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A New Approach to Gradient Ricci Solitons and Generalizations

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Abstract. This short note concerns firstly with two inequalities in the geometry of gradient Ricci solitons (g, f, λ) on a smooth manifold M. These inequalities provide some relationships between the curvature of the Riemannian metric g and the behavior of the scalar field f through two second order equations satisfied by the scalar λ. Secondly, we propose several generalizations of Ricci solitons to the setting of manifolds endowed with linear connections, not necessary of metric type. Thirdly, we express the usual Ricci solitons equation in terms of two Golab connections.

Let (M^n, g) be a n-dimensional Riemannian manifold endowed with a smooth function $f \in C^\infty(M)$. The scalar field f yields *the Hessian endomorphism*:

$$h_f : \mathcal{X}(M) \to \mathcal{X}(M), \quad h_f(X) = \nabla_X \nabla f \tag{1}$$

where ∇ is the Levi-Civita connection of g. Then we know the symmetry of *the Hessian tensor field* of f:

$$H_f(X, Y) := g(h_f(X), Y), \quad H_f(X, Y) = H_f(Y, X). \tag{2}$$

It follows the existence of a g-orthonormal frame $E = \{E_i\}_{i=1,\dots,n} \in \mathcal{X}(M)$ and the existence of the eigenvalues $\{\lambda_i\}_{i=1,\dots,n} \in C^\infty(M)$:

$$h_f(E_i) = \lambda_i E_i. \tag{3}$$

Hence we express all the geometric objects related to f in terms of the pair $(E, \{\lambda_i\})$ which we call *the spectral data* of f:

$$\nabla f = \sum_{i=1}^{n} E_i(f) E_i, \quad \|\nabla f\|_g^2 = \sum_{i=1}^{n} [E_i(f)]^2, \quad h_f(X) = \nabla_X \nabla f = \sum_{i=1}^{n} (\lambda_i X^i) E_i, \tag{4}$$

for $X = X^i E_i$. Also the Hessian and *the Laplacian* of f are:

$$H_f(X, Y) = \sum_{i=1}^{n} \lambda_i (X^i Y^i), \quad \Delta f := Tr_g H_f = \sum_{i=1}^{n} \lambda_i. \tag{5}$$

Suppose now that the triple $(g, f, \lambda \in \mathbb{R})$ is a *gradient Ricci soliton* on M, [4, p. 76]:

$$H_f + Ric + \lambda g = 0 \tag{6}$$

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where Ric is the Ricci tensor field of g. By considering the Ricci endomorphism $Q \in \mathcal{T}^1_1(M)$ provided by:

$$Ric(X, Y) = g(QX, Y) \tag{7}$$

we can express (6) as:

$$h_f + Q + \lambda I = 0 \tag{8}$$

with I being the Kronecker endomorphism. From (4) we get that Q is also of diagonal form with respect to the frame E:

$$Q(X) = -\sum_{i=1}^{n}(\lambda_i + \lambda)X^i E_i, \quad \|Q\|^2_g = \sum_{i=1}^{n}(\lambda_i + \lambda)^2. \tag{9}$$

By developing the second formula above we derive:

$$\|Ric\|^2_g = \sum_{i=1}^{n} \lambda_i^2 + 2\lambda \sum_{i=1}^{n} \lambda_i + n\lambda^2 = \|H_f\|^2_g + 2\lambda\Delta f + n\lambda^2. \tag{10}$$

Hence the scalar λ is a solution of the second order equation:

$$n\lambda^2 + 2\Delta f \cdot \lambda + (\|H_f\|^2_g - \|Ric\|^2_g) = 0 \tag{11}$$

which means the positivity:

$$0 \leq \Delta' := (\Delta f)^2 - n(\|H_f\|^2_g - \|Ric\|^2_g). \tag{12}$$

It follows a lower bound of the geometry of g in terms of f:

$$\|Ric\|^2_g \geq \|H_f\|^2_g - \frac{1}{n}(\Delta f)^2. \tag{13}$$

An "exotic" consequence is provided by the case of strict inequality in (12); more precisely, it follows that the data (g, f, λ) is doubled by the triple $(g, f, -\frac{2\Delta f}{n} - \lambda = \lambda + \frac{2R}{n})$ conform (16) below.

