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A Fixed Point Problem with Constraint Inequalities via a Contraction in Incomplete Metric Spaces

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Abstract. In the present paper, firstly, we review the notion of the SO-complete metric spaces. This notion let us to consider some fixed point theorems for single-valued mappings in incomplete metric spaces. Secondly, as motivated by the recent work of H. Baghani et al.(A fixed point theorem for a new class of set-valued mappings in R-complete (not necessarily complete) metric spaces, Filomat, 31 (2017), 3875–3884), we obtain the results of Ansari et al. [J. Fixed Point Theory Appl. (**2017**), 1145–1163] with very much weaker conditions. Also, we provide some examples show that our main theorem is a generalization of previous results. Finally, we give an application to the boundary value system for our results.

1. Introduction and preliminaries

The Banach contraction mapping principle is one of the pivotal results in fixed point theory which their conditions dropped by a large number of researchers(see [1, 7–9, 13]). Recently, Jleli and Samet [11] provided sufficient conditions for the existence of a fixed point of T satisfying the two constraint inequalities: $Ax \preceq_1 Bx$ and $Cx \preceq_2 Dx$, where $T : X \to X$ defined on a complete metric space equipped with two partial orders $'' \preceq_1 ''$ and $'' \preceq_2 ''$ and $A, B, C, D : X \to X$ are self-operators. In the other words, this problem containes: finding $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{1}$$

Ansari, Kumam and Samet in [2] proved that this problem has a unique solution without continuity of C and D.
Before presenting the main result obtained in [2], let us recall some concepts introduced in [11].

Definition 1.1. *[11] Let (X, d) be a metric space. A partial order $'' \preceq ''$ on X is d-regular if for any two sequences $\{u_n\}$ and $\{v_n\}$ in X, we have*

$$\lim_{n\to\infty} d(u_n, u) = \lim_{n\to\infty} d(v_n, v) = 0, u_n \preceq v_n \text{ for all } n \Longrightarrow u \preceq v,$$

where $(u, v) \in X \times X$.

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Definition 1.2. *[11] Let "* $\preceq_1$ *" and "* $\preceq_2$ *" be two partial orders on* X *and operators* $T, A, B, C, D : X \to X$ *be given. The operator* T *is called* $(A, B, C, D, \preceq_1, \preceq_2)$*-stable if*

$$x \in X,\ Ax \preceq_1 Bx \implies CTx \preceq_2 DTx.$$

Let Φ be the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(Φ_1) φ is a lower semicontinuous function;
(Φ_2) $\varphi^{-1}(\{0\}) = \{0\}$.

The main theorem presented in [2] is given by the following result.

Theorem 1.3. *Let* (X, d) *be a complete metric space endowed with two partial orders "* $\preceq_1$ *" and "* $\preceq_2$ *". Let operators* $T, A, B, C, D : X \to X$ *be given. Suppose that the following conditions are satisfied:*

(i) " $\preceq_i$ " *is d-regular,* $i = 1, 2$*;*
(ii) A, B *are continuous;*
(iii) there exists $x_0 \in X$ *such that* $Ax_0 \preceq_1 Bx_0$*;*
(iv) T *is* $(A, B, C, D, \preceq_1, \preceq_2)$*-stable;*
(v) T *is* $(C, D, A, B, \preceq_2, \preceq_1)$*-stable;*
(vi) there exsists $\varphi \in \Phi$ *such that*

$$Ax \preceq_1 Bx,\ Cy \preceq_2 Dy \Longrightarrow d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)).$$

Then the sequence $\{T^n x_0\}$ *converges to some* $x^* \in X$*, which is a unique solution to* (1).

In this paper, we address the following questions.
Q_1: Is it possible to remove the completness assumption of the space in Theorem 1.3?
Q_2: Is it possible to remove the continuity conditions of the mappings A and B in Theorem 1.3?
Q_3: Is condition (*vi*) have to satisfy all the x and y that $Ax \preceq_1 Bx$ and $Cy \preceq_2 Dy$ or not, we can limite it?

In future, we show that Theorem 1.3 is hold whenever X is not a complete metric space and condition (*iv*) is suffitient to satisfy more limited number x and y in X. For this purpose, we review the concept of orthogonal sets introduced in [4, 5, 10]. Also, we prove that continuity assumptions of the mappings A and B in Theorem 1.3 are not necessary. Finally, we give an application related to boundary value systems. For more application of fixed point theorem the reads can see [6, 12, 15, 16].
At first, we recall some important definitions.

Definition 1.4. *[3, 10] Let* $X \neq \emptyset$ *and* $\perp \subseteq X \times X$ *be a binary relation. If "* $\perp$ *" satisfies the following condition:*

$$\exists x_0\text{: } (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y),$$

then " $\perp$ *" is called an orthogonality relation and the pair* $(X, \perp)$ *an orthogonal set(briefly O-set).*

Note that in above definition, we say that x_0 is an orthogonal element. Also, we say that elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$.

Definition 1.5. *[3, 10] Let* $(X, \perp)$ *be an O-set. A sequence* $\{x_n\}$ *is called an orthogonal sequence(briefly, O-sequence) if*

$$(\forall n, x_n \perp x_{n+1}) \text{ or } (\forall n, x_{n+1} \perp x_n).$$

Next, we introduce the new type of sequences in *O*-sets.

