Filomat 32:9 (2018), 3297–3304
https://doi.org/10.2298/FIL1809297P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Stability Results for Suzuki Contractions with an Application to Initial Value Problems

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**Abstract.** Stability of fixed points for a wider class of mappings is studied in a metric space. The results obtained herein include a number of known results. An application to initial value problems is also discussed.

## 1. Introduction and Preliminaries

The interrelationship between the convergence of a sequence of contraction mappings $\{f_n\}$ and their fixed points $\{x_n\}$ on a metric space $(M, d)$, known as the stability (resp. continuity) of fixed points, is contained in a classical theorem of Bonsall [4, Theorem 1.5, p. 5]. This result (see also Sonneschein [15]) marks the beginning of the study on the stability of fixed points. Subsequently, Nadler, Jr. [9] and others have addressed mainly the problem of replacing the completeness of the space $M$ by the existence of fixed points and various relaxations on the contraction constant and contractive conditions (cf. [1, 10–14], among others). In 2006, Barbet and Nachi [3] (see also [2]) introduced new notions of convergence over a variable domain in a metric space and discussed the stability of fixed points for contraction mappings. These notions of convergence may be considered as weaker forms of their corresponding notions of pointwise and uniform convergence which were extensively used earlier to ensure the stability of fixed points. The results of Barbet and Nachi [3] have been further generalized in various settings by Mishra et al. [5–8].

On the other hand, a theorem of Suzuki [16] (see Theorem 1.1 below) is considered to be an interesting generalization of the well-known Banach contraction principle. In this paper, we use the Suzuki contraction combined with the new notions of convergence due to Barbet and Nachi [3] to study the stability of fixed points over a variable domain. The results obtained herein thus present stability results for a much wider class of mappings. An application to initial value problems is also discussed.

2010 *Mathematics Subject Classification*. Primary 47H10; 54H25.
*Keywords*. Metric space, Suzuki contraction, stability, initial value problem.
Received: 28 August 2017; Accepted: 25 December 2018
Communicated by Adrian Petrusel
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**Theorem 1.1.** *Let $(M,d)$ be a complete metric space and $f$ a mapping on $M$. Define a nondecreasing function $\theta : [0,1) \to (\frac{1}{2},1]$ by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2 \\ (1-r)r^{-2}, & \text{if } (\sqrt{5}-1)/2 \leq r \leq 2^{-\frac{1}{2}} \\ (1+r)^{-1}, & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

*Assume that there exists $r \in [0,1)$ such that*

$$\theta(r)d(u,fu) \leq d(u,v) \text{ implies } d(fu,fv) \leq r\, d(u,v) \tag{1.1}$$

*for all $u, v \in M$. Then $f$ has a unique fixed point $z \in M$.*

The mapping satisfying condition (1.1) is known as Suzuki contraction (see also [17]).

**Remark 1.2.** *We note that unlike the Banach contraction, the Suzuki contraction need not be continuous and that the contraction condition is required to hold only for certain points of the domain.*

## 2. Barbet-Nachi Type Convergence

Now onwards, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ the set of natural numbers, and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The following notions of convergence are due to Barbet and Nachi [3] and present generalizations of the well known notions of pointwise and uniform convergence.

Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings. Then $f_\infty$ is called a $(G)$-limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ or, equivalently $\{f_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property $(G)$, if the following condition holds:

**(G)** $Gr(f_\infty) \subset \liminf Gr(f_n)$: for every $u \in M_\infty$, there exist a sequence $\{u_n\}$ in $\prod_{n\in\mathbb{N}} M_n$ such that:

$$\lim_{n\to\infty} d(u_n,u) = 0 \text{ and } \lim_{n\to\infty} d(f_n u_n, f_\infty u) = 0,$$

where $Gr(f)$ stands for the graph of $f$.

The mapping $f_\infty$ is called a $(G^-)$-limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ or, equivalently $\{f_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property $(G^-)$ if the following condition holds:

**(G⁻)** $Gr(f_\infty) \subset \limsup Gr(f_n)$: for every $u \in M_\infty$, there exists a subsequence$\{u_{n_j}\}_{j\in\mathbb{N}}$ of $\{u_n\}$ in $\prod_{n\in\mathbb{N}} M_n$ such that:

$$\lim_{j\to\infty} d(u_{n_j},u) = 0 \text{ and } \lim_{j\to\infty} d(f_{n_j} u_{n_j}, f_\infty u) = 0.$$