Examples 1) (Gaussian soliton) We have $(M = \mathbb{R}^n, g_{can})$ and $f(x) = -\frac{\lambda}{2}\|x\|^2$. It results: $h_f = -\lambda I_n$ and $\Delta f = -n\lambda$. Since $\|H_f\|^2 = n\lambda^2$ the left hand side of (11) is:

$$n\lambda^2 + 2\Delta f \cdot \lambda + (\|H_f\|^2_g - \|Ric\|^2_g) = n\lambda^2 + 2(-n\lambda)\lambda + (n\lambda^2 - 0)$$

which is exactly zero. Also: $\Delta' = (n\lambda)^2 - n(n\lambda^2 - 0) = 0$ which means the uniqueness of λ and the equality case in (13): $0 = n\lambda^2 - \frac{(n\lambda)^2}{n}$.

2) (Einstein manifold) Let (M, g) be an Einstein manifold: $Ric = -\lambda g$. A function f with vanishing Hessian is called *Killing potential* in [8] since its gradient is a Killing vector field; in [18, p. 283] such a function is called *linear*. Hence: $\Delta f = \|H_f\|^2 = 0$ and $\|Ric\|^2 = n\lambda^2$ which yields the following value of the left hand side of (11):

$$n\lambda^2 + 2\Delta f \cdot \lambda + (\|H_f\|^2_g - \|Ric\|^2_g) = n\lambda^2 + 2(\cdot 0)\lambda + (0 - n\lambda^2)$$

which is exactly zero. Also: $\Delta' = 0^2 - n(0 - n\lambda^2) = n^2\lambda^2 \geq 0$ which means that (13) is the inequality $n\lambda^2 \geq 0 - \frac{0}{n}$ and the uniqueness of λ gives a steady soliton, equivalently g is Ricci-flat. We consider an interesting open problem to find the linear functions of a steady soliton and of a Ricci-flat metric.

3) A generalization of the first example is provided on a Ricci-flat manifold by a smooth function f satisfying a generalization of Hessian structures:

$$H_f = -\lambda g. \tag{14}$$

Then: $\Delta f = -n\lambda$ and $\|H_f\|^2 = n\lambda^2$ exactly as for the Gaussian soliton. Using Lemma 4.1. of [7, p. 1540] it results form (14) that ∇f is a particular *concircular vector field*: $h_f = -\lambda I$; hence $\lambda_1 = ... = \lambda_n = -\lambda$. If ∇f is without zeros it follows from Theorem 3.1. of [7, p. 1539] that (M, g) is locally a warped product with a 1-dimensional basis: $(M, g) = (I \times_\varphi (F^{n-1}, g_F))$. In fact: $\nabla f = \varphi(s)\frac{\partial}{\partial s}$ with $\varphi'(s) = -\lambda$ which means a affine

warping function: $\varphi(s) = -\lambda s + C$.
4) (Hamilton's cigar) The famous Hamilton's soliton is the steady soliton provided by the complete Riemannian geometry $(\mathbb{R}^2, g = \frac{1}{1+x^2+y^2} g_{can})$ and the potential function $f(x,y) = -\ln(1+x^2+y^2)$. The only non-zero components of the Hessian are $(H_f)_{11} = (H_f)_{22} = \frac{-2}{(1+x^2+y^2)^2}$ which yields the norm $\|H_f\|^2 = \frac{8}{(1+x^2+y^2)^2}$ and the Laplacian $\Delta f = \frac{-4}{1+x^2+y^2}$. The Gaussian curvature of g is $K(x,y) = \frac{2}{1+x^2+y^2}$ and $\|Ric\|^2 = 2K^2$. In conclusion, the inequality (13) becomes strictly: $\frac{8}{(1+x^2+y^2)^2} > \frac{8}{(1+x^2+y^2)^2} - \frac{16}{2(1+x^2+y^2)^2} = 0$. The spectral data of f is given by $(E_1 = (1+x^2+y^2)^{\frac{1}{2}}\frac{\partial}{\partial x}, E_2 = (1+x^2+y^2)^{\frac{1}{2}}\frac{\partial}{\partial y}, \lambda_1(x,y) = \lambda_2(x,y) = \frac{-2}{1+x^2+y^2})$. Since H_f is a negative definite billiniar form we have that f is strictly concave with respect to g.
5) (Cylinder shrinking soliton) Consider the Riemannian product $(M,g) = (\mathbb{S}^{n-1} \times \mathbb{R}, 2(n-2)g_S + dt^2)$ of the shrinking unit sphere for $n \geq 3$ with a line. We have:

$$Ric_g = (n-2)g_S = \frac{1}{2}(g - dt^2), \quad \|Ric\|^2 = \frac{n-1}{4}. \tag{15}$$

