\{3\}) = X$, $a(\{2,3\}) = \{2,3\}$, $a(\{1,2\}) = \{1,2\}$, $a(\{1,3\}) = \{1,3\}$ and finally $a(X) = X$.

Define b on Y by, $b(\emptyset) = \emptyset$, $b(\{2,3\}) = \{2,3\}$, $b(\{1,2\}) = \{1,2\}$, $b(\{1,3\}) = \{1,3\}$, $b(\{1,4\}) = \{1,4\}$, $b(\{4,2\}) = \{4,2\}$ and $b(A) = X$ if not.

Clearly, (Y, b) is a T_1-pretopological space.

Define q from X to Y by $q(x) = x$ for every x. One can see easily that q is a quasihomeomorphism which is not onto.

Definition 3.9. Let $f : (X, a) \longrightarrow (Y, b)$ be a p-continuous map. f is called *p-onto* if for any $y \in Y$ there exists $x \in X$ such that $pc(y) = pc(f(x))$.

For any functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ between two categories, the family of all arrows in **C** rendered invertible by F has some important applications. Note that, the class $D^{\perp}$ (the class of morphisms orthogonal to all X in D) is the collection of all morphisms of C rendered invertible by the functor F (i.e. $D^{\perp} = \{f \in hom_C : F(f) \text{ is an isomorphism of } D\}$)[1, Proposition 2.3]. The following result characterizes morphisms in **PreTop** rendered invertible by the functor $\mathbf{T_0}$.

Proposition 3.10. *Let $f : (X, a) \longrightarrow (Y, b)$ be a p-continuous map. Then, the following statements are equivalent:*

1. *f is a p-onto quasihomeomorphism;*
2. *$T_0(f)$ is a homeomorphism.*

Proof. (**1**) ⇒ (**2**) Since $T_0(f) \circ \mu_X = (\mu_Y \circ f)$, then using Remark 3.5, $T_0(f)$ is quasihomeomorphism.
$\mathbf{T_0}$ (f) is onto:
Let $\mu_Y(y) \in T_0((Y, b))$. Since f is p-onto then there is $x \in X$ such that $pc(f(x)) = pc(y)$ and then:
$\mu_Y(y) = \mu_Y(f(x)) = T_0(f)(\mu_X(x))$, so $T_0(f)$ is onto.
$\mathbf{T_0}$ (f) is one to one:
Let $\mu_X(x_1), \mu_X(x_2) \in T_0((X, a))$ such that $T_0(f)(\mu_X(x_1))) = T_0(f)(\mu_X(x_2)))$. Then,
$(T_0(f) \circ \mu_X)(x_1) = (T_0(f) \circ \mu_X)(x_2)$
$\Longrightarrow (\mu_Y \circ f)(x_1) = (\mu_Y \circ f)(x_2)$
$\Longrightarrow pc(f(x_1)) = pc(f(x_2))$
$\Longrightarrow pc(x_1) = pc(x_2)$ (*Since f is a quasihomeomorphism*)
$\Longrightarrow \mu_X(x_1) = \mu_X(x_2)$.
Then $T_0(f)$ is one to one.
As a bijective quasihomeomorphism, $T_0(f)$ is a homeomorphism.
(**2**) ⇒ (**1**) Since $\mu_X, \mu_Y, T_0(f)$ are quasihomeomorphisms and $T_0(f) \circ \mu_X = \mu_Y \circ f$ then, by Remark 3.5, f is a quasihomeomorphism.
Now to conclude it is sufficient to see that $T_0(f)$ is onto if and only if f is p-onto. □

Corollary 3.11. ([2, Theorem 2.4]) *Let $q : X \longrightarrow Y$ be a continuous map between two topological spaces. Then the following statements are equivalent:*

(i) q is a topologically onto quasihomeomorphism;

(ii) $T_0(q)$ is a homeomorphism.

Proof. It is sufficient to take the closure operator on X (respectively, on Y) and apply Proposition 3.10. □

4. Some New Separation Axioms

New separation axioms are presented in [2] in the category of Topological spaces. After that this concept is studied in different categories such as the category of ordered topological spaces in [8, 9] and the category of generalized topological spaces in [11].

