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Application of the $\exp(-\varphi)$-expansion method to the Pochhammer-Chree equation

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Abstract. In the present article, a direct approach, namely $\exp(-\varphi)$-expansion method, is used for obtaining analytical solutions of the Pochhammer-Chree equations which have a many of models. These solutions are expressed in exponential functions expressed by hyperbolic, trigonometric and rational functions with some parameters. Recently, many methods were attempted to find exact solutions of nonlinear partial differential equations, but it seems that the $\exp(-\varphi)$-expansion method appears to be efficient for finding exact solutions of many nonlinear differential equations.

1. Introduction

Many physical phenomena can be modeled by nonlinear partial differential equations (NLPDEs), such as plasma physics, solid state physics, fluid mechanics, optimal fiber, electro magnetics, chemical physics, propagation of shallow water waves, fluid dynamics, and so on. The exact solutions of nonlinear differential equations have an important role in the investigation of nonlinear physical phenomena. Therefore, many researchers have studied mathematical physics[17] and exact solutions of NLPDEs. Recently, many methods have been used for solving NLPDEs analytically, such as the first integral method [1], the sin-cos-function method [2, 7], the homogeneous balance method [8], the $\frac{G'}{G}$-expansion method[9, 15], the exp-function method [12], the tanh-function method [3], the tan-expansion method [13], the simplest equation method [10, 20], the Hirota′s bilinear method [22], and so on. These methods consider a useful scheme for analytic solutions of a wide class of nonlinear differential equations describing real physical problems. Many solutions types from the most nonlinear equations can be obtained with the above methods. Recently, some new methods were introduced, among which is $\exp(-\varphi)$-expansion method [18, 19]. In this method,

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the travelling wave solutions of nonlinear equations can be expressed by a polynomial in $\exp(-\varphi)$, where $\varphi = \varphi(\xi)$ satisfies the following ordinary differential equation:

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda, \tag{1}$$

where μ, λ are constants. The $\exp(-\varphi)$-expansion method is one of the powerful methods by which the exact and appropriate analytical solutions to nonlinear equations can be obtained. The aim of this article is to investigate the $\exp(-\varphi)$-expansion method to solve the Pochhammer-Chree equation of the form [14]:

$$u_{tt} - u_{xxtt} - (\alpha u + \beta u^{n+1} + \gamma u^{2n+1})_{xx} = 0, \ n \geq 1, \tag{2}$$

where α, β and γ are constants. The Eq.(2) illustrates a nonlinear plan of longitudinal wave propagation of elastic bars [16, 21]. The phenomenon of dispersion is the reason wherefore waves with different wavelengths move with several speeds in the similar material. This phenomenon becomes visible in a cylindrical rod when the radius of the rod is analogous to the wavelength of the wave spread in it [23]. Pochhammer and Chree introduced the accurate formulation of the wave equation in an infinite long cylindrical rod of circular cross-section [6]. The Pochhammer-Chree equation is concluded along the formulation of the motion equation in the diffusion of a sinusoidal wave sequence in an infinite long rod of circular cross-section, when the motion equation is converted into cylindrical coordinates, and the boundary conditions for traction-free surfaces are applied. A first model for $\alpha = 0, \beta = -\frac{1}{2}$ and $\gamma = 0$ was investigated by Parker [5], and solitary wave solutions were obtained for $n = 1, 2$ and 4. Furthermore, a second model for $\alpha = 1, \beta = \frac{1}{n+1}$ and $\gamma = 0$ was studied in [11, 21], and solitary wave solutions were obtained. Moreover, a third model was studied in [16, 23] for $n = 1$ and 2, where kinks solutions and explicit solitary wave solutions were obtained. The extended $\frac{G'}{G}$-expansion method was employed for Eq.(2) with all possible cases of γ by Jin-Ming Zuo [14]. The tanh-coth and the sin-cos-methods for kinks, solitons, and periodic solutions for the Pochhammer-Chree equations were applied [4].

The rest of this article is organized as follows. In section 2, we recall the methodology of the $\exp(-\varphi)$-expansion method. In section 3, we extend the application of the $\exp(-\varphi)$-expansion method to construct analytical solutions for the nonlinear Pochhammer-Chree equation. Finally, conclusions are summarized in section 4.

