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Almost Sure Limit Theorem for the Order Statistics of Stationary Gaussian Sequences

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Abstract. In this paper, by using a new comparison inequality for order statistics of Gaussian variables, we proved an almost sure central limit theorem for extreme order statistics of stationary Gaussian sequences with covariance r_n under the condition $r_n \log n(\log\log n)^{1+\varepsilon} = O(1)$ for some $\varepsilon > 0$. A similar result on intermediate order statistics is also proved for stationary Gaussian sequences. The obtained results improve some of the existing results.

1. Introduction

The almost sure central limit theorem (ASCLT) has been first introduced independently by Brosamler (1988) and Schatte (1988) for partial sum, and then it become an intensively studied subject. Fahrner and Stadtmüller (1998) and independently Cheng et al. (1998) investigated the ASCLT for the maxima $M_n = \max_{k\le n} X_k$ of independent random variables and showed that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}(a_n(M_n - b_n) \le x) = G(x) \ \ a.s. \tag{1}$$

for any $x \in \mathbb{R}$ under the conditions that

$$\lim_{n\to\infty} P(a_n(M_n - b_n) \le x) = G(x) \tag{2}$$

with real sequences $a_n > 0, b_n \in \mathbb{R}, n \ge 1$ and a non-degenerate distribution $G(x)$, where $\mathbf{1}$ denotes the indicator function.

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Csáki and Gonchigdanzan (2002) extended (1) to stationary Gaussian case. Let $X_1, X_2, \cdots$ be a sequence of stationary Gaussian random variables with covariance function $r_n = EX_1X_{n+1}$ satisfying

$$r_n \log n(\log\log n)^{1+\varepsilon} = O(1) \tag{3}$$

for some $\varepsilon > 0$. Csáki and Gonchigdanzan (2002) showed that condition (3) is enough for (1). For more work on this topic, we refer to Chen and Lin (2006) and Peng and Nadarajah (2011) for the non-stationary Gaussian case, Tan (2013) for continuous time Gaussian process, Tan and Wang (2014) and Wu (2017) for Gaussian random field.

It is also of interest to extend (1) to order statistics. The pioneers in this direction are Stadtmüller (2002), Peng and Qi (2003) who studied the ASCLT for intermediate and central order statistics of i.i.d. random variables. Hörmann (2005) provided a relative simple proof for ASCLT for order statistics. Especially for extreme order statistics i.e., for some fixed $k \in \mathbb{N}$ they showed that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}(a_n(M_n^{(k)} - b_n) \le x) = G(x) \sum_{s=0}^{k-1} \frac{(-\log G(x))^s}{s!} \quad a.s. \tag{4}$$

for any $x \in \mathbb{R}$ provided that (2) held, where $M_n^{(k)}$ denotes the k-th maximum of $X_1, \ldots, X_n$. Dudziński (2009) extended (4) to stationary Gaussian sequences provided that the covariance function of the sequence satisfies the following condition, i.e.,

$$\sum_{i=\lfloor n^{1/k^\beta} \rfloor}^{\infty} |r_i| \le O(1) \frac{1}{n^{k-1-1/k^{\beta-1}+1/k^\beta}} \tag{5}$$

for some $\beta > 1$, where $\lfloor x \rfloor$ denotes the integral part of x. ASCLT for intermediate order statistics was also obtained under condition (5) and some other conditions. By studying the exceedance point processes of some stationary sequences, Tan (2015) proved that (4) still holds under some long range dependence conditions. As an application to stationary Gaussian case, it is shown that the following convergence rate on the covariance function is enough, i.e.,

$$r_n = O(|n|^{-(1+\varepsilon)})$$

with some $\varepsilon > 0$, but the spectral of the Gaussian sequence $\{X_n\}_{n\ge1}$ should be bounded below. In this paper, we show that (4) holds for stationary Gaussian sequence with covariance function satisfying condition (3), which completes the work of Dudziński (2009) and Tan (2015). The ASCLT for intermediate order statistics from stationary Gaussian sequence is also studied.

In the following part of this paper, let $X_1, X_2, \cdots$ be a sequence of stationary Gaussian random variables with covariance function $r_n = EX_1X_{n+1}$ and denote by $M_n^{(1)} \ge M_n^{(2)} \ge, \cdots, \ge M_n^{(n)}$ the order statistics of $X_1, X_2, \cdots, X_n$.