Definition 1.6. *[14] Let* $(X, \perp)$ *be an O-set. A sequence* $\{x_n\}$ *is called a strongly orthogonal sequence(briefly, SO-sequence) if*

$$(\forall n, k;\ \ x_n \perp x_{n+k}) \text{ or } (\forall n, k;\ \ x_{n+k} \perp x_n).$$

It is obvious that every SO-sequence is an O-sequence. The following example shows that the converse is not true in general.

Example 1.7. *Let $X = \mathbb{N} \cup \{0\}$. Suppose $x \bot y$ iff $xy = 0$. Define the sequence $\{x_n\}$ as follows:*

$$x_n = \begin{cases} 0 & n = 2k, \ for\ some\ k \in \mathbb{N} \cup \{0\}, \\ n & n = 2k+1, \ for\ some\ k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Then for all $n \in \mathbb{N} \cup \{0\}$, $x_n \bot x_{n+1}$, but x_{2n+1} is not orthogonal to x_{4n+1}. Therefore $\{x_n\}$ is an O-sequence which is not SO-sequence.

Definition 1.8. *[3, 10] Let $(X, \bot, d)$ be an orthogonal metric space ($(X, \bot)$ is an O-set and (X, d) is a metric space). X is said to be orthogonal complete(briefly, O-complete) if every Cauchy O-sequence is convergent.*

Definition 1.9. *[14] Let $(X, \bot, d)$ be an orthogonal metric space. X is said to be strongly orthogonal complete(briefly, SO-complete) if every Cauchy SO-sequence is convergent.*

Clearly, every O-complete metric space is SO-complete. In the next example X is SO-complete but it is not O-complete.

Example 1.10. *Let $X = \{\sqrt{2}\} \cup \{\frac{1}{2n}\}_{n>1}$ with the Euclidean metric. Define orthogonal relation " $\bot$ " as follows:*

$$x \bot y \iff \frac{x}{y} \notin \mathbb{N} - \{1\} \quad and \quad x \geq y.$$

Clearly, X is O-set with $x_0 = \sqrt{2}$. Obviously, X is SO-complete metric space. But X is not O-complete metric space. Because the Cauchy O-sequence $x_n = 1/2n$ in X is not convergent in X.

Definition 1.11. *[3, 10] Let $(X, \bot, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is orthogonal continuous(briefly, O-continuous) in $a \in X$ if for each O-sequence $\{a_n\}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is O-continuous on X if f is O-continuous in each $a \in X$.*

Definition 1.12. *[14] Let $(X, \bot, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is strongly orthogonal continuous(briefly, SO-continuous) in $a \in X$ if for each SO-sequence $\{a_n\}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is SO-continuous on X if f is SO-continuous in each $a \in X$.*

It is easy to see that every continuous mapping is O-continuous and every O-continuous mapping is SO-continuous. The following example shows that the converse is not true in general.

Example 1.13. *Let $X = [0, 1]$ with the Euclidean metric. Assume " $\bot$ " is the orthogonal relation in Example 1.7. Define $f : X \to X$ by*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1], \\ x & x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Notice that f is not continuous but we can see that f is SO-continuous. If $\{x_n\}$ is a SO-sequence in X which converges to $x \in X$. Applying definition " $\bot$ " we obtain $x_n = 0$. This implies that $1 = f(x_n) \to f(x) = 1$. To see that f is not O-continuous, consider the sequence

$$x_n = \begin{cases} 0 & n = 2k+1, \ for\ some\ k \in \mathbb{N} \cup \{0\}, \\ \frac{\sqrt{3}}{k} & n = 2k, \ for\ some\ k \in \mathbb{N} \cup \{0\}. \end{cases}$$

It's clear that $x_n \to 0$ while the sequence $\{f(x_n)\}$ is not convergent to $f(0)$.

Definition 1.14. *Let $(X, \perp, d)$ be an orthogonal metric space. Then X is said to be $\perp$-regular if for each SO-sequence $\{x_n\}$ with $x_n \to x$ for some $x \in X$, we conclude that*

$$(\forall n; \quad x_n \perp x) \quad or \quad (\forall n; \quad x \perp x_n).$$

Definition 1.15. *Let $(X, \perp, d)$ be an orthogonal metric space. We say that a partial order $'' \leq ''$ on X is $d_\perp$-regular if for each two SO-sequences $\{u_n\}$ and $\{v_n\}$ in X, we have*

$$\lim_{n\to\infty} d(u_n, u) = \lim_{n\to\infty} d(v_n, v) = 0, u_n \leq v_n \quad \text{for all } n \Longrightarrow u \leq v,$$

where $(u, v) \in X \times X$.

It is easy to see that every partial order $'' \leq ''$ which is d-regular also is $d_\perp$-regular but the converse is not true in general.

Example 1.16. *Let $X = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \cdots, \frac{1}{n+1}, \frac{n+1}{n+2}, \cdots\}$. Define partial order $'' \leq ''$ on X as follows:*

$$x \leq y \Longleftrightarrow (x = y = 1) \quad \text{or} \quad (y \neq 1 \quad and \quad x \leq y).$$

We claim that $'' \leq ''$ is not d-regular.
For this purpose, we consider two sequences $t_n = \{\frac{n+1}{n+2}\}$ and $t'_n = \{\frac{1}{n+1}\}$. We have $\lim_{n\to\infty} d(t_n, 1) = \lim_{n\to\infty} d(t'_n, 0) = 0, t'_n \leq t_n$ for all n but $0 \leq 1$. Now for all $x, y \in X$ define $x \perp y$ if and only if either $x = 0$ or $x \leq y \leq \frac{1}{2}$. Then $(X, \perp)$ is an O-set with orthogonal element $x_0 = 0$ and also it is $d_\perp$-regular.