For $f(x,t) = \frac{t^2}{4}$ we obtain a shrinking gradient Ricci soliton with $\lambda = -\frac{1}{2}$ from $H_f = \frac{1}{2}dt^2$; also: $\Delta f = \frac{1}{2}$ and the spectral part of the spectral data of f is given by $(\lambda_1 = ... = \lambda_{n-1} = 0, \lambda_n = \frac{1}{2})$. Then (13) becomes the strict inequality: $\frac{n-1}{4} > \frac{1}{4} - \frac{1}{4n}$. □

A new second order equation, similar to (11), follows from a well-known formula from the theory of gradient Ricci solitons, [4, p. 79]:

$$\Delta f + R + n\lambda = 0 \tag{16}$$

obtained by tracing (6); here R is the scalar curvature of g. Hence the companion equation of (11) is:

$$n\lambda^2 + 2R\lambda + (\|Ric\|_g^2 - \|H_f\|_g^2) = 0. \tag{17}$$

The new inequality is then:

$$0 \leq \Delta' := R^2 - n(\|Ric\|_g^2 - \|H_f\|_g^2) \tag{18}$$

and it results a lower bound of the behavior of f in terms of geometry of g:

$$\|H_f\|_g^2 \geq \|Ric\|_g^2 - \frac{R^2}{n}. \tag{19}$$

We remark that (13) and (19) can be unified in the double inequality:

$$\|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2 \leq \|Ric\|_g^2 \leq \|H_f\|_g^2 + \frac{R^2}{n} \tag{20}$$

and the simultaneous equalities hold if and only if: $H_f = -Ric$ with $\Delta f = R = \lambda = 0$; hence f is a harmonic map on a steady gradient Ricci soliton. Moreover, with the Lemma 11.14 of [4, p. 86-87] it results that: $H_f = -Ric = 0$ and then f is a linear function on the Ricci-flat (M,g).

Examples revisited 1) (Gaussian soliton) The inequality (19) becomes: $n\lambda^2 \geq 0$.
2) (Einstein manifold) (19) becomes an equality: $0 = n\lambda^2 - \frac{(n\lambda)^2}{n}$.
3) (Hamilton soliton) (19) is the strict inequality: $\frac{8}{(1+x^2+y^2)^2} > \frac{2\cdot 4}{(1+x^2+y^2)^2} - \frac{4^2}{2(1+x^2+y^2)^2} = 0$. We remark that this inequality is the same as that discussed in the example of Hamilton soliton above but we point out that the zero is obtained from different terms.
4) (Cylinder shrinking soliton) Since $R = \frac{n-1}{2}$ we get the following form of (19): $\frac{1}{4} > \frac{1}{4} - \frac{(n-1)^2}{4n^2}$.
5) (Constant scalar curvature) Suppose that R is a constant. From the formula (3.26) of [10] we have: $\|Ric\|_g^2 = -\lambda R$ and then the inequality (19) becomes: $\|H_f\|^2 \geq -(\lambda R + \frac{R^2}{n})$=constant. □

Remark 1 An unified proof of the double inequality (20) is provided by the following relation satisfied by a gradient Ricci soliton:

$$\|H_f\|_g^2 - \frac{(\Delta f)^2}{n} = \|Ric\|_g^2 - \frac{R^2}{n} = \frac{1}{n}\sum_{i \neq j}(\lambda_i - \lambda_j)^2 \tag{NormGRicSols}$$

and is important to point out that this equation does not involves the scalar λ. In other words, (*NormGRicSols*) is a universal formula of the gradient Ricci solitons. With the Corollary 8.2 of [1, p. 452] we get the norm of the Hessian:

$$\|H_f\|_g^2 = \frac{1}{n}\left[(\Delta f)^2 - R^2\right] + \frac{1}{2}\left[\nabla f(R) - \Delta R\right] - R\lambda \tag{\textit{HessGRicSols}}$$

and for the case of constant scalar curvature the middle term of the right hand side is zero; this middle term is $\frac{1}{2}\Delta_f R$ with Δ_f the *weighted* (or *drifting*) *Laplacian with respect to* f.
We can generalize the relation (*NormGRicSols*) to a linear algebraic setting: let (V^n, g) be a n-dimensional Euclidean linear space and A, B two g-symmetric endomorphisms with the spectrum $(\lambda_1, ..., \lambda_n)$ and $(\rho_1, ..., \rho_n)$ respectively. Then the condition:

$$n\left[\|A\|_g^2 - \|B\|_g^2\right] = (Tr_g A)^2 - (Tr_g B)^2 \tag{\textit{AlgebraicSols}}$$

yields the spectral relation:

$$\sum_{i\neq j}\left(\lambda_i - \lambda_j\right)^2 = \sum_{i\neq j}\left(\rho_i - \rho_j\right)^2. \tag{\textit{SpectAlgebraicSols}}$$