Further, the mapping $f_\infty$ is called a $(H)$-limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ or, equivalently $\{f_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property $(H)$ if the following condition holds:

**(H)** For each sequence $\{u_n\}$ in $\prod_{n\in\mathbb{N}} M_n$, there exists a sequence $\{v_n\}$ in $M_\infty$ such that:

$$\lim_{n\to\infty} d(u_n,v_n) = 0 \text{ and } \lim_{n\to\infty} d(f_n u_n, f_\infty v_n) = 0.$$

**Remark 2.1.** *The sequential form of limits as indicated in $(G)$ and $(G^-)$ are obtained by using the definitions of sequences of sets and the graph of a function.*

**Remark 2.2.** *Properties of these limits and the inter relationship between the above notions of convergence and the well-known classical notions of pointwise and uniform convergences are discussed by Barbet and Nachi [3] and are also captured briefly in Mishra and Pant [6].*

### 3. Stability results for ($G$) and ($G^-$)-convergences

We begin with the following theorem which is our first stability result.

**Theorem 3.1.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(G)$ such that for all $n \in \mathbb{N}$, $f_n$ is a Suzuki contraction with the same coefficient $r \in [0,1)$ i.e., $f_n$ satisfies (1.1) for all $n \in \mathbb{N}$ and $r \in [0,1)$. If for all $n \in \overline{\mathbb{N}}$, $u_n$ is a fixed point of $f_n$, then the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to $u_\infty$.*

*Proof.* Let $u_n$ be a fixed point of $f_n$ for each $n \in \overline{\mathbb{N}}$. Since the property (G) holds and $u_\infty \in M_\infty$, there exists a sequence $\{v_n\}$ in $\prod_{n\in\mathbb{N}} M_n$ such that $v_n \to u_\infty$ and $f_n v_n \to f_\infty u_\infty$. Therefore

$$\begin{aligned} d(u_n, u_\infty) &= d(f_n u_n, f_\infty u_\infty) \\ &\leq d(f_n u_n, f_n v_n) + d(f_n v_n, f_\infty u_\infty). \end{aligned}$$

Since $u_n$ is a fixed point of $f_n$ for each $n \in \overline{\mathbb{N}}$, $\theta(r) d(u_n, f_n u_n) = 0 \leq d(u_n, v_n)$ for any $r \in [0,1)$. Now, by (1.1), we get

$$\begin{aligned} d(u_n, u_\infty) &\leq r d(u_n, v_n) + d(f_n v_n, f_\infty u_\infty) \\ &\leq r[d(u_n, u_\infty) + d(v_n, u_\infty)] + d(f_n v_n, f_\infty u_\infty). \end{aligned}$$

Taking $n \to \infty$, we obtain

$$\begin{aligned} \lim_{n\to\infty} d(u_n, u_\infty) &\leq \lim_{n\to\infty} r\,[d(u_n, u_\infty) + d(v_n, u_\infty)] \\ &= \lim_{n\to\infty} r d(u_n, u_\infty). \end{aligned}$$

Thus $\lim_{n\to\infty} (1-r) d(u_n, u_\infty) \leq 0$. Since $r < 1$, we get $\lim_{n\to\infty} d(u_n, u_\infty) = 0$. □

The following result in [3, Theorem 2] directly follows from the above theorem.

**Corollary 3.2.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(G)$ such that for all $n \in \mathbb{N}$, $f_n$ is a contraction from $M_n$ into $M$ with the same coefficient $r \in [0,1)$. If for all $n \in \overline{\mathbb{N}}$, $u_n$ is a fixed point of $f_n$, then the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to $u_\infty$.*

**Example 3.3.** *Let $M = [1,2]$ be endowed with the usual metric $d(x,y) = |x-y|$. Then $(M,d)$ is a metric space. Let $\{M_n = [1, 1+\frac{1}{n}]\}$ be a family of nonempty subsets of $M$ for each $n \in \overline{\mathbb{N}}$. Define a family of mappings $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ by*

$$f_n u = 1 + \frac{1}{1+nu} \text{ for } u \in M_n.$$

*It is easy to see that $M_\infty = \{1\}$. Define $f_\infty u = 1$ for all $u \in M_\infty$. Then there exist the sequence $\{u_n = 1 + \frac{1}{2n}\}$ in $M_n$ for $n \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} d(u_n, 1) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(f_n u_n, f_\infty 1) = 0.$$