Definition 1.17. *[3, 10] Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $T(x) \perp T(y)$.*

Proposition 1.18. *Let $(X, \perp, d)$ be an O-set with orthogonal element x_0 and $T : X \to X$ be $\perp$-preserving. Let $\{x_n\}$ be Picard iterative sequence with initial point x_0 in X, i.e. $x_n = T^n x_0$. Then $\{x_n\}$ is a SO-sequence.*

Proof. From the definition of orthogonal element x_0, we have

$$x_0 \perp Tx_0 = x_1, \ \ x_0 \perp T^2x_0 = x_2, \ \ldots \ , x_0 \perp T^n x_0 = x_n, \ \cdots,$$

or

$$x_1 = Tx_0 \perp x_0, \ \ x_2 = T^2x_0 \perp x_0, \ \ldots \ , x_n = T^n x_0 \perp x_0, \ \cdots.$$

Also, since T is $\perp$-preserving, we have

$$x_1 = Tx_0 \perp T^2x_0 = x_2, x_1 = Tx_0 \perp T^3x_0 = x_3, \ldots, x_1 \perp x_{n+1}, \cdots,$$

or

$$x_2 = T^2x_0 \perp Tx_0 = x_1, x_3 = T^3x_0 \perp Tx_0 = x_1, \ldots, x_{n+1} \perp x_1, \cdots.$$

Continuing this process, we have

$$x_n = T^n x_0 \perp T^{n+1}x_0 = x_{n+1}, x_n = T^n x_0 \perp T^{n+2}x_0 = x_{n+2}, \ldots, x_n \perp x_{n+k}, \ldots,$$

or

$$x_{n+1} = T^{n+1}x_0 \perp T^n x_0 = x_n, x_{n+2} = T^{n+2}x_0 \perp T^n x_0 = x_n, \ldots, x_{n+k} \perp x_n, \ldots.$$

Therefore, we see that

$$(\forall n, k; \quad x_n \perp x_{n+k}) \quad or \quad (\forall n, k; \quad x_{n+k} \perp x_n).$$

□

2. The main results

In the following theorem, which is our main result, we weaken assumptions (*ii*) and (*vi*) of Theorem 1.3. Moreover, we show that under our assumptions, (1) has a unique solution. This gives a partial answer to Q_1, Q_2 and Q_3.

Theorem 2.1. *Let $(X, \perp, d)$ be an SO-complete metric space(not necessarily complete) with orthogonal element x_0. Let " $\preceq_1$ " and " $\preceq_2$ " be two partial order over X. Also, let operators $T, A, B, C, D : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) " $\preceq_i$ " *is $d_\perp$-regular, $i = 1, 2$ and T is $\perp$-preserving;*

(ii) *A, B are SO-continuous;*

(iii) *$Ax_0 \preceq_1 Bx_0$ and X is $\perp$-regular;*

(iv) *T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable;*

(v) *T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;*

(vi) *there exits $\varphi \in \Phi$ such that for each $\perp$-comparable elements $x, y \in X$*

$$(Ax \preceq_1 Bx \text{ and } Cy \preceq_2 Dy) \Longrightarrow d(Tx, Ty) \leqslant d(x, y) - \varphi(d(x, y)).$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$ which is a solution to (1). *Moreover, x^* is the unique solution of* (1).

Proof. Consider the sequence $\{x_n\}$ defined by $x_n = T^n x_0$, $n = 0, 1, 2, \cdots$. Applying Proposition 1.18, $\{x_n\}$ is a SO-sequence. Applying (*iii*), we have

$$Ax_0 \preceq_1 Bx_0.$$

On the other hand, since T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, we have

$$Ax_0 \preceq_1 Bx_0 \quad \Longrightarrow \quad CTx_0 \preceq_2 DTx_0,$$

that is, $Cx_1 \preceq_2 Dx_1$. Hence

$$Ax_0 \preceq_1 Bx_0 \text{ and } Cx_1 \preceq_2 Dx_1.$$

Since T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable,

$$Cx_1 \preceq_2 Dx_1 \Longrightarrow ATx_1 \preceq_1 BTx_1,$$

that is, $Ax_2 \preceq_1 Bx_2$.
Continuing this process, by induction, we get

$$Ax_{2n} \preceq_1 Bx_{2n} \text{ and } Cx_{2n+1} \preceq_2 Dx_{2n+1}, \quad n = 0, 1, 2, \cdots . \tag{2}$$

Since $\{x_n\}$ is SO-sequence, applying (2) and (*vi*), we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1})). \tag{3}$$

for each $n \in \mathbb{N}$. This implies that $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$ for all $n \in \mathbb{N}$. Then $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence and bounded below. Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{4}$$

Let $r > 0$. Applying (3), we have

$$d(x_{n+1}, x_n) + \varphi(d(x_n, x_{n-1})) \leq d(x_n, x_{n-1}), \quad n = 0, 1, 2, 3, \cdots .$$