Another proof of (*NormGRicSols*) is given by the following direct consequence of the gradient Ricci solitons equation (6):

$$H_f - \frac{\Delta f}{n} g = -\left(Ric - \frac{R}{n} g\right) =: Einst(g) \tag{\textit{GRicSols}}$$

and the fact that an orthogonal decomposition of *Ric* gives: $\|Ric\|_g^2 = \frac{R^2}{n} + \|Ric - \frac{R}{n} g\|_g^2$ followed by a similar orthogonal decomposition for H_f. The right hand side is exactly the Einstein tensor field of g and hence to a function $f \in C^\infty(M)$ we can associate its *Einstein tensor with respect to* g:

$$Einst_g(f) := H_f - \frac{\Delta f}{n} g. \tag{\textit{EinsteinFunction}}$$

The vanishing of $Einst_g(f)$ is involved in Brinkmann (1925) characterization of g being a (locally) warped product metric, [18, p. 129].
We finish this remark with the fact that the inequality:

$$\|H_f\|_g^2 \geq \frac{1}{n}(\Delta f)^2 \tag{\textit{HessIneq}}$$

can be derived from *the Gallot-Meyer inequality* for a p-form ω, [13]:

$$\|\nabla\omega\|_g^2 \geq \frac{1}{p+1}\|d\omega\|_g^2 + \frac{1}{n-p+1}\|\delta\omega\|_g^2 \tag{\textit{GallotMeyer}}$$

by considering $\omega = df$. Here δ is the co-differential operator and for a function f its Laplacian is: $-\Delta f = \delta df$. The equality in (*GallotMeyer*) is provided by conformal Killing forms. □

In the second part of this note we connect the above considerations with a study of my former Ph. D supervisor, Academician Radu Miron. More precisely, let $\tilde{\nabla}$ a linear connection different to the Levi-Civita connection of g. Let also the 1-form $\eta = df$ be the g-dual of $\xi = \nabla f$. The data $(M, g, \tilde{\nabla}, \eta)$ is called *Weyl space* in [16] if g is $\tilde{\nabla}$-*recurrent* with the factor η:

$$\tilde{\nabla} g = \eta \otimes g. \tag{21}$$

Hence, we arrive at a generalization of Ricci solitons in the framework of manifolds endowed with a linear connection:

Definition 1 Let $(M, \tilde{\nabla})$ be given. A triple $(g, \xi \in \mathcal{X}(M), \lambda)$ is a $\tilde{\nabla}$-*Ricci soliton* if:

$$\tilde{\nabla}\xi + Q + \lambda I = 0. \tag{22}$$

Let $\eta = \xi^\flat$ be the g-dual form of ξ. The data $(g, \xi, \lambda, \mu \in \mathbb{R})$ is a $(\tilde{\nabla}, \eta)$-*Ricci soliton* if:

$$\tilde{\nabla}\xi + Q + \lambda I + \mu\eta \otimes \xi = 0. \tag{23}$$

More generally, let $(M, \tilde{\nabla}, F \in \mathcal{T}^1_1(M))$ be given. The pair (ξ, λ) will be a $(\tilde{\nabla}, F)$-*soliton* if:

$$\tilde{\nabla}\xi + F + \lambda I = 0 \tag{24}$$

and the triple (ξ, λ, μ) is a $(\tilde{\nabla}, F, \eta)$-*soliton* if:

$$\tilde{\nabla}\xi + F + \lambda I + \mu\eta \otimes \xi = 0. \tag{25}$$

On this way we propose a study of classes of solitons, maybe more adapted to Hermitian/Kähler geometry ($F^2 = -I$) and para-Hermitian/para-Kähler geometry ($F^2 = I$) by using some linear connections adapted to these settings like the Chern and Bismut complex connections, [9].

As a first example, we consider the Vaisman geometry following [2]. Let (M^{2n}, J, g) be a Hermitian manifold complex dimension n and Ω its fundamental 2-form given by $\Omega(X, Y) = g(X, JY)$ for any vector fields $X, Y \in \Gamma(TM)$. Recall from [12, p. 1] that (M, J, g, Ω) is a *locally conformal Kähler manifold* (l.c.K) if there exists a closed 1-form $\omega \in \Gamma(T^0_1(M))$ such that: $d\Omega = \omega \wedge \Omega$. In particular, M is called *strongly non-Kähler* if ω is without singularities i.e. $\omega \neq 0$ everywhere; hence we consider $2c = \|\omega\|$ and $u = \omega/2c$ the corresponding 1-form. Since ω is called *the Lee form* of M the vector field $U = u^\sharp$ will be called *the Lee vector field*. Consider also the unit vector field $V = JU$, *the anti-Lee vector field*, as well as its dual form $v = V^\flat$, so: $u(V) = v(U) = 0, v = -u \circ J, u = v \circ J$.