*Therefore $f_\infty$ is a G-limit of $\{f_n\}$. Now, for each $n \in \mathbb{N}$ and $u, v \in M_n$*

$$\begin{aligned} d(f_n u, f_n v) &= \frac{n|u-v|}{(1+nu)(1+nv)} \\ &\leq \frac{n}{(1+n)(1+n)}|u-v| \\ &= r|u-v| \\ &= r d(u,v), \end{aligned}$$

*where* $r = \frac{n}{(1+n)^2} < 1$ *for each* $n \in \mathbb{N}$. *Hence,* $f_n$ *is contraction for each* $n \in \mathbb{N}$. *Also* $\frac{n-1+\sqrt{(n-1)^2+8n}}{2n}$ *is a fixed point of* $f_n$ *in* $M_n$ *for each* $n \in \mathbb{N}$. *Finally,*

$$\lim_{n\to\infty} \frac{n-1+\sqrt{(n-1)^2+8n}}{2n} = 1.$$

*Therefore the sequence of fixed points* $\left\{\frac{n-1+\sqrt{(n-1)^2+8n}}{2n}\right\}_{n\in\mathbb{N}}$ *converges to* $1 = f_\infty 1$, *and all the conditions of Corollary 3.2 are satisfied.*

When $M_n = M$ for all $n \in \overline{\mathbb{N}}$, $M$ is complete and $r \in (0, 1)$, then we get the following result of Bonsall [4, Theorem 1.2, p. 5] as a consequence of Theorem 3.1.

**Corollary 3.4.** *Let* $(M, d)$ *be a complete metric space, and* $\{f_n : M \to M\}$ *be a family of mappings such that for all* $u, v \in M$ *and* $n \in \mathbb{N}$,

$$\theta(r)d(u, f_n \leq d(u, v) \text{ implies } d(f_n u, f_n v) \leq rd(u, v).$$

*Suppose the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *converges pointwise to* $f_\infty$. *Then for all* $n \in \overline{\mathbb{N}}$, $f_n$ *has a unique fixed point* $u_n$ *and the sequence* $\{u_n\}_{n\in\mathbb{N}}$ *converges to* $u_\infty$.

The following theorem proves the existence of a fixed point for a $(G)$-limit of a sequence of Suzuki contractions.

**Theorem 3.5.** *Let* $\{M_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of a metric space* $(M, d)$ *and* $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ *be a family of mappings satisfying the property (G) such that for any* $n \in \mathbb{N}$, $f_n$ *is a Suzuki contraction with the same coefficient* $r \in [0, 1)$. *Assume that for any* $n \in \mathbb{N}$, $u_n$ *is a fixed point of* $f_n$. *Then*

$$\begin{aligned} f_\infty \text{ admits a fixed point} \quad &\Leftrightarrow \quad \{u_n\} \text{ converges and } \lim_{n\to\infty} u_n \in M_\infty \\ &\Leftrightarrow \quad \{u_n\} \text{ admits a subsequence converging to a point of } M_\infty. \end{aligned}$$

*Proof.* The necessary part follows from Theorem 3.1. To prove the sufficiency, let $\{u_{n_j}\}$ be a subsequence of $\{u_n\}$ such that $\lim_{j\to\infty} u_{n_j} = u_\infty \in M_\infty$. By the property (G), there exists a sequence $\{v_n\}$ in $\prod_{n\in\mathbb{N}} M_n$ such that $v_n \to u_\infty$ and $f_n v_n \to f_\infty u_\infty$ as $n \to \infty$. For any $j \in \mathbb{N}$, we have

$$d(u_\infty, f_\infty u_\infty) \leq d(u_\infty, u_{n_j}) + d(f_{n_j} u_{n_j}, f_{n_j} v_{n_j}) + d(f_{n_j} v_{n_j}, f_\infty u_\infty). \tag{3.1}$$

Since for any $r \in [0, 1)$, $\theta(r)d(u_{n_j}, f_{n_j} u_{n_j}) \leq d(u_{n_j}, v_{n_j})$ for all $n_j$, by (1.1)

$$d(f_{n_j} u_{n_j}, f_{n_j} v_{n_j}) \leq rd(u_{n_j}, v_{n_j})$$

Now, from (3.1)

$$\begin{aligned} d(u_\infty, f_\infty u_\infty) &\leq d(u_\infty, u_{n_j}) + rd(u_{n_j}, v_{n_j}) + d(f_{n_j} v_{n_j}, f_\infty u_\infty) \\ &\leq d(u_\infty, u_{n_j}) + r[d(u_{n_j}, u_\infty) + d(v_{n_j}, u_\infty)] + d(f_{n_j} v_{n_j}, f_\infty u_\infty). \end{aligned}$$