The definition given in [2] can be stated in the category **PreTop**. Hence, a pretopological space is said to be a $T_{(0,1)}$ (respectively, $T_{(0,2)}, T_{(0,D)}, T_{(0,1^k)}$)-pretopological space if its T_0-reflection is a T_1 (respectively, T_2, T_D, T_1^k)-pretopological space. In this part we will find some characterizations for such spaces. Regarding topological spaces as a particular case of pretopological spaces, all results given next are a generalization of some results given in [2] in the case of topological spaces. Note that if there is no confusion we can write $T_{(0,i)}$-space instead of $T_{(0,i)}$-pretopological space.

By Remark 2.16 (3), we can deduce easily the following result:

Proposition 4.1. $T_{(0,2)} \Longrightarrow T_{(0,1^k)} \Longrightarrow T_{(0,1)} \Longrightarrow T_{(0,D)}$.

Let us start by characterizing $T_{(0,D)}$-pretopological spaces.

Theorem 4.2. *Let (X, a) be a pretopological space. Then the following statements are equivalent:*

1. *(X, a) is a $T_{(0,D)}$-pretopological space;*
2. *For all $x \in X$ the subset $pc(x) \setminus \{y \mid pc(y) = pc(x)\}$ is p-closed.*

Proof. It is sufficient to see that

$$\mu_X^{-1}(pc(\mu_X(x)) \setminus \{\mu_X(x)\}) = pc(x) \setminus \{y \mid pc(y) = pc(x)\}.$$

□

Let X be a topological space and a its closure operator. Then (X, a) is a pretopological space satisfying $pc(x) = \overline{\{x\}}$ and thus $pc(x) \setminus \{y \mid pc(y) = pc(x)\}$ is exactly the subset $\gamma(X)$ defined in [2, Remark 3.2]. Hence the following corollary is straightforward.

Corollary 4.3. ([2, Theorem 3.3]) *Let X be a topological space. Then the following statements are equivalent:*

1. *X is a $T_{(0,D)}$ space;*
2. *For each $x \in X$, $\gamma(X)$ is a closed subset of X.*

Now, we give the characterization of $T_{(0,1)}$-pretopological spaces.

Theorem 4.4. *Let (X, a) be a pretopological space. The following statements are equivalent :*

1. *(X, a) is a $T_{(0,1)}$-pretopological space.*
2. *For each $x, y \in X$ such that $pc(x) \neq pc(y)$, there is a p-closed subset containing x which does not contain y.*
3. *For any $x, y \in X$, $x \in pc(y) \Longrightarrow y \in pc(x)$.*
4. *For each $x \in X$ and each subset $A \subseteq X$, if $pc(x) \cap a(A) \neq \emptyset$ then $x \in a(A)$.*
5. *For each subset $A \subseteq X$, if $x \in i(A)$ then $pc(x) \subseteq i(A)$.*

Proof. (1) ⇒ (2) : Suppose $x, y \in X$ and $pc(x) \neq pc(y)$. Then $\mu_X(x) \neq \mu_X(y)$. Since $({}^X/_\sim, \tilde{a})$ is T_1, then there exists a p-closed set $\tilde{a}(B)$ containing $\mu_X(x)$ and not containing $\mu_X(y)$, so that $\mu_X^{-1}(\tilde{a}(B))$ is a p-closed subset of (X, a) which contain x and does not contain y.

(2) ⇒ (3) : Let $x, y \in X$ such that $x \in pc(y)$. Then x belongs to every p-closed subset of X containing y. Now, suppose that $y \notin pc(x)$. Then $pc(x) \neq pc(y)$ and, by (2), there exists a p-closed subset of X containing y not containing x, which leads to a contradiction.

(3) ⇒ (4) : Let x be a point in X and A be a subset of X such that $pc(x) \cap a(A) \neq \emptyset$. If $y \in pc(x) \cap a(A)$, then $y \in pc(x)$ and thus by (3), $x \in pc(y)$ which means that x belongs to every p-closed subset of X containing y. In particular $x \in a(A)$.

(4) ⇒ (5) : Let x be a point in X and A be a subset of X such that $x \in i(A)$. If $pc(x) \nsubseteq i(A)$, then $pc(x) \cap (i(A))^c \neq \emptyset$. So, by (4), $x \in (i(A))^c$ which is impossible.