Therefore,

$$\liminf_{n\to\infty}(d(x_{n+1},x_n)+\varphi(d(x_n,x_{n-1})))\leq\liminf_{n\to\infty}(d(x_n,x_{n-1})).$$

Applying (4) and the lower semi-continuity of φ, we have

$$r+\varphi(r)\leq r.$$

This is a contradiction, since $\varphi(r)>0$. Thus

$$\lim_{n\to\infty} d(x_{n+1},x_n)=0. \tag{5}$$

Now, we show that $\{x_n\}$ is a Cauchy SO-sequence. Suppose that $\{x_n\}$ is not a Cauchy SO-sequence. Then, there exists some $\varepsilon>0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k, we have

$$n(k)>m(k)>k,\quad d(x_{m(k)},x_{n(k)})\geq\varepsilon,\quad d(x_{m(k)},x_{n(k)-1})<\varepsilon. \tag{6}$$

To prove (6), suppose that

$$\sum\nolimits_k=\{m\in\mathbb{N};\quad \exists m_k\geq k\ ,\ d(x_m,x_{m_k})\geq\epsilon\ ,\ m>m_k>k\}.$$

Obviously, $\sum_k\neq\emptyset$ and $\sum_k\subseteq\mathbb{N}$. Then by the well ordering principle, the minimum element of $\sum_k$ exists and denoted by n_k, and clearly (6) holds. Applying (6), we deduce that

$$\varepsilon\leq d(x_{m(k)},x_{n(k)})\leq d(x_{m(k)},x_{n(k)-1})\ +\ d(x_{n(k)-1},x_{n(k)})<\varepsilon\ +\ d(x_{n(k)-1},x_{n(k)}).$$

Let $k\to\infty$ and using (5), we have

$$\lim_{k\to\infty} d(x_{n(k)},x_{m(k)})=\varepsilon. \tag{7}$$

Triangle inequality, implies that

$$|d(x_{n(k)+1},x_{m(k)})-d(x_{m(k)},x_{n(k)})|\leq d(x_{n(k)+1},x_{n(k)}).$$

Applying (5) and (7), as $k\to\infty$, we have

$$\lim_{k\to\infty} d(x_{n(k)+1},x_{m(k)})=\varepsilon. \tag{8}$$

Similarly,

$$\lim_{k\to\infty} d(x_{n(k)},x_{m(k)-1})=\varepsilon, \tag{9}$$

and also

$$\lim_{k\to\infty} d(x_{n(k)+1},x_{m(k)+1})=\varepsilon. \tag{10}$$

We see that, for all k, there exists $i(k)\in\{0,1\}$ such that

$$n(k)-m(k)+i(k)\equiv 1(2).$$

Now, applying (2), for all $k>1$, we deduce that

$$Ax_{n(k)}\preceq_1 Bx_{n(k)}\ \text{ and }\ Cx_{m(k)-i(k)}\preceq_2 Dx_{m(k)-i(k)},$$

or

$$Ax_{m(k)-i(k)}\preceq_1 Bx_{m(k)-i(k)}\ \text{ and }\ Cx_{n(k)}\preceq_2 Dx_{n(k)}.$$

Now, applying (vi), for $k > 1$, we conclude that

$$d(x_{n(k)+1}, x_{m(k)-i(k)+1}) = d(Tx_{n(k)}, Tx_{m(k)-i(k)}) \leq d(x_{n(k)}, x_{m(k)-i(k)}) - \varphi(d(x_{n(k)}, x_{m(k)-i(k)})). \tag{11}$$

Define

$$\Lambda = \{k > 1 : i(k) = 0\} \text{ and } \Delta = \{k > 1 : i(k) = 1\},$$

and investigate the following two cases:
Cace1. $|\Lambda| = \infty$.
Applying (11), for $k \in \Lambda$, we have

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{n(k)}, x_{m(k)}) - \varphi(d(x_{n(k)}, x_{m(k)})).$$

Therefore

$$\liminf_{k\to\infty}(d(x_{n(k)+1}, x_{m(k)+1}) + \varphi(d(x_{n(k)}, x_{m(k)})) \leq \liminf_{k\to\infty} d(x_{n(k)}, x_{m(k)}).$$

Applying (7), (10) and lower semi-continuity of φ, we have

$$\varepsilon + \varphi(\varepsilon) \leq \varepsilon.$$

This is a contradiction, since $\varphi(\varepsilon) > 0$. Hence $\varepsilon = 0$.
Cace2. $|\Lambda| < \infty$.
Therefore, $|\Delta| = \infty$. Applying (11), we have

$$d(x_{n(k)+1}, x_{m(k)}) + \varphi(d(x_{n(k)}, x_{m(k)-1})) \leq d(x_{n(k)}, x_{m(k)-1}), \quad k \in \Delta.$$

Hence

$$\liminf_{k\to\infty}(d(x_{n(k)+1}, x_{m(k)}) + \varphi(d(x_{n(k)}, x_{m(k)-1}))) \leq \liminf_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}).$$

Applying (8), (9) and lower semi-continuity of φ, we deduce that

$$\varepsilon + \varphi(\varepsilon) \leq \varepsilon,$$

which is a contradiction, since $\varphi(\varepsilon) > 0$. Thus $\varepsilon = 0$. Therefore $\{x_n\}$ is a Cauchy SO-sequence. Since $(X, \perp, d)$ is SO-complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{12}$$