For the extreme order statistics, Theorem 5.3.1 of Leadbetter et al. (1983) provided the following result.

Theorem 1.1. *Assume that the covariance function r_n of the stationary Gaussian sequence $\{X_n, n \ge 1\}$ satisfies*

$$r_n \log n \to 0 \text{ as } n \to \infty.$$

If moreover, the numerical sequences u_n fulfills the relation

$$n(1 - \Phi(u_n)) \to \tau, \text{ for } 0 < \tau < \infty \text{ as } n \to \infty \tag{6}$$

then we have

$$\lim_{n\to\infty} P(M_n^{(k)} \le u_n) = e^{-\tau} \sum_{s=0}^{k-1} \frac{\tau^s}{s!} \tag{7}$$

for some fixed $k \in \mathbb{N}$. As a direct conclusion of (7), we have, if,

$$a_n = (2\log n)^{1/2}, \ \ b_n = a_n - (\log\log n + \log 4\pi)/(2(2\log n)^{1/2}), \tag{8}$$

then

$$\lim_{n\to\infty} P(a_n(M_n^{(k)} - b_n) \le x) = \exp(-e^{-x}) \sum_{s=0}^{k-1} \frac{(e^{-x})^s}{s!} \tag{9}$$

for any $x \in \mathbb{R}$.

Now, let k_n be integers such that $1 \le k_n \le n$ for each n. Then if $k_n \to \infty$ but $k_n/n \to 0$, $\{M_n^{(k_n)}\}$ is called a sequence of intermediate order satistics and $\{k_n\}$ an intermediate rank sequence. Define $\theta = \theta(k_n)$ by

$$\theta = \inf\{\theta' : k_n = O(n^{\theta'})\}.$$

For the intermediate order statistics from stationary Gaussian sequence, Watts et al. (1982) proved the following result.

Theorem 1.2. *Assume that the covariance function r_n of the stationary Gaussian sequence $\{X_n, n \ge 1\}$ satisfies*

$$r_n = O(n^{-\rho}) \ \ \text{for some} \ \ \rho > \max\{3\theta/2, 2(2 - 1/\theta)\}, \tag{10}$$

and suppose that in addition $k_n/(\log n)^{2/\rho} \to \infty$. Then

$$\lim_{n\to\infty} P(\alpha_n(M_n^{(k_n)} - \beta_n) \le x) = \Phi(x), \tag{11}$$

for any $x \in \mathbb{R}$, where α_n and β_n are defined by $\Phi(\beta_n) = 1 - k_n/n$ and $\alpha_n = n\Phi'(\beta_n)/\sqrt{k_n}$ and $\Phi(x)$ stands for the standard normal distribution function.

In this paper, we extend Theorems 1.1 and 1.2 to the almost sure limit sure version. As a by-product, we show that condition (10) can be weakened, if we assume that k_n does not increase too faster.

2. Main results

Now we state our main results. The first result is about the ASCLT for extreme order statistics.

Theorem 2.1. *Let $X_1, X_2, \cdots$ be a standardized stationary Gaussian sequence with covariance function $r_n = E(X_1X_{n+1})$ satisfying (3), i.e.,*

$$r_n \log n(\log\log n)^{1+\varepsilon} = O(1)$$

for some $\varepsilon > 0$. Then:

(i). If the numerical sequence u_n fulfills (6), we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}(M_n^{(k)} \le u_n) = e^{-\tau} \sum_{s=0}^{k-1} \frac{(\tau)^s}{s!} \ \ a.s. \tag{12}$$

for some fixed $k \in \mathbb{N}$.

(ii). If a_n, b_n are defined as in (8), we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}(a_n(M_n^{(k)} - b_n) \le x) = \exp(-e^{-x}) \sum_{s=0}^{k-1} \frac{(e^{-x})^s}{s!} \ \ a.s. \tag{13}$$

for any $x \in \mathbb{R}$ and some fixed $k \in \mathbb{N}$

Remark 2.1. Under the same conditions, Theorem 2.1 extends the main result of Csáki and Gonchigdanzan (2002) to the k−th maxima. Theorem 2.1 also improves the results of Dudziński (2009) and Tan (2015).

For the intermediate order statistics, we have the following result.