Our setting is provided by the particular case of strongly non-Kähler l.c.K. manifolds, called *Vaisman manifolds*, and given by the parallelism of ω with respect to the Levi-Civita connection ∇ of g. Hence c is a positive constant and the Lemma 2 of [17] gives the covariant derivative of V with respect to any $X \in \Gamma(TM)$:

$$\nabla_X V = c[u(X)V - v(X)U - JX]. \tag{26}$$

It follows a class of general solitons provided by:

Proposition 1 *Let $(M, J, g, c = 1)$ be a Vaisman manifold and the linear connection:*

$$\tilde{\nabla} := \nabla - [u \otimes I + v \otimes J]. \tag{27}$$

Then $(V, 0 \in \mathbb{R})$ is a $(\tilde{\nabla}, J)$-soliton. Moreover, g is recurrent with respect to $\tilde{\nabla}$ with the factor $2u$:

$$\tilde{\nabla} g = 2u \otimes g \tag{28}$$

but $\tilde{\nabla}$ is not the corresponding Weyl connection since it is not torsion-free:

$$\tilde{T} = I \otimes u + J \otimes v - u \otimes I - v \otimes J. \tag{29}$$

As second example we consider $\tilde{\nabla}$ as being exactly the Weyl connection (21) of the pair (g, η). Its expression is well-known:

$$\tilde{\nabla} = \nabla - \frac{1}{2}\eta \otimes I - \frac{1}{2}I \otimes \eta + \frac{1}{2}g \otimes \xi \tag{30}$$

and we derive:

Proposition 2 *Let $(M, g, \eta, \tilde{\nabla})$ be a Weyl geometry with η of constant norm and endowed with an endomorphism F. Then $(\xi = \eta^\sharp, \lambda, \mu)$ is a $(\tilde{\nabla}, F, \eta)$-soliton if and only if $(\xi, \lambda - \frac{\|\xi\|^2_g}{2}, \mu + \frac{1}{2})$ is a (∇, F, η)-soliton with ∇ the usual Levi-Civita connection of g.*

As third example, we consider the vector field ξ as being *torse-forming* on the Riemannian manifold (M, g):

$$\nabla\xi = fI + \gamma \otimes \xi \tag{31}$$

for a smooth function $f \in C^\infty(M)$ and a 1-form $\gamma \in \Omega^1(M)$. Note that torse-forming vector fields appear in many areas of differential geometry and physics as is pointed out in [14] and are natural generalizations of concircular vector fields. We get immediately:

Proposition 3 *Suppose that ξ is a special torse-forming vector field having f a constant function and $\gamma = \eta = \xi^\flat$. Then (g, ξ, λ) is a Ricci soliton if and only if g is an eta-Einstein metric*:

$$Q = -(\lambda + f)I - \eta \otimes \xi. \tag{32}$$

As fourth example, let (M, g, ∇) be a hypersurface of the Riemannian manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\nabla})$ and A its shape operator.

Definition 2 The pair $(\xi \in \mathcal{X}(M), \lambda)$ is *a shape soliton on* (M, g) if:

$$\nabla\xi + A + \lambda I = 0. \tag{33}$$

Let $\eta = \xi^\flat$ be the g-dual form of ξ. The data $(\xi, \lambda, \mu \in \mathbb{R})$ is a *η-shape soliton* if:

$$\nabla\xi + A + \lambda I + \mu\eta \otimes \xi = 0. \tag{34}$$

Remark 2 For example, if M is eta-umbilical i.e. A is of eta-type which means that it is has two eigenvalues: $A = \sigma I + (\rho - \sigma)\eta \otimes \xi$ then the above condition (33) yields that ξ is torse-forming:

$$\nabla\xi = -(\lambda + \sigma)I + (\sigma - \rho)\eta \otimes \xi. \tag{35}$$

Hopf hypersurfaces of eta-umbilical type are studied in [6, p. 60]. Also, the CR submanifolds of maximal CR dimension in complex projective space have eta-type shape operators as is pointed out in [11, p. 190]. □

We go further with a generalization to the setting of statistical structures of [3] provided by data $(M, g, \tilde{\nabla}, \tilde{\nabla}^*)$ where $\tilde{\nabla}, \tilde{\nabla}^*$ is a pair of torsion-free *dual connections* on (M, g):

$$Z(g(X, Y)) = g(\tilde{\nabla}_Z X, Y) + g(X, \tilde{\nabla}^*_Z Y) \tag{36}$$

for any vector fields X, Y, Z. We introduce:

Definition 3 i) The statistical manifold $(M, g, \tilde{\nabla}, \tilde{\nabla}^*)$ is called *Ricci-symmetric* if the Ricci tensor field $\tilde{Q}$ of $\tilde{\nabla}$ (equivalently, $\tilde{Q}^*$ of $\tilde{\nabla}^*$ by Corollary 9.5.3 of [3, p. 267]) is symmetric.
ii) The pair (ξ, λ) is a *statistical soliton* for the Ricci-symmetric statistical manifold $(M, g, \tilde{\nabla}, \tilde{\nabla}^*)$ if the triple (g, ξ, λ) is both $\tilde{\nabla}$-Ricci and $\tilde{\nabla}^*$-Ricci soliton.

Remark 3 Since the Levi-Civita connection of g is the arithmetic mean of the pair $(\tilde{\nabla}, \tilde{\nabla}^*)$:

$$2\nabla = \tilde{\nabla} + \tilde{\nabla}^* \tag{37}$$

it follows that a statistical soliton is an usual Ricci soliton. □

A new generalization eliminates the scalar λ. Recall that given the pair $(\tilde{\nabla}, F)$ as in Definition 1 *the exterior covariant derivative of F with respect to* $\tilde{\nabla}$ is:

$$(d^{\tilde{\nabla}}F)(X, Y) := (\tilde{\nabla}_X F)Y - (\tilde{\nabla}_Y F)X + F(\tilde{T}(X, Y)) \leftrightarrow d^{\tilde{\nabla}}F := \tilde{\nabla}F \otimes I - I \otimes \tilde{\nabla}F + F \circ \tilde{T}. \tag{38}$$

Since (24) is expressed as:

$$d^{\tilde{\nabla}}\xi + F + \lambda I = 0 \tag{24Bis}$$

we introduce:

Definition 4 Let $(M, \tilde{\nabla}, F)$ be given with a symmetric $\tilde{\nabla}$. The vector field ξ will be a $(\tilde{\nabla}, F)$*-weak soliton* if:

$$d^{\tilde{\nabla}} \circ d^{\tilde{\nabla}}(\xi) + d^{\tilde{\nabla}}F = 0. \tag{39}$$

Let $\eta = \xi^\flat$ be the g-dual form of ξ. The data $(\xi, \mu \in \mathbb{R})$ is a $(\tilde{\nabla}, F, \eta)$*-weak soliton* if:

$$d^{\tilde{\nabla}} \circ d^{\tilde{\nabla}}(\xi) + d^{\tilde{\nabla}}F + \mu(2d\eta \otimes \xi + \tilde{\nabla}\xi \otimes \eta - \eta \otimes \tilde{\nabla}\xi) = 0. \tag{40}$$

Remark 4 i) The terms of the equation of Ricci solitons are symmetric tensor fields while the terms of equation (40) are skew-symmetric tensor fields.
ii) For the Levi-Civita connection ∇:

$$d^{\nabla} \circ d^{\nabla}(Z) = Riem(\cdot, \cdot)Z, \tag{41}$$

while Lemma 2 of [15, p. 182] gives:

$$d^{\nabla}Q = -d_2 \circ \delta^{\nabla} Riem. \tag{42}$$

with the right hand side:

$$(d_2 \circ \delta^{\nabla} Riem)(X, Y) = \sum_{k=1}^{n} (\nabla_{e_k} Riem)(X, Y, e_k) \tag{43}$$

for an arbitrary orthonormal field $e = \{e_k\}_{k=1,\dots,n}$ on (M, g). Hence ξ is a Ricci weak-soliton if and only if:

$$Riem(\cdot, \cdot)\xi = d_2 \circ \delta^{\nabla} Riem(\cdot, \cdot). \tag{44}$$

iii) Let F be *recurrent* with respect to the Levi-Civita connection i.e. there exists a 1-form η such that:

$$\nabla F := \eta \otimes F. \tag{45}$$

Then ξ is a (∇, F)-weak soliton if:

$$Riem(X, Y)\xi + \eta(X)F(Y) - \eta(Y)F(X) = 0 \tag{46}$$

for all vector fields X, Y. In particular, if η is the g-dual of ξ then (46) means:

$$||\xi||^2 F(X) = Riem(X, \xi)\xi + \eta(X)F(\xi). \tag{47}$$

Let us remark that the necessary orthogonality relation:

$$g(Riem(X, Y)\xi, \xi) = 0 \tag{48}$$

restricts (46) to the identity:

$$\eta(X)g(F(Y), \xi) = \eta(Y)g(F(X), \xi) \tag{49}$$

which holds for F given by (47) as well as for a symmetric or skew-symmetric F satisfying $F(\xi) = 0$.
iv) Let η be the g-dual of ξ and define: $F^{\xi} := \eta \otimes \xi$. Then ξ is a (∇, F^{ξ})-weak soliton if:

$$Riem(\cdot, \cdot)\xi + 2d\eta \otimes \xi + \nabla\xi \otimes \eta - \eta \otimes \nabla\xi = 0. \tag{50}$$