Since $\{x_n\}$ is SO-sequence, we deduce that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are SO-sequences. Applying the SO-continuity of A and B, and (12), we deduce that

$$\lim_{n\to\infty} d(Ax_{2n}, Ax^*) = \lim_{n\to\infty} d(Bx_{2n}, Bx^*) = 0.$$

Since " $\preceq_1$ " is $d_\perp$-regular, (2) implies that

$$Ax^* \preceq_1 Bx^*. \tag{13}$$

Since X is $\perp$-regular, then

$$(\forall n; \quad x_{2n+1} \perp x^*) \quad \text{or} \quad (\forall n; \quad x^* \perp x_{2n+1}).$$

Applying (2), (13) and (vi), we obtain that

$$d(Tx^*, Tx_{2n+1}) \leq d(x^*, x_{2n+1}) - \varphi(d(x^*, x_{2n+1})), \quad n = 0, 1, 2 \ldots.$$

The triangle inequality implies that

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_{2n+1}) + d(Tx_{2n+1}, x^*) \\ &\leq d(x^*, x_{2n+1}) - \varphi(d(x_{2n+1}, x^*)) + d(x_{2n+2}, x^*). \end{aligned}$$

Hence

$$\liminf_{k\to\infty}(d(Tx^*, x^*) + \varphi(d(x^*, x_{2n+1}))) \leq \liminf_{k\to\infty}(d(x^*, x_{2n+1}) + d(x_{2n+2}, x^*)).$$

The lower semi-continuity of φ, $\varphi(0) = 0$ and (12) imply that

$$d(x^*, Tx^*) = 0,$$

that is

$$Tx^* = x^*. \tag{14}$$

Since T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, applying (13), we have

$$CTx^* \preceq_2 DTx^*,$$

and also (14) implies that

$$Cx^* \preceq_2 Dx^*. \tag{15}$$

Applying (13), (14) and (15), we deduce that x^* is a solution of (1). We show that x^* is unique. For this purpose, let $y^* \in X$ be another solution of (1), that is

$$Ty^* = y^*, \quad Ay^* \preceq_1 By^*, \quad Cy^* \preceq_2 Dy^* \quad \text{and} \quad d(x^*, y^*) > 0. \tag{16}$$

Since x_0 is an orthogonal element, by the definition of orthogonality, we have

$$x_0 \perp y^* \quad or \quad y^* \perp x_0.$$

Since T is $'' \perp ''$ preserving, then

$$x_{2n} = T^{2n}x_0 \perp T^{2n}y^* = y^* \quad or \quad y^* = T^{2n}y^* \perp T^{2n}x_0 = x_{2n}. \tag{17}$$

Applying (2), (17), (16) and (vi), we have

$$d(Tx_{2n}, Ty^*) \leq d(x_{2n}, y^*) - \varphi(d(x_{2n}, y^*)).$$

Therefore

$$d(x_{2n+1}, y^*) + \varphi(d(x_{2n}, y^*)) \leq d(x_{2n}, y^*). \tag{18}$$

Since φ is lower semi-continuous, we deduce that

$$d(x^*, y^*) + \varphi(d(x^*, y^*)) \leq d(x^*, y^*).$$

This is a contradiction. Therefore $x^* = y^*$ and x^* is the unique solution of (1). □

3. Particular cases

Now, we consider some special cases, where in our result deduce several well-known fixed point theorems of the existing literature.
In Theorem 2.1, by setting $\preceq_1=\preceq_2=\preceq$, $C = B$ and $D = A$, we get a generalization of Corollary 3.1 of [11].

Corollary 3.1. *Let $(X, \perp, d)$ be a SO-complete metric space(not necessarily complete) with orthogonal element x_0. Let "$\preceq$" be a certain partial order over X. Also, let operators $T, A, B : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) "$\preceq$" is $d_\perp$-regular and T is $\perp$-preserving;
(ii) A, B are SO-continuous;
(iii) $Ax_0 \preceq Bx_0$ and X is $\perp$-regular;
(iv) for all $x \in X$, we have

$$Ax \preceq Bx \implies BTx \preceq ATx;$$

(v) for all $x \in X$, we have

$$Bx \preceq Ax \implies ATx \preceq BTx;$$

(vi) there exists $\varphi \in \Phi$ such that for each $\perp$-comparable elements $x, y \in X$

$$(Ax \preceq Bx \text{ and } By \preceq Ay) \implies d(Tx, Ty) \leqslant d(x, y) - \varphi(d(x, y)).$$

Then
(1) *The sequence $\{T^n x_0\}$ converges to $x^* \in X$ satisfying $Ax^* = Bx^*$.*
(2) *The point $x^* \in X$ is a unique solution to following problem*

$$\begin{cases} x = Tx, \\ Ax = Bx. \end{cases}$$

By setting $A = D = I_x$ and $C = B$ we get a generalization of Corollary 3.2 of [11].