Theorem 2.2. *Let $X_1, X_2, \cdots$ be a standardized stationary Gaussian sequence with covariance function $r_n = E(X_1 X_{n+1})$ satisfying*

$$r_n = O(n^{-\rho}) \ \text{ for some } \ \rho > 0,$$

and suppose that in addition $k_n \to \infty$ and $\log k_n \ll (\log n)^{1-\varepsilon}$ for some $\varepsilon > 0$. Then:

(i). If the numerical sequence v_n fulfills

$$n\Phi(v_n)(1-\Phi(v_n)) \to \infty \ \text{ and } \ \frac{k_n - n(1-\Phi(v_n)}{n\Phi(v_n)(1-\Phi(v_n))} \to \tau \tag{14}$$

as $n \to \infty$ for some fixed constant τ, we have

$$\lim_{n\to\infty} P(M_n^{(k_n)} \le v_n) = \Phi(\tau) \tag{15}$$

and

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}(M_n^{(k_n)} \le v_n) = \Phi(\tau) \ \ a.s. \tag{16}$$

(ii). If α_n and β_n are defined by $\Phi(\beta_n) = 1 - k_n/n$ and $\alpha_n = n\Phi'(\beta_n)/\sqrt{k_n}$, we have

$$\lim_{n\to\infty} P(\alpha_n(M_n^{(k_n)} - \beta_n) \le x) = \Phi(x) \tag{17}$$

and

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}(\alpha_n(M_n^{(k_n)} - \beta_n) \le x) = \Phi(x) \ \ a.s. \tag{18}$$

for any $x \in \mathbb{R}$.

Remark 2.2. (i). This result does not need the condition on the constant ρ as in Theorem 1.2, since here we assume that k_n does not increase too fast, i.e, $\log k_n \ll (\log n)^{1-\varepsilon}$ for some $\varepsilon > 0$. Note that the condition $k_n/(\log n)^{2/\rho} \to \infty$ is replaced by $k_n \to \infty$ as $n \to \infty$. Thus, Theorem 2.2 extends the main results of Dudziński (2009).

(ii). Especially, we can choose the normalized constants α_n and β_n as

$$\alpha_n = \left(\frac{2\log(n/k_n)}{k_n}\right)^{1/2} \ \text{ and } \ \beta_n = (2\log(n/k_n))^{1/2} - \frac{\log\log(n/k_n) + \log 4\pi}{2(2\log(n/k_n))^{1/2}}.$$

(iii). The results in Theorem 2.1 and 2.2 can be extended to more general weight sequences, ie., the sequence n^{-1} and $\log N$ can be replaced by such as $d_n = n^{-1}\exp(\ln^{\alpha} n)$ and $D_N = \sum_{n=1}^{N} d_n$, respectively. See Wu (2017) and the references therein for more details. Since the proof of the general case is similar with that of Theorem 2.1 and 2.2, we omit the details.

3. Proofs of the main results

Before giving the proofs, we state and prove several lemmas which will be used in the proofs of our main results. Let $Y_1, Y_2, \cdots$, be an associated independent sequence of $X_1, X_2, \cdots$, i.e., a sequence of independent standard normal random variables, and we denote by $\widetilde{M}_n^{(1)} \ge \widetilde{M}_n^{(2)} \ge, \cdots, \ge \widetilde{M}_n^{(n)}$ the order statistics of $Y_1, Y_2, \cdots, Y_n$. For $n - m > k$, let $M_{m,n}^{(k)}$ and $\widetilde{M}_{m,n}^{(k)}$ be the k-th maximum of $X_{m+1}, \ldots, X_n$ and $Y_{m+1}, \ldots, Y_n$, respectively. As usual, $a_n \ll b_n$ means $\limsup_{n\to\infty} |a_n/b_n| < +\infty$. K will denote a constant whose value will change from line to line.

Lemma 3.1. *Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of uniformly bounded random variables, i.e., there exists some $M \in (0, \infty)$ such that $|\xi_k| \le M$ a.s. for all $k \in N$. If*

$$Var\left(\sum_{n=1}^{N} \frac{1}{n}\xi_n\right) \ll (\log N)^2 (\log\log N)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$, then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n}(\xi_n - E\xi_n) = 0 \ \ a.s.$$

Proof: See Lemma 3.1 of Csáki and Gonchigdanzan (2002).