Now passing over to the limit as $j \to \infty$, we deduce that $f_\infty u_\infty = u_\infty$. □

**Remark 3.6.** *Under the assumptions of Theorem 3.5, and if,*

1. $\liminf_{n\to\infty} M_n \subset M_\infty$ *(i.e., the limit of any convergent sequence* $\{z_n\} \in \prod_{n\in\mathbb{N}} M_n$ *is in* $M_\infty$ *), then* $f_\infty$ *admits a fixed point* $\Leftrightarrow \{u_n\}$ *converges.*
2. $\limsup_{n\to\infty} M_n \subset M_\infty$ *(i.e., the cluster point of any sequence* $\{z_n\} \in \prod_{n\in\mathbb{N}} M_n$ *is in* $M_\infty$*), then* $f_\infty$ *admits a fixed point* $\Leftrightarrow \{u_n\}$ *admits a convergent subsequence.*

**Proposition 3.7.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(G)$ and such that, for any $n \in \mathbb{N}$, $f_n$ is a Suzuki contraction with the same coefficient $r \in [0,1)$. Then $f_\infty$ is a Suzuki contraction with the same coefficient $r$.*

*Proof.* Given two points $u$ and $v$ in $M_\infty$, by the property $(G)$, there exist two sequences $\{u_n\}$ and $\{v_n\}$ in $\prod_{n\in\mathbb{N}} M_n$ converging respectively to $u$ and $v$ such that the sequences $\{f_n u_n\}$ and $\{f_n v_n\}$ converge respectively to $f_\infty u$ and $f_\infty v$. Suppose for each $n \in \mathbb{N}$, $f_n$ is a Suzuki contraction with the same coefficient $r \in [0,1)$. Then

$$\begin{aligned}\theta(r)d(u, f_\infty u) &\leq \theta(r)[d(u,u_n) + d(u_n, f_n u_n) + d(f_n u_n, f_\infty u)]\\ &\leq \theta(r)[d(u,u_n) + d(f_n u_n, f_\infty u)] + d(u_n, v_n).\end{aligned}$$

Since $u_n \to u$, $v_n \to v$ and $f_n u_n \to f_\infty u$ as $n \to \infty$, we get

$$\theta(r)d(u, f_\infty u) \leq d(u,v).$$

Also,

$$\begin{aligned}d(f_\infty u, f_\infty v) &\leq d(f_\infty u, f_n u_n) + d(f_n u_n, f_n v_n) + d(f_n v_n, f_\infty v)\\ &\leq d(f_\infty u, f_n u_n) + r d(u_n, v_n) + d(f_n v_n, f_\infty v).\end{aligned}$$

Taking $n \to \infty$, we get $d(f_\infty u, f_\infty v) \leq r d(u,v)$ and the conclusion holds. $\square$

In the next result, a sufficient condition is given in order that the two notions of convergence become equivalent.

**Proposition 3.8.** ([3, Proposition 4]). *Let $X$ be a subset of a metric space $(M,d)$ and $\{f_n : X \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying property $(G)$ such that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous on $X$. Then $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to $f_\infty$.*

The following result which presents an analogue of [3, Proposition 4] follows from Proposition 3.7.

**Corollary 3.9.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying property $(G)$ such that, for any $n \in \mathbb{N}$, $f_n$ is a contraction with the same coefficient $r \in [0,1)$. Then $f_\infty$ is a contraction with the coefficient $r$.*

Under a compactness assumption, the existence of a fixed point of the $(G)$- limit mapping can be obtained from the existence of fixed points of the Suzuki contractions $f_n$.

**Theorem 3.10.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(G)$ and such that, for any $n \in \mathbb{N}$, $f_n$ is a Suzuki contraction with the same coefficient $r \in [0,1)$. Assume that $\limsup_{n\to\infty} M_n \subset M_\infty$ and $\bigcup_{n\in\mathbb{N}} M_n$ is relatively compact. If for any $n \in \mathbb{N}$, $f_n$ admits a fixed point $u_n$, then the $(G)$-limit mapping $f_\infty$ admits a fixed point $u_\infty$ and the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to $u_\infty$.*

*Proof.* Let $u_n$ be the fixed point of $f_n$ for each $n \in \mathbb{N}$. From the compactness condition, there exists a convergent subsequence $\{u_{n_j}\}$ of $\{u_n\}$. Now, by Remark 3.6, $f_\infty$ admits a fixed point $u_\infty$ and by Theorem 3.5, the sequence $\{u_n\}$ converges to $u_\infty$. $\square$

As a consequence of Theorem 3.10 and Remark 2.1, we have the following result in [3, Theorem 7].