Corollary 3.2. *Let $(X, \perp, d)$ be a SO-complete metric space(not necessarily complete) with orthogonal element x_0. Let "$\preceq$" be a certain partial order over X. Also, let operators $T, B : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) "$\preceq$" is $d_\perp$-regular and T is $\perp$-preserving;
(ii) B is SO-continuous;
(iii) $x_0 \preceq Bx_0$ and X is $\perp$-regular;
(iv) for all $x \in X$, we have

$$x \preceq Bx \implies BTx \preceq Tx;$$

(v) for all $x \in X$, we have

$$Bx \preceq x \implies Tx \preceq BTx;$$

(vi) there exists $\varphi \in \Phi$ such that for each $\perp$-comparable elements $x, y \in X$

$$(x \preceq Bx \text{ and } By \preceq y) \implies d(Tx, Ty) \leqslant d(x, y) - \varphi(d(x, y)).$$

Then
(1) *The sequence $\{T^n x_0\}$ converges to $x^* \in X$ satisfying $x^* = Tx^*$.*
(2) *The point $x^* \in X$ is a unique solution of following problem*

$$\begin{cases} x = Tx, \\ x = Bx. \end{cases}$$

By setting $C = B = T$ and $A = D = I_x$, we obtain a generalization of Corollary 3.4 of [11].

Corollary 3.3. *Let $(X, \perp, d)$ be a SO-complete metric space(not necessarily complete) with orthogonal element x_0. Let $'' \preceq ''$ be a certain partial order over X. Also, let operator $T : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) $'' \preceq ''$ is $d_\perp$-regular and T is $\perp$-preserving;
(ii) T is SO-continuous;
(iii) $x_0 \preceq Tx_0$ and X is $\perp$-regular;
(iv) for all $x \in X$, we have
$$x \preceq Tx \implies T^2x \preceq Tx;$$
(v) for all $x \in X$, we have
$$Tx \preceq x \implies Tx \preceq T^2x;$$
(vi) there exists $\varphi \in \Phi$ such that for each $\perp$-comparable elements $x, y \in X$
$$(x \preceq Tx \text{ and } Ty \preceq y) \Longrightarrow d(Tx, Ty) \leqslant d(x, y) - \varphi(d(x, y)).$$

Then
(1) *The sequence $\{T^n x_0\}$ converges to $x^* \in X$ satisfying $x^* = Tx^*$.*
(2) *The point $x^* \in X$ is a unique fixed point of T.*

4. Some examples

Now, we illustrate our main results by the following examples.

Example 4.1. *Let $X = (-2, 3)$. Suppose that*
$$x \perp y \Longleftrightarrow (x = 0) \text{ or } (-1 \leq x \leq y \leq 1 \text{ and } y \neq 0).$$
Then $(X, \perp)$ is an O-set with orthogonal element $x_0 = 0$. Clearly, X with the Euclidean metric is not a complete metric space, but it is SO-complete(In fact, if $\{x_k\}$ is an arbitrary Cauchy SO-sequence in X, either there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $\{x_{k_n}\} = 0$ for all $n \geq 1$ or there exists a monotone subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $-1 \leq x_{k_n} \leq 1$ for all $n \geq 1$. It follows that $\{x_{k_n}\}$ converges to a point $x \in [-1, +1] \subseteq X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent.).
We see that X is $\perp$-regular. We take $\preceq_1 = \preceq_2 = \preceq$. Let $T : X \to X$ be the mapping defined by
$$T(x) = \begin{cases} 0 & x < 1 \\ -1/2 & x = 1 \\ 1 & x > 1. \end{cases}$$
We show that T is $\perp$-preserving. For all $x, y \in X$ such that $x \perp y$, we consider the following cases:
Case1. If $x < 1$, then $Tx = 0$. Thus $Tx \perp Ty$.
Case2. If $x = 1$, then we have $y = 1$ and so $Tx \perp Ty$.
Case3. If $x > 1$, there is not $y \in X$ such that $x \perp y$.
Therefore T is $\perp$-preserving.
Consider the mappings $A, B, C, D : X \to X$ defined by $Ax = 0$, $Cx = x$,
$$B(x) = \begin{cases} 1 & x \leq 1 \\ -x & x > 1, \end{cases}$$

and

$$D(x) = \begin{cases} 1 - x & x < 0 \\ -x/2 & x \geq 0. \end{cases}$$

Obviously, " $\preceq_i$ *" is* $d_\perp$*-regular,* $i = 1, 2$*. Moreover, A and B are SO-continuous mappings. If for some* $x \in X$*, we have*

$$Ax \leq Bx,$$

then $x \leq 1$*, which yields*

$$Tx = 0 \text{ or } Tx = -1/2.$$

If $Tx = 0$*, we have*

$$CT(x) = C(0) = 0 = D(0) = DT(x).$$

On the other hand, if $Tx = -1/2$*, we obtain*

$$CT(x) = C(-1/2) = -1/2 \leq 3/2 = D(-1/2) = DT(x).$$

Thus T is $(A, B, C, D, \preceq_1, \preceq_2)$*-stable. If for some* $x \in X$*, we have*

$$Cx \leq Dx,$$

then $x \leq 0$*, which yields* $Tx = 0$*. Therefore*

$$AT(x) = A(0) = 0 \leq 1 = B(0) = BT(x).$$

Thus T is $(C, D, A, B, \preceq_2, \preceq_1)$*-stable. For all* $(x, y) \in X \times X$*, we have*

$$Ax \preceq_1 Bx, \quad Cy \preceq_2 Dy \quad \Longrightarrow (x \leq 1 \quad \text{and} \quad y \leq 0).$$