The following lemma is from Dębicki et al. (2017) which plays a crucial role in the proofs of Theorems 2.1 and 2.2.

Lemma 3.2. *Denote by $\mathcal{X} = (X_{il})_{d\times n}$ and $\mathcal{Y} = (Y_{il})_{d\times n}$ two random arrays with $N(0,1)$ components, and let $(\sigma_{il,jk}^{(1)})_{dn\times dn}$ and $(\sigma_{il,jk}^{(0)})_{dn\times dn}$ be the covariance matrices of $\mathcal{X}$ and $\mathcal{Y}$, respectively, with $\sigma_{il,jk}^{(1)} := EX_{il}X_{jk}$ and $\sigma_{il,jk}^{(0)} := EY_{il}Y_{jk}, 1 \le i, j \le d, 1 \le l, k \le n$. Furthermore, define $(M_{1n}^{(r)}(X), \cdots, M_{dn}^{(r)}(X))$ to be the r-th order statistics vector generated by $\mathcal{X}$ as follows*

$$M_{in}^{(1)}(X) = \max_{1\le l\le n} X_{il} \ge \cdots \ge M_{in}^{(r)}(X) \ge \cdots \ge \min_{1\le l\le n} X_{il} = M_{in}^{(n)}(X), \quad 1 \le i \le d.$$

Similarly, we write $(M_{1n}^{(r)}(Y), \cdots, M_{dn}^{(r)}(Y))$ which is generated by $\mathcal{Y}$. Then for any real numbers $u_1, \ldots, u_d$, and any $1 \le r \le n$

$$\begin{aligned}&\left|P\Big(M_{in}^{(r)}(X) \le u_i, i = 1, \ldots, d\Big) - P\Big(M_{in}^{(r)}(Y) \le u_i, i = 1, \ldots, d\Big)\right|\\ &\le K \sum_{\substack{1\le i,j\le d\\ 1\le l,k\le n}} |\arcsin(\sigma_{il,jk}^{(1)}) - \arcsin(\sigma_{il,jk}^{(0)})| \exp\left(-\frac{u_i^2 + u_j^2}{2(1+\rho_{il,jk})}\right),\end{aligned}$$

where $\rho_{il,jk} = \max\{|\sigma_{il,jk}^{(0)}|, |\sigma_{il,jk}^{(1)}|\}$.

Lemma 3.3. *Under the conditions of Theorem 2.1, we have for $m + k < n$*

$$E|\mathbf{1}(M_{m,n}^{(k)} \le u_n) - \mathbf{1}(M_n^{(k)} \le u_n)| \ll k\frac{m}{n} + 2(\log\log n)^{-(1+\varepsilon)}$$

and

$$|Cov(\mathbf{1}(M_m^{(k)} \le u_m), \mathbf{1}(M_{m,n}^{(k)} \le u_n))| \ll (\log\log n)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$.

Proof. Obviously, we have

$$\begin{aligned}&E|\mathbf{1}(M_{m,n}^{(k)} \le u_n) - \mathbf{1}(M_n^{(k)} \le u_n)|\\ &= P(M_{m,n}^{(k)} \le u_n) - P(M_n^{(k)} \le u_n)\\ &\le |P(M_{m,n}^{(k)} \le u_n) - P(\widetilde{M}_{m,n}^{(k)} \le u_n)| + |P(M_n^{(k)} \le u_n) - P(\widetilde{M}_n^{(k)} \le u_n)| + |P(\widetilde{M}_{m,n}^{(k)} \le u_n) - P(\widetilde{M}_n^{(k)} \le u_n)|\\ &=: A_{n,1} + A_{n,2} + A_{n,3}.\end{aligned}$$