**Corollary 3.11.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(G)$ such that for any $n \in \mathbb{N}$, $f_n$ is a contraction with the same coefficient $r \in [0,1)$. Assume that $\limsup_{n\to\infty} M_n \subset M_\infty$ and $\bigcup_{n\in\mathbb{N}} M_n$ is relatively compact. If for any $n \in \mathbb{N}$, $f_n$ admits a fixed point $u_n$, then the $(G)$-limit mapping $f_\infty$ admits a fixed point.*

Now we present a stability result for a sequence of mappings $\{f_n\}$ satisfying the property $(G^-)$.

**Theorem 3.12.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of Suzuki contraction mappings (with the same coefficient $r \in [0,1)$) satisfying the property $(G^-)$. If for any $n \in \overline{\mathbb{N}}$, $u_n$ is a fixed point of $f_n$, then $u_\infty$ is a cluster point of the sequence $\{u_n\}_{n\in\mathbb{N}}$.*

*Proof.* By the property $(G^-)$, there exists a sequence $\{v_n\}$ in $\prod_{n\in\mathbb{N}} M_n$ which has a subsequence $\{v_{n_j}\}$ such that $v_{n_j} \to u_\infty$ and $f_{n_j}v_{n_j} \to f_\infty u_\infty$ as $j \to \infty$. We have

$$d(u_{n_j}, u_\infty) \le d(f_{n_j}u_{n_j}, f_{n_j}v_{n_j}) + d(f_{n_j}v_{n_j}, f_\infty u_\infty).$$

Since for any $r \in [0,1)$, $\theta(r)d(u_{n_j}, f_{n_j}u_{n_j}) \le d(u_{n_j}, v_{n_j})$ by (1.1), the above inequality reduces to

$$\begin{aligned} d(u_{n_j}, u_\infty) &\le rd(u_{n_j}, v_{n_j}) + d(f_{n_j}v_{n_j}, f_\infty u_\infty) \\ &\le r[d(u_{n_j}, u_\infty) + d(v_{n_j}, u_\infty)] + d(f_{n_j}v_{n_j}, f_\infty u_\infty) \\ &= \frac{1}{1-r}[rd(v_{n_j}, u_\infty) + d(f_{n_j}v_{n_j}, f_\infty u_\infty)] \end{aligned}$$

Thus $\{u_{n_j}\}$ converges to $u_\infty$, the fixed point of $f_\infty$. □

The following result in [3, Theorem 8] follows from Theorem 3.12.

**Corollary 3.13.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ a family of contraction mappings satisfying the property $(G^-)$. If for any $n \in \overline{\mathbb{N}}$, $u_n$ is a fixed point of $f_n$, then $u_\infty$ is a cluster point of the sequence $\{u_n\}_{n\in\mathbb{N}}$.*

## 4. Stability Results for (*H*)-Convergence

Now, we present another stability result using the $(H)$-convergence.

**Theorem 4.1.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(H)$ such that $f_\infty$ is a Suzuki contraction. If for any $n \in \overline{\mathbb{N}}$, $u_n$ is a fixed point of $f_n$, then the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to $u_\infty$.*

*Proof.* By property $(H)$, there exists a sequence $\{v_n\}$ in $M_\infty$ such that $d(u_n, v_n) \to 0$ and $d(f_nu_n, f_\infty v_n) \to 0$. We have

$$d(u_n, u_\infty) \le d(f_nu_n, f_\infty v_n) + d(f_\infty v_n, f_\infty u_\infty).$$

Since for any $r \in [0,1)$, $\theta(r)d(u_\infty, f_\infty u_\infty) \le d(v_n, u_\infty)$ by (1.1), we have

$$\begin{aligned} d(u_n, u_\infty) &\le d(f_nu_n, f_\infty v_n) + rd(v_n, u_\infty) \\ &\le d(f_nu_n, f_\infty v_n) + r[d(v_n, u_n) + d(u_n, u_\infty)] \\ &= \frac{1}{1-r}[d(f_nu_n, f_\infty v_n) + rd(v_n, u_n)]. \end{aligned}$$

Thus $\lim\limits_{n\to\infty} d(u_n, u_\infty) = 0$ and hence the conclusion follows. □

The following result in [3, Theorem 11] follows directly from the above theorem.