2. The methodology of exp(−φ)-expansion method

In this section, we illustrate the basic idea of the $\exp(-\varphi)$-expansion method for obtaining exact solutions of NLPDEs. For a given partial differential equation in a form

$$N(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{3}$$

where $u = u(x, t)$, and N is a polynomial in $u = u(x, t)$ and its various partial derivatives. We take the travelling wave transformation

$$\xi = x - ct, \tag{4}$$

where c is a nonzero constant to be determined later. Substituting (4) into (3), we reduce (3) to the following ordinary differential equation:

$$\tilde{N}(u, u', u'', u''', \dots) = 0 \tag{5}$$

to a polynomial $\tilde{N}$. Here prime denotes the derivative with respect to ξ. Exact solutions for this equation can be constructed as a finite series

$$u(\xi) = \sum_{i=0}^{m} A_i (\exp(-\varphi(\xi)))^i, \tag{6}$$

where A_i ($A_m \neq 0$) are constants to be determined later, $\varphi = \varphi(\xi)$ satisfies the following ordinary differential equation:

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda. \tag{7}$$

We know the Eq.(7) has been following special solutions:

- case 1. Hyperbolic function solutions. (When $\lambda^2 - 4\mu > 0$, $\mu \neq 0$.)

$$\varphi_1(\xi) = \mathrm{Ln}\left(\frac{-\lambda - \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + \xi_0)\right)}{2\mu}\right). \tag{8}$$

- case 2. Trigonometric function solutions. (When $\lambda^2 - 4\mu < 0$, $\mu \neq 0$.)

$$\varphi_2(\xi) = \mathrm{Ln}\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}(\xi + \xi_0)\right) - \lambda}{2\mu}\right). \tag{9}$$

- case 3. Hyperbolic function solutions. (When $\lambda^2 - 4\mu > 0$, $\lambda \neq 0$, $\mu = 0$.)

$$\varphi_3(\xi) = -\mathrm{Ln}\left(\frac{\lambda}{\cosh(\lambda(\xi + \xi_0)) + \sinh(\lambda(\xi + \xi_0)) - 1}\right). \tag{10}$$

- case 4. Rational function solutions.(When $\lambda^2 - 4\mu = 0$, $\lambda \neq 0$, $\mu \neq 0$.)

$$\varphi_4(\xi) = \mathrm{Ln}\left(-\frac{2\left(\lambda\left(\xi + \xi_0\right) + 2\right)}{\lambda^2\left(\xi + \xi_0\right)}\right). \tag{11}$$

- case 5. (When $\lambda^2 - 4\mu = 0$, $\lambda = 0$, $\mu = 0$.)

$$\varphi_5(\xi) = \mathrm{Ln}\left(\xi + \xi_0\right). \tag{12}$$

Here ξ_0 is an integration constant. Now the main steps of the $\exp(-\varphi)$-expansion method is to obtain exact solutions of NLPDEs that can be determined as follows:

- Step (1). By considering the homogeneous balance between the highest order derivatives and the highest nonlinear terms of $u(x)$ in Eq.(5), we can obtain the positive integer m in (6).
- Step (2). By substituting (6) with Eq.(7) into (5) and collecting all terms with the same powers of $\exp(-\varphi)$ together, the left hand side of Eq.(5) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of A_m ($m = 0, 1, 2, ..., n$), c, λ, μ.
- Step (3). Solving the system of algebraic equations and then substituting the results and the general solutions of (8)-(12) into (6), it gives travelling wave solutions of (5).

3. Application of exp(−φ)-expansion method to NLPDEs

Here, we use the $\exp(-\varphi)$-expansion method to obtain the travelling wave solutions for the Pochhammer-Chree equation (2). We will conduct our analysis by examining some possible cases of γ, β.

3.1. Case I: $\gamma = 0, \beta \neq 0$.

We now consider the Pochhammer-Chree equation for $\gamma = 0, \beta \neq 0$,

$$u_{tt} - u_{xxtt} - (\alpha u + \beta u^{n+1})_{xx} = 0. \tag{13}$$

Using the travelling wave transformation $\xi = x - ct$, and integrating twice Eq.(13), we have

$$(c^2 - \alpha)u - c^2 u'' - \beta u^{n+1} = 0. \tag{14}$$

By the balancing procedure according to Step (1), between u'' and u^{n+1}, we have:

$$m + 2 = m(n + 1),$$

therefore, we get $m = \frac{2}{n}$. Now m should be an integer. Therefore, we use a transformation as follows:

$$u = \varphi^{\frac{1}{n}}. \tag{15}$$

Using (15), Eq.(14) can be written to

$$\left(c^2 - \alpha\right) n^2 \varphi^2 - nc^2 \varphi\varphi'' - c^2(1 - n)(\varphi')^2 - n^2 \beta \varphi^3 = 0. \tag{16}$$