By Lemma 3.2 for the case $d = 1$ and Lemma 2.1 of Csáki and Gonchigdanzan (2002), we have

$$A_{n,1} \le Kn \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_n^2}{1+|r_j|}\right) \ll (\log\log n)^{-(1+\varepsilon)}$$

and

$$A_{n,2} \le Kn \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_n^2}{1+|r_j|}\right) \ll (\log\log n)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$. By Lemma 1 of Peng et al. (2009), we have

$$A_{n,3} \le P(\widetilde{M}_{m,n}^{(k)} \ne \widetilde{M}_n^{(k)}) \le k\frac{m}{n},$$

which completes the first assertion of the lemma. For the second assertion, by Lemma 3.2 for case $d = 2$ and Lemma 2.1 of Csáki and Gonchigdanzan (2002) again, we have

$$\begin{aligned} |Cov(\mathbf{1}(M_m^{(k)} \le u_m), \mathbf{1}(M_{m,n}^{(k)} \le u_n))| &= |P(M_m^{(k)} \le u_m, M_{m,n}^{(k)} \le u_n) - P(M_m^{(k)} \le u_m)P(M_{m,n}^{(k)} \le u_n)| \\ &\le \sum_{i=1}^{m}\sum_{j=m+1}^{n} |r_{j-i}| \exp\left(-\frac{u_m^2+u_n^2}{2(1+|r_{j-i}|)}\right) \\ &\le m\sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_m^2+u_n^2}{2(1+|r_j|)}\right) \ll (\log\log n)^{-(1+\varepsilon)}. \end{aligned}$$

The proof of the lemma is complete.

Lemma 3.4. *Under the conditions of Theorem 2.2, we have*

$$n\sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_n^2}{1+|r_j|}\right) \ll n^{-\epsilon}$$

and

$$\sup_{1\le m\le \frac{n}{k_n}-1} m\sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_m^2+v_n^2}{2(1+|r_j|)}\right) \ll n^{-\epsilon'}$$

for some $\epsilon, \epsilon' > 0$.

Proof. By the conditions of (14), it is easy to see that

$$k_n/n \sim 1-\Phi(v_n) \sim (2\pi)^{-1/2} v_n^{-1} \exp(-\frac{v_n^2}{2})$$

and taking logarithms gives

$$v_n \sim \sqrt{2\log(n/k_n)}$$

so that

$$\exp(-\frac{v_n^2}{2}) \sim 2\sqrt{\pi}(k_n/n)(\log(n/k_n))^{1/2}.$$

Let

$$\delta = \sup_{n\ge 1}|r_n|, \quad \delta_m = \sup_{n\ge m}|r_n|.$$

It is easy to see that since $r_n = O(n^{-\rho})$, we must have $\delta < 1$ and $\delta_n = O(n^{-\rho})$. Thus, we can chose γ such that $0 < \gamma < \frac{2}{1+\delta} - 1$. We have

$$\begin{aligned} n\sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_n^2}{1+|r_j|}\right) &= n\sum_{j=1}^{\lfloor n^\gamma \rfloor} |r_j| \exp\left(-\frac{v_n^2}{1+|r_j|}\right) + n\sum_{j=\lfloor n^\gamma\rfloor+1}^{n} |r_j| \exp\left(-\frac{v_n^2}{1+|r_j|}\right) \\ &=: B_{n,1} + B_{n,2}. \end{aligned}$$

For $B_{n,1}$, noting that, by assumption $\log k_n \ll (\log n)^{1-\varepsilon}$, $k_n \ll n^{\lambda}$ for any $\lambda > 0$, we have

$$\begin{aligned} B_{n,1} &\leq n\sum_{j=1}^{\lfloor n^{\gamma}\rfloor} \exp\left(-\frac{v_n^2}{1+\delta}\right) \\ &\ll n^{1+\gamma-2/(1+\delta)}(k_n)^{2/(1+\delta)}(\log(n/k_n))^{2/(1+\delta)} \\ &\leq n^{-\epsilon_1} \end{aligned}$$

for some $\epsilon_1 > 0$. For $B_{n,2}$, since $\delta_n \leq Kn^{-\rho}$ by $r_n = O(n^{-\rho})$, $v_n^2\delta_{n^{\gamma}} \leq K(\log n)n^{-\rho\gamma} < \varepsilon$ for any $\varepsilon > 0$, and hence

$$\begin{aligned} B_{n,2} &\leq n\delta_{n^{\gamma}} \exp\left(-v_n^2\right) \sum_{j=\lfloor n^{\gamma}\rfloor+1}^{n} \exp\left(\frac{v_n^2|r_j|}{1+|r_j|}\right) \\ &\ll n^2\delta_{n^{\gamma}} \exp\left(-v_n^2\right)\exp\left(v_n^2\delta_{n^{\gamma}}\right) \\ &\ll \delta_{n^{\gamma}}k_n^2\log(n/k_n) \\ &\leq n^{-\epsilon_2} \end{aligned}$$