**Corollary 4.2.** *Let $\{M_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(M,d)$ and $\{f_n : M_n \to M\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property $(H)$ such that $f_\infty$ is a $k$ -contraction. If for any $n \in \overline{\mathbb{N}}$, $u_n$ is a fixed point of $f_n$ then the sequence $\{u_n\}$ converges to $u_\infty$.*

## 5. Applications

Inspired by S. B. Nadler, Jr. [9], we present an application of our results to an initial value problem.

**Proposition 5.1.** *Let $D$ be an open subset of $\mathbb{R}^2$, $(a,b) \in D$ and $K > 0$ be real number. Assume that:*

**(a)** *$\{f_i\}$ is a sequence of real valued continuous functions defined on $D$ such that $|f_i(x,y)| \leq K$ for all $(x,y) \in D$ with a G-limit $f$, a continuous function on $D$.*

**(b)** *the set*

$$C = \{(x,y) : |x-a| \leq p \ \text{ and } \ |y-b| \leq K|x-a|\},$$

*is a subset of $D$ with $p > 0$.*

**(c)** *for every pair of real valued functions $g$ and $h$*

$$\theta(kp)|g(x) - T_i g(x)| \leq |g(x) - h(x)| \text{ implies } |f_i(x, g(x)) - f_i(x, h(x))| \leq k|g(x) - h(x)|, \tag{5.1}$$

*where $kp \in [0,1)$ and $T_i$ is defined by*

$$T_i(g)x := b + \int_a^x f_i(t, g(t))dt. \tag{5.2}$$

*Then the sequence $\{y_i\}$ converges uniformly on $I = [a-p, a+p]$ to $y_0$, where for each $i \in \mathbb{N}$, $y_i$ is the unique solution on $I$ of the initial value problem*

$$y'(x) = f_i(x, y(x)); \quad y(a) = b.$$

*Proof.* Let $M$ be the set of all real valued continuous functions on $I$ with graph lying in $C$ and with Lipschitz constant $\leq K$. Then $M$ with the supremum metric $d$ is a compact metric space. For each each $g \in M$, define

$$T(g)x = b + \int_a^x f(t, g(t))dt, \quad x \in I.$$

From (5.2) for each $i \in \mathbb{N}$

$$\begin{aligned}
|T_i(g)x - T_i(h)x| &\leq \int_a^x |f_i(t, g(t)) - f_i(t, h(t))|dt \\
&\leq k \int_a^x |g(t)) - h(t)|dt \\
&\leq k \sup_{t \in [a,x]} |g(t) - h(t)| \int_a^x dt \\
&\leq kp \sup_{t \in [a,x]} |g(t) - h(t)|.
\end{aligned}$$

From the above inequality and (5.1), we get

$$\theta(kp)d(g(x), T_i g(x)) \leq d(g(x), h(x)) \text{ implies } d(T_i(g)x, T_i(h)x) \leq kpd(g(x), h(x)).$$

Thus $T_i$ is a Suzuki contraction on $M$ for each $i \in \mathbb{N}$. Now, Proposition 3.7 implies that $T$ is also a Suzuki contraction on $M$. For each $g \in X$, $x \in I$ and $i \in \mathbb{N}$,

$$T_i(g)x - T(g)x = \int_a^x [f_i(t, g(t)) - f(t, g(t))]dt.$$

Since $f$ is the $G$-limit of $f_i$, the sequence of integrands converges to zero and is uniformly bounded by $2K$. The Lebesgue bounded convergence theorem guarantees that the sequence of integrals on the R.H.S. goes to 0 as $i \to \infty$. Therefore $T(g)$ is the $G$-limit of $T_i(g)$ on $I$. Now by Proposition 3.8, $G$-limit is equivalent to pointwise limit. It is easy to see that $T_i(g)$ is uniformly continuous on $I$ for each $i \in \mathbb{N}$ and hence the sequence $\{T_i(g)\}$ is equicontinuous on the compact set $I$. Therefore the sequence $\{T_i(g)\}$ converges uniformly to $T(g)$ on $I$. Hence the sequence $\{T_i\}$ converges pointwise to $T$ on $M$. By Theorem 3.10 the sequence $\{y_i\}$ where $y_i$ is the unique fixed point of $T_i$ for each $i \in \mathbb{N}$, converges to the fixed point $y_0$ of $T$. The result follows since these fixed points are the unique solutions of the initial value problem. □

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