Balancing $\varphi\varphi''$ and φ^3, we have $m + (m + 2) = 3m$. Therefore, we get $m = 2$.
Also, balancing $(\varphi')^2$ and φ^3, we have $2(m + 1) = 3m$. Therefore, we get $m = 2$.
Therefore, the solution of (16) can be expressed by a polynomial in $\exp(-\varphi)$ as follows:

$$\varphi(\xi) = A_0 + A_1 \exp(-\varphi(\xi)) + A_2(\exp(-\varphi(\xi)))^2, \quad A_2 \neq 0, \tag{17}$$

where φ is the solution of Eq.(7). Substituting (17) into (16) and making use of Eq.(7) and equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for $A_0, A_1, A_2, c, \lambda, \mu$. Solving obtained system using *Mathematica*, we obtain

$$\textit{case}\,1: \quad A_0 = 0,\ A_1 = -\frac{2\alpha\lambda(\mu + 1)(n + 2)}{\beta(n^2 - \lambda^2)},\ A_2 = -\frac{2\alpha(\mu + 1)^2(n + 2)}{\beta(n^2 - \lambda^2)}, \quad \beta\left(n^2 - \lambda^2\right) \neq 0, \tag{18}$$

$$\textit{case}\,2: \quad A_0 = 0,\ A_1 = -\frac{2c^2\lambda(\mu + 1)}{\beta},\ A_2 = -\frac{c^2\lambda^2(\mu + 1)^2}{2\beta}, \quad \alpha = 0,\ \lambda = \pm 2,\ \beta \neq 0, \tag{19}$$

$$\textit{case}\,3: \quad A_0 = 0,\ A_1 = -\frac{2c^2(\mu + 1)(\lambda + 2)}{\beta\lambda},\ A_2 = -\frac{2c^2(\mu + 1)^2(\lambda + 2)}{\beta\lambda^2}, \quad \alpha = 0,\ n = \lambda \geq 1,\ \lambda^2 - 4 \neq 0. \tag{20}$$

Therefore, substituting (18) (case 1) in (17) and using the general solutions of Eq.(7) according to (8)-(12), we obtain solutions of Eq.(16) as follows:

$$
\begin{aligned}
\varphi_1(\xi) &= \frac{4\alpha\mu(\mu+1)(n+2)}{\beta\,(n^2-\lambda^2)}\ \frac{\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda^2-2\mu(\mu+1)}{\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda\right)^2};\\
&\quad \lambda^2-4\mu>0,\ \mu\neq 0,\\
\varphi_2(\xi) &= -\frac{4\alpha\mu(\mu+1)(n+2)}{\beta\,(n^2-\lambda^2)}\ \frac{\lambda\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)-\lambda^2+2\mu(\mu+1)}{\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)\right)^2};\\
&\quad \lambda^2-4\mu<0,\ \mu\neq 0,\\
\varphi_3(\xi) &= -\frac{2\alpha\lambda^2(n+2)}{\beta\,(n^2-\lambda^2)}\ \frac{\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))}{(\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))-1)^2};\\
&\quad \lambda^2-4\mu>0,\ \lambda\neq 0,\ \mu=0,\\
\varphi_4(\xi) &= -\frac{\alpha\lambda^3(\mu+1)(n+2)(\xi+\xi_0)(\lambda(\mu-1)(\xi+\xi_0)-4)}{2\beta(\lambda(\xi+\xi_0)+2)^2\,(n^2-\lambda^2)};\quad \lambda^2-4\mu=0,\ \lambda\neq 0,\ \mu\neq 0,\\
\varphi_5(\xi) &= -\frac{2\alpha(n+2)}{\beta n^2(\xi+\xi_0)^2};\quad \lambda^2-4\mu=0,\ \lambda=0,\ \mu=0.
\end{aligned}
$$

Now, substituting (19) (case 2) in (17) and using the general solutions of Eq.(7) according to (8)-(12), we obtain solutions of Eq.(16) as follows:

$$
\begin{aligned}
\varphi_6(\xi) &= -\frac{2c^2\lambda\mu(\mu+1)}{\beta}\ \frac{\lambda\left(\mu^2+\mu-2\right)-2\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)}{\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda\right)^2};\\
&\quad \lambda^2-4\mu>0,\ \mu\neq 0,\\
\varphi_7(\xi) &= -\frac{2c^2\lambda\mu(\mu+1)}{\beta}\ \frac{2\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)+\lambda\left(\mu^2+\mu-2\right)}{\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)\right)^2};\\
&\quad \lambda^2-4\mu<0,\ \mu\neq 0,\\
\varphi_8(\xi) &= -\frac{c^2\lambda^2}{2\beta}\ \frac{\lambda^2+4\sinh(\lambda(\xi+\xi_0))+4\cosh(\lambda(\xi+\xi_0))-4}{(\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))-1)^2};\\
&\quad \lambda^2-4\mu>0,\ \lambda\neq 0,\ \mu=0,\\
\varphi_9(\xi) &= -\frac{c^2\lambda^3(\mu+1)(\xi+\xi_0)\left(\lambda^3(\mu+1)(\xi+\xi_0)-8\lambda(\xi+\xi_0)-16\right)}{8\beta(\lambda(\xi+\xi_0)+2)^2};\\
&\quad \lambda^2-4\mu=0,\ \lambda\neq 0,\ \mu\neq 0.
\end{aligned}
$$

Finally, substituting (20) (case 3) in (17) and using the general solutions of Eq.(7) according to (8)-(12), we obtain solutions of Eq.(16) as follows:

$$
\begin{aligned}
\varphi_{10}(\xi) &= \frac{4c^2(\lambda+2)\mu(\mu+1)}{\beta\lambda^2}\,\frac{\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda^2-2\mu(\mu+1)}{\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda\right)^2};\\
&\quad \lambda^2-4\mu>0,\ \mu\neq 0,\\
\varphi_{11}(\xi) &= \frac{4c^2(\lambda+2)\mu(\mu+1)}{\beta\lambda^2}\,\frac{-\lambda\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)+\lambda^2-2\mu(\mu+1)}{\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)\right)^2};\\
&\quad \lambda^2-4\mu<0,\ \mu\neq 0,\\
\varphi_{12}(\xi) &= -\frac{2c^2(\lambda+2)}{\beta}\,\frac{\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))}{(\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))-1)^2};\\
&\quad \lambda^2-4\mu>0,\ \lambda\neq 0,\ \mu=0,\\
\varphi_{13}(\xi) &= -\frac{c^2\lambda(\lambda+2)(\mu+1)(\xi+\xi_0)(\lambda(\mu-1)(\xi+\xi_0)-4)}{2\beta(\lambda(\xi+\xi_0)+2)^2};\quad \lambda^2-4\mu=0,\ \lambda\neq 0,\ \mu\neq 0.
\end{aligned}
$$

3.2. Case II: $\beta=0,\gamma\neq 0$.

We next study the Pochhammer-Chree equation for $\beta=0,\gamma\neq 0$. Using the travelling wave transformation $\xi=x-ct$, and integrating twice we have:

$$(c^2-\alpha)u-c^2u''-\gamma u^{2n+1}=0. \tag{21}$$

By the balancing procedure according to Step (1) we get $m=\frac{1}{n}$. Therefore, proceeding as before, we use a transformation as follows:

$$u=\varphi^{\frac{1}{n}}. \tag{22}$$

Using (22), Eq.(21) can be written to

$$\left(c^2-\alpha\right)n^2\varphi^2-nc^2\varphi\varphi''-c^2(1-n)(\varphi')^2-n^2\gamma\varphi^4=0. \tag{23}$$

Balancing $\varphi\varphi''$ and φ^4 gives $m=1$. Therefore, the solution of (23) can be expressed by a polynomial in $\exp(-\varphi)$ as follows:

$$\varphi(\xi)=A_0+A_1\exp(-\varphi(\xi)),\quad A_1\neq 0, \tag{24}$$

where φ is the solution of Eq.(7). Substituting (24) into (23) and making use of Eq.(7) and equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for $A_0, A_1, c, \lambda, \mu$. Solving obtained system using *Mathematica*, we obtain

$$\textit{case}\,1:\quad A_0=\frac{c}{\sqrt{\gamma}},\ A_1=\frac{c\lambda(\mu+1)(5\lambda^2-14)}{24\sqrt{\gamma}},\quad \lambda=\pm i\sqrt{2},\ \alpha=0,\ n=-\frac{\lambda^2}{2}, \tag{25}$$

$$\textit{case}\,2:\quad A_0=\frac{\sqrt{c^2-\alpha}}{\sqrt{\gamma}},\ A_1=\frac{(\mu+1)(-10\alpha+c^2(5\lambda^2-14))}{24\sqrt{\gamma(c^2-\alpha)}},\quad \lambda^2+2\neq 0,\ n=1,\ c=\pm\frac{\sqrt{2\alpha}}{\sqrt{\lambda^2+2}},\ \gamma\neq 0. \tag{26}$$