for some $\epsilon_2 > 0$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we get the desired result. Similarly, write

$$\begin{aligned} &\sup_{1\leq m\leq \frac{n}{k_n}-1} m\sum_{j=1}^{n}|r_j|\exp\left(-\frac{v_m^2+v_n^2}{2(1+|r_j|)}\right) \\ &\leq \sup_{1\leq m\leq \frac{n}{k_n}-1} m\sum_{j=1}^{\lfloor n^{\gamma}\rfloor}|r_j|\exp\left(-\frac{v_m^2+v_n^2}{2(1+|r_j|)}\right) + \sup_{1\leq m\leq \frac{n}{k_n}-1} m\sum_{j=\lfloor n^{\gamma}\rfloor+1}^{n}|r_j|\exp\left(-\frac{v_m^2+v_n^2}{2(1+|r_j|)}\right) \\ &=: B_{n,3} + B_{n,4}. \end{aligned}$$

For the first term, recalling that $k_n \ll n^{\lambda}$ for any $\lambda > 0$, we have

$$\begin{aligned} B_{n,3} &\leq \sup_{1\leq m\leq \frac{n}{k_n}-1} m\sum_{j=1}^{\lfloor n^{\gamma}\rfloor}\exp\left(-\frac{v_m^2+v_n^2}{2(1+\delta)}\right) \\ &\ll \sup_{1\leq m\leq \frac{n}{k_n}-1} mn^{\gamma}(k_m/m)^{1/(1+\delta)}(k_n/n)^{1/(1+\delta)}(\log(m/k_m))^{1/2(1+\delta)}(\log(n/k_n))^{1/2(1+\delta)} \\ &\leq n^{1+\gamma-2/(1+\delta)}(k_n)^{2/(1+\delta)}(\log(n/k_n))^{1/(1+\delta)} \\ &\leq n^{-\epsilon_3} \end{aligned}$$

for some $\epsilon_3 > 0$. For $B_{n,4}$, we have

$$\exp\left(\frac{(v_m^2+v_n^2)\delta_{n^{\gamma}}}{2}\right) \leq \exp\left(v_n^2\delta_{n^{\gamma}}\right) \leq K$$

Thus,

$$\begin{aligned} B_{n,4} &\leq \sup_{1\leq m\leq \frac{n}{k_n}-1} m\delta_{n^{\gamma}}\exp\left(-\frac{v_m^2+v_n^2}{2}\right)\sum_{j=n^{\gamma}}^{n}\exp\left(\frac{(v_m^2+v_n^2)|r_j|}{2(1+|r_j|)}\right) \\ &\leq \sup_{1\leq m\leq \frac{n}{k_n}-1} mn\delta_{n^{\gamma}}\exp\left(-\frac{v_m^2+v_n^2}{2}\right)\exp\left(\frac{(v_m^2+v_n^2)\delta_{n^{\gamma}}}{2}\right) \\ &\ll \sup_{1\leq m\leq \frac{n}{k_n}-1} mn\delta_{n^{\gamma}}(k_n/n)(k_m/m)(\log(n/k_n))^{1/2}(\log(m/k_m))^{1/2} \\ &\leq \delta_{n^{\gamma}}k_n^2(\log(n/k_n)) \\ &\leq n^{-\epsilon_4} \end{aligned}$$

for some $\epsilon_4 > 0$. Setting $\epsilon' = \min\{\epsilon_3, \epsilon_4\}$, we get the desired result.

Lemma 3.5. *Under the conditions of Theorem 2.2, we have for* $m \le \frac{n}{k_n} - 1$

$$E|\mathbf{1}(M_{m,n}^{(k_n)} \le v_n) - \mathbf{1}(M_n^{(k_n)} \le v_n)| \ll k_n \frac{m}{n - k_n} + 2n^{-\epsilon}$$

and

$$Cov(\mathbf{1}(M_m^{(k_m)} \le v_m), \mathbf{1}(M_{m,n}^{(k_n)} \le v_n)) \ll n^{-\epsilon'}.$$