Therefore, either

$$(x < 1 \quad \text{and} \quad y \leq 0) \Longrightarrow (Tx, Ty) = (0, 0),$$

or

$$(x = 1 \quad \text{and} \quad y \leq 0) \Longrightarrow (Tx, Ty) = (-1/2, 0).$$

Thus

$$Ax \preceq_1 Bx \quad \text{and} \quad Cy \preceq_2 Dy \implies d(Tx, Ty) \leqslant d(x, y) - \varphi(d(x, y)),$$

where $\varphi(t) = t/3$*,* $t \geq 0$*. Applying Theorem 2.1, (1) has unique solution* $x^* = 0$*.*
Note that, the mappings B, C, D and T are not continuous and (X, d) *is not a complete metric space.*

Example 4.2. *Let* $X = \mathbb{Q}$*. Suppose that*

$$x \perp y \Longleftrightarrow (x = 0) \text{ or } (y = 1/n, \ n \in \mathbb{N}).$$

Then $(X, \perp)$ *is an O-set with orthogonal element* $x_0 = 1/2$*. Clearly,* $\mathbb{Q}$ *with the Euclidean metric is not a complete metric space, but it is SO-complete. In fact, if* $\{x_k\}$ *is an arbitrary Cauchy SO-sequence in X, either there exists a subsequence* $\{x_{k_n}\}$ *of* $\{x_k\}$ *for which* $\{x_{k_n}\} = 0$ *for all* $n \geq 1$ *or there exists a monotone subsequence* $\{1/n_k\}$ *of* $\{1/n\}$ *for which* $1/n_k \to 0$ *as* $k \to \infty$*. It follows that* $\{1/n_k\}$ *converges to* $0 \in X$*. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follow that* $\{x_k\}$ *is convergent.*
We see that X is $\perp$*-regular. We take* $\preceq_1 = \preceq_2 = \leq$ *. Let* $T : X \to X$ *be the mapping defined by*

$$T(x) = \begin{cases} -1/2 & x \in \mathbb{Q} \cap \{x \leq -1\} \\ 0 & x \in \mathbb{Q} \cap \{-1 < x \leq 0\} \\ 1/2 & x \in \mathbb{Q} \cap \{x > 0\}. \end{cases}$$

Observed that T is $\perp$*-preserving. Let* $x \perp y$*. Then we have two cases:*

(1) If $x = 0$, since $Tx = 0$, then for each $y \in X$, we have $Tx \perp Ty$.

(2) If $x \neq 0$, then for each $n_0 \in \mathbb{N}$ such that $y = 1/n_0$. Since $T(1/n_0) = 1/2$, then we have $Tx \perp Ty$.

Consider the mappings $A, B, C, D : X \to X$ defined by $Cx = 1$,

$$A(x) = \begin{cases} x^2 + 1 & x \in \mathbb{Q} \cap \{x \geq -1\} \\ 1 & x \in \mathbb{Q} \cap \{x < -1\}, \end{cases}$$

$$B(x) = \begin{cases} 5x/2 & x \in \mathbb{Q} \cap \{x \geq 0\} \\ -1 & x \in \mathbb{Q} \cap \{x < 0\}, \end{cases}$$

and

$$D(x) = \begin{cases} x + 1 & x \in \mathbb{Q} \cap \{x > 0\} \\ -1 & x \in \mathbb{Q} \cap \{x \leq 0\}. \end{cases}$$

Obviously, " $\preceq_i$ " is $d_\perp$-regular, $i = 1, 2$. Moreover, A and B are SO-continuous mappings. If for some $x \in X$, we have $Ax \leq Bx$, then $x \in \mathbb{Q} \cap [1/2, 2]$, which yields $Tx = 1/2$. Therefore

$$CT(x) = C(1/2) = 1 \leq 3/2 = D(1/2) = DT(x).$$

Thus T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable. If for some $x \in X$, we have $Cx \leq Dx$, then $x \in \mathbb{Q} \cap (0, +\infty)$, which yields $Tx = 1/2$. Therefore

$$AT(x) = A(1/2) = 5/4 = B(1/2) = BT(x).$$

Thus T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable. Also, for all $(x, y) \in X \times X$, we have

$$\begin{aligned} Ax \preceq_1 Bx, Cy \preceq_2 Dy &\Longrightarrow (x \in \mathbb{Q} \cap [1/2, 2] \ \text{ and } \ y \in \mathbb{Q} \cap (0, +\infty)) \\ &\Longrightarrow (Tx, Ty) = (1/2, 1/2). \end{aligned}$$

Therefore,

$$Ax \preceq_1 Bx \quad \text{and} \quad Cy \preceq_2 Dy \implies d(Tx, Ty) \leqslant d(x, y) - \varphi(d(x, y)),$$

where $\varphi(t) = \frac{1}{3}t$, $t \geq 0$. Applying Theorem 2.1, (1) has unique solution $x^* = 1/2$.
Note that, the mappings A, B, D and T are not continuous and (X, d) is not a complete metric space.