Therefore, substituting (25) (case 1) in (24) and using the general solutions of Eq.(7) according to (8)-(12), we obtain solutions of Eq.(23) as follows:

$$\begin{aligned}
\varphi_1(\xi) &= \frac{\sqrt{c}}{\sqrt{\gamma}} - \frac{c\lambda\left(5\lambda^2-14\right)\mu(\mu+1)}{12\sqrt{\gamma}\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda\right)}; \\
&\quad \lambda^2-4\mu>0,\ \mu\neq 0, \\
\varphi_2(\xi) &= \frac{\sqrt{c}}{\sqrt{\gamma}} - \frac{c\lambda\left(5\lambda^2-14\right)\mu(\mu+1)}{12\sqrt{\gamma}\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)\right)}; \\
&\quad \lambda^2-4\mu<0,\ \mu\neq 0, \\
\varphi_3(\xi) &= \frac{\sqrt{c}}{\sqrt{\gamma}} + \frac{c\left(5\lambda^2-14\right)\lambda^2}{24\sqrt{\gamma}(\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))-1)}; \\
&\quad \lambda^2-4\mu>0,\ \lambda\neq 0,\ \mu=0, \\
\varphi_4(\xi) &= \frac{\sqrt{c}}{\sqrt{\gamma}} - \frac{c\lambda^3\left(5\lambda^2-14\right)(\mu+1)(\xi+\xi_0)}{48\sqrt{\gamma}(\lambda(\xi+\xi_0)+2)}; \quad \lambda^2-4\mu=0,\ \lambda\neq 0,\ \mu\neq 0.
\end{aligned}$$

Finally, substituting (26) (case 2) in (24) and using the general solutions of Eq.(7) according to (8)-(12), we obtain solutions of Eq.(23) as follows:

$$\begin{aligned}
\varphi_5(\xi) &= \frac{\sqrt{c^2-\alpha}}{\sqrt{\gamma}} + \frac{\mu(\mu+1)\left(10\alpha+c^2\left(14-5\lambda^2\right)\right)}{12\sqrt{\gamma(c^2-\alpha)}\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(\xi+\xi_0)\sqrt{\lambda^2-4\mu}\right)+\lambda\right)}; \\
&\quad \lambda^2-4\mu>0,\ \mu\neq 0, \\
\varphi_6(\xi) &= \frac{\sqrt{c^2-\alpha}}{\sqrt{\gamma}} + \frac{\mu(\mu+1)\left(10\alpha+c^2\left(14-5\lambda^2\right)\right)}{12\sqrt{\gamma\,(c^2-\alpha)}\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(\xi+\xi_0)\sqrt{4\mu-\lambda^2}\right)\right)}; \\
&\quad \lambda^2-4\mu<0,\ \mu\neq 0, \\
\varphi_7(\xi) &= \frac{\sqrt{c^2-\alpha}}{\sqrt{\gamma}} + \frac{\lambda\left(c^2\left(5\lambda^2-14\right)-10\alpha\right)}{24\sqrt{\gamma\,(c^2-\alpha)}(\sinh(\lambda(\xi+\xi_0))+\cosh(\lambda(\xi+\xi_0))-1)}; \\
&\quad \lambda^2-4\mu>0,\ \lambda\neq 0,\ \mu=0, \\
\varphi_8(\xi) &= \frac{\sqrt{c^2-\alpha}}{\sqrt{\gamma}} - \frac{\lambda^2(\mu+1)(\xi+\xi_0)\left(c^2\left(5\lambda^2-14\right)-10\alpha\right)}{48(\lambda(\xi+\xi_0)+2)\sqrt{\gamma\,(c^2-\alpha)}}; \quad \lambda^2-4\mu=0,\ \lambda\neq 0,\ \mu\neq 0, \\
\varphi_9(\xi) &= \frac{\sqrt{c^2-\alpha}}{\sqrt{\gamma}} + \frac{-14c^2-10\alpha}{24(\xi+\xi_0)\sqrt{\gamma\,(c^2-\alpha)}}; \quad \lambda^2-4\mu=0,\ \lambda=0,\ \mu=0.
\end{aligned}$$

4. Concluding remarks

In this paper, we obtained exact travelling wave solutions for the nonlinear Pochhammer-Chree equation. These exact solutions are composed from the hyperbolic function solutions, rational function solutions and trigonometric function solutions as well, and they are very useful in many circumstances. The applied method to get these solutions was the $\exp(-\varphi)$-expansion method. It was concluded that the $\exp(-\varphi)$-expansion method is a direct and a powerful method for solving nonlinear equations of mathematical physics. In the future the direct method could be used to solve nonlinear systems of mathematical systems (e.g. [17]).

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