Proof. Obviously, we have

$$\begin{aligned}
&E|\mathbf{1}(M_{m,n}^{(k_n)} \le v_n) - \mathbf{1}(M_n^{(k_n)} \le v_n)| \\
&= P(M_{m,n}^{(k_n)} \le v_n) - P(M_n^{(k_n)} \le v_n) \\
&\le |P(M_{m,n}^{(k_n)} \le v_n) - P(\widetilde{M}_{m,n}^{(k_n)} \le v_n)| + |P(M_n^{(k_n)} \le v_n) - P(\widetilde{M}_n^{(k_n)} \le v_n)| \\
&\quad + |P(\widetilde{M}_{m,n}^{(k_n)} \le v_n) - P(\widetilde{M}_n^{(k_n)} \le v_n)| \\
&=: C_{n,1} + C_{n,2} + C_{n,3}.
\end{aligned}$$

By Lemmas 3.2 and 3.4, we have

$$C_{n,1} \le Kn \sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_n^2}{1 + |r_j|}\right) \ll n^{-\epsilon}$$

and

$$C_{n,2} \le Kn \sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_n^2}{1 + |r_j|}\right) \ll n^{-\epsilon}.$$

By Lemma 1 of Stadtmüller (2002), we have

$$C_{n,3} \le k_n \frac{m}{n - k_n},$$

which completes the first assertion of the lemma. For the second assertion, By Lemmas 3.2 and 3.4 again, we have

$$\begin{aligned}
|Cov(\mathbf{1}(M_m^{(k_n)} \le v_m), \mathbf{1}(M_{m,n}^{(k_n)} \le v_n))| &= |P(M_m^{(k_n)} \le v_m, M_{m,n}^{(k_n)} \le v_n) - P(M_m^{(k_n)} \le v_m)P(M_{m,n}^{(k_n)} \le v_n)| \\
&\le \sum_{i=1}^{m} \sum_{j=m+1}^{n} |r_{j-i}| \exp\left(-\frac{v_m^2 + v_n^2}{2(1 + |r_{j-i}|)}\right) \\
&\le m \sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_m^2 + v_n^2}{2(1 + |r_j|)}\right) \ll n^{-\epsilon'}.
\end{aligned}$$

The proof of the lemma is complete.

Proof of Theorem 2.1. We first prove (i). Let $\eta_n = \mathbf{1}(M_n^{(k)} \le u_n) - P(M_n^{(k)} \le u_n)$. Notice that $(\eta_n)_{n=1}^{\infty}$ is a sequence of bounded random variables with $Var(\eta_n) \le 1$. We first show

$$\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \eta_n = 0, \quad a.s. \tag{19}$$

Using Lemma 3.1, we only need to show

$$Var\left(\sum_{n=1}^{N} \frac{1}{n} \eta_n\right) \ll (\log N)^2 (\log\log N)^{-(1+\varepsilon)}. \tag{20}$$

We have,

$$\begin{aligned} Var\left(\sum_{n=1}^{N}\frac{1}{n}\eta_n\right) &= E\left(\sum_{n=1}^{N}\frac{1}{n}\eta_n\right)^2 \\ &= \sum_{n=1}^{N}\frac{E\eta_n^2}{n^2}+2\sum_{1\le m<n\le N}\frac{E(\eta_m\eta_n)}{nm} \\ &=: L_{N,1}+2L_{N,2}. \end{aligned}$$

Clearly

$$L_{N,1}=\sum_{n=1}^{N}\frac{1}{n^2}E\eta_n^2\le\sum_{n=1}^{N}\frac{1}{n^2}=O(1).$$

For $L_{N,2}$, for $n>m+k$, by Lemma 3.3, we have

$$\begin{aligned} \left|E(\eta_m\eta_n)\right| &= \left|Cov(\mathbf{1}(M_m^{(k)}\le u_m),\mathbf{1}(M_n^{(k)}\le u_n)\right| \\ &\le \left|Cov(\mathbf{1}(M_m^{(k)}\le u_m),[\mathbf{1}(M_n^{(k)}\le u_n)-\mathbf{1}(M_{m,n}^{(k)}\le u_n)])\right|+\left|Cov(\mathbf{1}(M_m^{(k)}\le u_m),\mathbf{1}(M_{m,n}^{(k)}\le u_n))\right| \\ &\le 2E\left|\mathbf{1}(M_n^{(k)}\le u_n)-\mathbf{1}(M_{m,n}^{(k)}\le u_n)\right|+\left|Cov(\mathbf{1}(M_m^{(k)}\le u_m),\mathbf{1}(M_{m,n}^{(k)}\le u_n))\right| \\ &\ll \frac{m}{n}+(\log\log n)^{-(1+\varepsilon)}. \end{aligned}$$