For example, if $(M^{2n+1}, g, \varphi, \xi)$ is a Kenmotsu manifold then ([5]): $\nabla\xi = I - \eta \otimes \xi$, $d\eta = 0$ and the curvature satisfies $Riem(\cdot, \cdot)\xi = \eta \otimes I - I \otimes \eta$; hence the Reeb vector field of a Kenmotsu manifold is a (∇, F^{ξ})-weak soliton.
v) If the almost contact manifold $(M^{2n+1}, g, \varphi, \xi)$ is a Sasakian manifold then we have $Riem(\cdot, \cdot)\xi = I \otimes \eta - \eta \otimes I$ and $\nabla\varphi = g \otimes \xi - I \otimes \eta$ which implies:

$$d^{\nabla}\varphi = \eta \otimes I - I \otimes \eta \tag{51}$$

and hence the Reeb vector field ξ is a (∇, φ)-weak soliton. If $(M^{2n+1}, g, \varphi, \xi)$ is a cosymplectic manifold then $\nabla\xi = 0$ and $\nabla\varphi = 0$ which means that the Reeb vector field ξ is also a (∇, φ)-weak soliton. □

We can extend the generalization of Definition 4 to the setting of vector bundles. Let $\pi : E \to M$ be a real vector bundle of rank k over the manifold M. We use the notation $\Omega^p(\pi) := \Gamma(\pi) \otimes \Omega^p(M)$ for the $C^{\infty}(M)$-module of π-valued p-forms on M. Fix a covariant derivative ∇ on π i.e. a tensorial map $\nabla : \Omega^0(\pi) := \Gamma(\pi) \to \Omega^1(\pi)$ satisfying the Leibniz rule. Its curvature is $R^{\nabla} \in \Omega^2(End(\pi))$. Fix also an arbitrary $C \in \Omega^2(\pi)$. We introduce:

Definition 5 The section $s \in \Gamma(\pi)$ is a (∇, C)-*soliton* if:

$$R^{\nabla}(\cdot, \cdot)s + C(\cdot, \cdot) = 0. \tag{52}$$

In order to express locally the equation (52) let $h = (U, u^\alpha; 1 \leq \alpha \leq n)$ be a local chart on M. Suppose that $E|_U := \pi^{-1}(U)$ has a trivialization $\{s_i \in \Gamma(E|_U); 1 \leq i \leq k\}$. Then ∇ has the local coefficients Γ given by:

$$\nabla^U_{\frac{\partial}{\partial u^\alpha}} s_i = \Gamma^j_{\alpha i} s_j \tag{53}$$

and hence:

$$R^\nabla\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right)s_i = R^j_{\alpha\beta i} s_j. \tag{54}$$

Also:

$$C\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) = C^j_{\alpha\beta} s_j. \tag{55}$$

In conclusion, if $s = X^i s_i$ then (52) is the linear system:

$$R^j_{\alpha\beta i} X^i + C^j_{\alpha\beta} = 0 \tag{56}$$

with $1 \leq j \leq k$ and $1 \leq \alpha, \beta \leq n$. The k expressions (56) are skew-symmetric in the indices α, β.

In the third part of this note we return to the usual notion of Ricci soliton and we introduce two linear connections adapted to this setting. Consider again the triple $(M^n, g, \lambda \in \mathbb{R})$. This triple defines a map from the Lie algebra of vector fields to the $C^\infty(M)$-module of g-symmetric covariant tensor fields of second order:

$$\lambda - RicSol : \Gamma(TM) \to \mathcal{T}^0_{2,s}(M), \quad \lambda - RicSol(\xi) := \mathcal{L}_\xi g + 2Ric + 2\lambda g. \tag{57}$$

More detailed, for any vector field X, Y:

$$\lambda - RicSol(\xi)(X, Y) = g(Q^\lambda_\xi X, Y) + g(X, Q^\lambda_\xi Y) \tag{58}$$

with:

$$Q^\lambda_\xi := \nabla\xi + Q + \lambda I \tag{59}$$

and hence (g, ξ, λ) is a Ricci soliton if and only if ξ belongs to the kernel of $\lambda - RicSol$ considered as real operator.