(5) ⇒ (1) : Let x be in X. To prove that $pc(\{\mu_X(x)\}) = \{\mu_X(x)\}$, consider $y \in X$ such that $\mu_X(y) \in pc(\{\mu_X(x)\})$. Then $\mu_X(y)$ belongs to every p-closed subset of $T_0(X)$ containing $\mu_X(x)$. That is, y belongs to every p-closed subset of X containing x and thus $y \in pc(x)$.

Now, suppose that $x \notin pc(y)$, so $x \in (pc(y))^c$. Hence by hypothesis, $pc(x) \subseteq (pc(y))^c$ which means that $pc(x) \cap pc(y) = \emptyset$ which is impossible because it contains y. Thus $x \in pc(y)$ and hence $pc(x) = pc(y)$ which means that $\mu_X(y) = \mu_X(x)$ as desired. □

Corollary 4.5. ([2, Theorem 3.5]) *Let (X, τ) be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,1)}$-space;*

(ii) *For each open subset U of X and each $x \in U$, we have $\overline{\{x\}} \subseteq U$;*

(iii) *For each $x \in X$ and each closed subset C of X such that $\overline{\{x\}} \cap C \neq \emptyset$, we have $x \in C$.*

Theorem 4.6. *Let (X, a) be a pretopological space. Then the following statements are equivalent:*

1. *(X, a) is a $T_{(0,1^k)}$-pretopological space;*
2. *For each $x \in X$ the subset $\{y \in X \mid pc(y) = pc(x)\}$ is p-closed.*

Proof. (1) $\Longrightarrow$ (2) Let x be in X. Since $T_0(X)$ is a T_1^k-space. Then by Remarks 2.16 (1), $\widetilde{a}(\{\mu_X(x)\}) = \{\mu_X(x)\}$. So $\mu_X^{-1}(\{\mu_X(x)\})$ is a p-closed subset of X. Therefore $\{y \in X \mid pc(y) = pc(x)\}$ is p-closed.

(2) $\Longrightarrow$ (1) Let x be in X. By hypothesis, the subset $\{y \in X \mid pc(y) = pc(x)\}$ is p-closed in X which means that $\mu_X^{-1}(\{\mu_X(x)\})$ is a p-closed subset of X. Then $\widetilde{a}(\{\mu_X(x)\}) = \{\mu_X(x)\}$ and consequently, by Remarks 2.16 (1), (X, a) is a $T_{(0,1^k)}$-pretopological space. □

Theorem 4.7. *Let (X, a) be a pretopological space. Then the following statements are equivalent :*

1. *(X, a) is a $T_{(0,2)}$-pretopological space.*
2. *For each $x, y \in X$ such that $pc(x) \neq pc(y)$ there are two disjoint p-closed subsets in X containing respectively x and y.*

Proof. (1) $\Rightarrow$ (2) : Let $x, y \in X$ such that $pc(x) \neq pc(y)$. Then $\mu_X(x) \neq \mu_X(y)$. Since $(X/\sim, \tilde{a})$ is T_2, there exist two p-closed subsets of $T_0(X)$ $\tilde{a}\,(B)$ and $\tilde{a}\,(B')$ containing respectively $\mu_X(x)$ and $\mu_X(y)$. Hence $\mu_X^{-1}\,(\tilde{a}\,(B))$ and $\mu_X^{-1}\,(\tilde{a}\,(B'))$ are two disjoint p-closed subsets of X containing respectively x and y.

(2) $\Rightarrow$ (1) : Let $\mu_X(x) \neq \mu_X(y) \in X/\sim$. Then $pc(x) \neq pc(y)$. Using (2) , there exists two disjoint p-closed subsets $a\,(A)$ and $a\,(A')$ in X containing respectively x and y. Hence $\mu_X\,(a\,(A))$ and $\mu_X\,(a\,(A'))$ are two disjoint p-closed subsets in $T_0(X)$ containing respectively $\mu_X(x)$ and $\mu_X(y)$. Therefore $(X/\sim, \tilde{a})$ is T_2 and (X, a) is a $T_{(0,2)}$ pretopological space. □

Corollary 4.8. ([2, Theorem 3.12]) *Let (X, τ) be a topological space. Then the following statements are equivalent:*

(i) X is a $T_{(0,2)}$-space;

(ii) For each $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there are two disjoints open subsets U and V in X with $x \in U$ and $y \in V$.

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