and then we conclude that

$$\begin{aligned} L_{N,2} &\ll \sum_{\substack{1\le m<n\le N\\ n>m+k}}\frac{1}{mn}\left(\frac{m}{n}+(\log\log n)^{-(1+\varepsilon)}\right)+\sum_{\substack{1\le m<n\le N\\ n\le m+k}}\frac{1}{mn} \\ &\le \sum_{n=1}^{N}\frac{1}{n}+\sum_{n=3}^{N}\frac{\log n}{n}(\log\log n)^{-(1+\varepsilon)}+\sum_{m=1}^{N}\frac{1}{m} \\ &\ll (\log N)^2(\log\log N)^{-(1+\varepsilon)}. \end{aligned}$$

Thus, (20) holds. Note that Theorem 1.1 implies

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}P(M_n^{(k)}\le u_n)=e^{-\tau}\sum_{s=0}^{k-1}\frac{(\tau)^s}{s!}\ \ a.s., \tag{21}$$

and then the first assertion of Theorem 2.1 follows from (19) and (21).

(ii). (ii) is a special case of (i), thus we omit the proof.

Proof of Theorem 2.2. (i). We give first the proof of (15). We have

$$\begin{aligned} |P(M_n^{(k_n)}\le v_n)-\Phi(x)| &\le |P(M_n^{(k_n)}\le v_n)-P(\widetilde{M}_n^{(k_n)}\le v_n)|+|P(\widetilde{M}_n^{(k_n)}\le v_n)-\Phi(x)| \\ &=: D_{n,1}+D_{n,2}. \end{aligned}$$

By Lemmas 3.2 and 3.4, we have

$$D_{n,1}\le Kn\sum_{j=1}^{n}|r_j|\exp\left(-\frac{v_n^2}{1+|r_j|}\right)\to 0$$

as $n\to\infty$. Recall that $v_n=\alpha_n^{-1}x+\beta_n$ and k_n is an intermediate rank sequence, from Theorem 2.5.2 of Leadbetter et al. (1983), $D_{n,2}\to 0$ as $n\to\infty$, see also Wu (1966). This completes the proof of (15).

Since the proof for (16) is similar to that of Theorem 2.1, we only give the arguments for $L_{N,2}$. By Lemma 3.5 and using the condition $\log k_n \ll (\log n)^{1-\varepsilon}$ for some $\varepsilon > 0$, we have

$$\begin{aligned}
L_{N,2} &\leq \sum_{\substack{1\le m<n\le N \\ m\le \frac{n}{k_n}-1}} \frac{1}{mn}(k_n\frac{m}{n-k_n} + n^{-\epsilon} + n^{-\epsilon'}) + \sum_{\substack{1\le m<n\le N \\ m> \frac{n}{k_n}-1}} \frac{1}{mn} \\
&\leq \sum_{n=k_n+1}^{N}\sum_{m=1}^{\lfloor n/k_n\rfloor} \frac{k_n}{n(n-k_n)} + \sum_{n=1}^{N}\sum_{m=1}^{n}\frac{1}{mn^{1+\epsilon}} + \sum_{n=1}^{N}\sum_{m=1}^{n}\frac{1}{mn^{1+\epsilon'}} + \sum_{n=1}^{N}\sum_{m=\lfloor n/k_n\rfloor}^{n}\frac{1}{mn} \\
&\leq \sum_{n=k_n+1}^{N}\frac{1}{n-k_n} + \sum_{n=1}^{N}\frac{\log n}{n^{1+\epsilon}} + \sum_{n=1}^{N}\frac{\log n}{n^{1+\epsilon'}} + \sum_{n=1}^{N}\frac{\log k_n}{n} \\
&\leq 3\log N + (\log N)^{2-\varepsilon}.
\end{aligned}$$

Thus, (20) holds. Note that (15) implies

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}P(M_n^{(k_n)} \leq v_n) = \Phi(x) \ \ a.s.,$$

and then (16) follows from (19).

(ii). Since (ii) is a special case of (i), the proof is omitted.

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