We recall that a triple $(g, \xi, F \in \mathcal{T}^1_1(M))$ yields a special type of linear connections, called *Golab* or *quarter-symmetric* ([9, p. 9]): $\bar{\nabla}$ is Golab if it is metrical and its torsion is:

$$\bar{T} = F \otimes \eta - \eta \otimes F \tag{60}$$

with η the 1-form g-dual to ξ. Recall also the usual notation of Riemannian geometry from [18, p. 82]; two vector fields X, Y gives the g-skew-symmetric endomorphism $X \wedge_g Y$ given by:

$$(X \wedge_g Y)Z := g(X, Z)Y - g(Y, Z)X. \tag{61}$$

Then, if the above endomorphism F is g-symmetric then the Golab connection exists and is unique provided by:

$$\bar{\nabla}(\xi, F) = \bar{\nabla} := \nabla + \xi \wedge_g F \tag{62}$$

which means:

$$\bar{\nabla}_X := \nabla_X + \xi \wedge_g F(X). \tag{63}$$

A more detailed formula is ([9, p. 9]):

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)FX - g(FX, Y)\xi. \tag{64}$$

Returning to the triple (g, λ, ξ) of equation (59) we associate two Golab connections:

$$\bar{\nabla}^1 := \bar{\nabla}(\xi, Q), \quad \bar{\nabla}^2 = \bar{\nabla}(\xi, Q + \lambda I) = \bar{\nabla}^1 + \lambda[I \otimes \eta - g]\xi. \tag{65}$$

The Golab covariant derivatives of ξ are:

$$\bar{\nabla}^1\xi = \nabla\xi + \|\xi\|^2 Q - Ric(\cdot,\xi)\xi, \quad \bar{\nabla}^2\xi = \nabla\xi + \|\xi\|^2(Q+\lambda I) - [Ric(\cdot,\xi)+\lambda\eta]\xi \tag{66}$$

and then the endomorphism Q^λ_ξ of (59) is expressed through Golab derivatives as:

$$Q^\lambda_\xi = \bar{\nabla}^1\xi + (1-\|\xi\|^2)Q + \lambda I + Ric(\cdot,\xi)\xi = \bar{\nabla}^2\xi + (1-\|\xi\|^2)(Q+\lambda I) + [Ric(\cdot,\xi)+\lambda\eta]\xi. \tag{67}$$

We conclude with:

Proposition 4 *(g,ξ,λ) is a Ricci soliton on M if and only if at least one endomorphism below:*

$$\bar{\nabla}^1\xi + (1-\|\xi\|^2)Q + \lambda I + Ric(\cdot,\xi)\xi, \quad \bar{\nabla}^2\xi + (1-\|\xi\|^2)(Q+\lambda I) + [Ric(\cdot,\xi)+\lambda\eta]\xi \tag{68}$$

is g-skew-symmetric. In particular, if (ξ,λ) is $(\bar{\nabla}^1, (1-\|\xi\|^2)Q + Ric(\cdot,\xi)\xi)$-soliton in the sense of equation (24) *then (g,ξ,λ) is a Ricci soliton.*

Remark 5 i) The 1-form $Ric(\cdot,\xi)$ is the g-dual of the vector field $Q(\xi)$. For a steady Ricci soliton we have $\bar{\nabla}^1 = \bar{\nabla}^2$ and if, in addition, ξ belongs to the kernel of Q then the endomorphism $\bar{\nabla}^1\xi + (1-\|\xi\|^2)Q$ is skew-symmetric.
ii) In defining the above Golab connections we restrict to symmetric terms from the expression (59) of Q^λ_ξ. The entire endomorphism Q^λ_ξ is symmetric for the example of a *self-torse-forming* ξ, namely a torse-forming vector field for which γ of (31) is exactly η:

$$Q^\lambda_\xi = (f+\lambda)I + Q + \eta\otimes\xi. \tag{69}$$

iii) Returning to the map $\lambda - RicSol$ we have the identity:

$$\lambda - RicSol(\xi)(\xi,\xi) = 2\eta(Q^\lambda_\xi(\xi)) \tag{70}$$

and then (g,ξ,λ) is a Ricci solitons if and only if $Q^\lambda_\xi(\xi)\in Ann(\eta)$. □

The last remark yields a new generalization of Ricci solitons:

Definition 6 Let (M,g) be endowed with a non-skew-symmetric endomorphism F. Then the vector field ξ not belonging to the kernel of F is a *(g,F)-soliton* if:

$$F(\xi)\in Ann(\eta) \tag{71}$$

with η being the g-dual of ξ.

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