5. Application for boundary value differential systems

Let $X = \{u \in C[0, 1] : u(t) \geq 0, \forall t \in [0, 1]\}$ endowed with the metric d induced by sup-norm. Consider the following boundary value system

$$\begin{cases} u^{(4)}(t) - \lambda f(t, u(t)) = 0, & \text{for } 0 < t < 1, \\ u^{(4)}(t) - \lambda g(t, u(t)) = 0, & \text{for } 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1), \end{cases} \tag{19}$$

where $0 < \lambda < 1$ is constant and $f, g : [0, 1] \times [0, \infty) \longrightarrow [0, \infty)$ are continuous functions for which:

(C_1) $g(t, u)$ is decreasing related to the second variable.

(C_2) (i) For all $u \in X$, we have

$$u(t) \leq \lambda \int_0^1 \left[\int_0^1 k(t,s)k(s,x)g(x,u(x))dx \right] ds \;\Rightarrow\; g(t,u(t)) \leq f(t,u(t)),$$

where $k : [0,1] \times [0,1] \longrightarrow [0,1]$ denotes the Green's function for the boundary value system (19) and is explicitly given by

$$k(t,s) = \begin{cases} t(1-s) & 0 \leq t \leq s \leq 1 \\ s(1-t) & 0 \leq s \leq t \leq 1. \end{cases}$$

(ii) For all $u \in X$, we have

$$\lambda \int_0^1 \left[\int_0^1 k(t,s)k(s,x)g(x,u(x))dx \right] ds \leq u(t) \;\Rightarrow\; f(t,u(t)) \leq g(t,u(t)).$$

(C_3) For all $u, v \in X$ with $u(t)v(t^{'}) \leq \max\{v(t), v(t^{'})\}$, for each $t, t^{'} \in [0,1]$, we have

$$\left(f(t,u(t))f(t^{'},v(t^{'})) \leq \frac{1}{\lambda} f(t,v(t)), \forall t,t^{'} \in [0,1] \right) \;or\; \left(f(t,u(t))f(t^{'},v(t^{'})) \leq \frac{1}{\lambda} f(t^{'},v(t^{'})), \forall t,t^{'} \in [0,1] \right).$$

(C_4) For all $u, v \in X$ with $u(t)v(t) \leq v(t)$, for each $t \in [0,1]$, we have

$$|f(t,u(t)) - f(t,v(t))| \leq \frac{\|u-v\|}{A},$$

where $\|u\| = \max_{t\in[0,1]} u(t)$ and $A = \max_{0\leq 1} \int_0^1 \int_0^1 k(t,s)k(s,x)dxds$.

Theorem 5.1. *Let the above conditions are satisfied. Then the boundary value system* (19) *has a unique positive solution.*

Proof. We define two operator equations $T, B : X \to X$ as follow:

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 \left[\int_0^1 k(t,s)k(s,x)f(x,u(x))dx \right] ds, \\ Bu(t) &= \lambda \int_0^1 \left[\int_0^1 k(t,s)k(s,x)g(x,u(x))dx \right] ds. \end{aligned} \tag{20}$$

We know that the boundary value system has a unique positive solution if and only if T and B have a unique common fixed point in X. We consider the following orthogonality relation in X:

$$u \perp v \iff u(t)v(t^{'}) \leq \max\{v(t), v(t^{'})\}, \tag{21}$$

for all $t, t^{'} \in [0,1]$ and $u, v \in X$. Since (X,d) is a complete metric space, then $(X, \perp, d)$ is SO-complete. We take $\preceq_1 = \preceq_2 = \preceq$. From definition, " $\preceq$ " is $d_\perp$-regular and X is $\perp$-regular. Clearly, B is SO-continuous. Now, we prove the following four steps to complete the proof.

Step1: T is $\perp$- preserving. Let $u, v \in X$ with $u \perp v$. We must show that

$$Tu(t)Tv(t^{'}) \leq \max\{T(v(t)), T(v(t^{'}))\},$$

for all $t, t' \in [0,1]$. Applying (20), we have

$$Tu(t)Tv(t') = \lambda^2 \int_0^1 \left[\int_0^1 \left[\int_0^1 \left[\int_0^1 k(t,s)k(s,x)k(t',s')k(s',x')f(x,u(x))f(x',v(x'))dx \right] dx' \right] ds \right] ds'.$$

Applying (C_3), we have two cases:

(1). $f(t,u(t))f(t',v(t')) \leq \frac{1}{\lambda} f(t,v(t))$. Applying definition of k, we have

$$\begin{aligned} Tu(t)Tv(t') &\leq \lambda^2 \int_0^1 \left[\int_0^1 \left[\int_0^1 \left[\int_0^1 k(t,s)k(s,x)k(t',s')k(s',x')\frac{1}{\lambda} f(x,v(x))dx \right] dx' \right] ds \right] ds' \\ &\leq \lambda^2 \frac{1}{\lambda} \int_0^1 \left[\int_0^1 \left[\int_0^1 \left[\int_0^1 k(t,s)k(s,x)f(x,v(x))dx \right] dx' \right] ds \right] ds' \\ &= \lambda \int_0^1 \left[\int_0^1 k(t,s)k(s,x)f(x,v(x))dx \right] ds \\ &= T(v(t)) \\ &\leq \max\{T(v(t)), T(v(t'))\}. \end{aligned}$$

(2). $f(t,u(t))f(t',v(t')) \leqslant \frac{1}{\lambda} f(t',v(t'))$. Applying definition of k, we have

$$\begin{aligned} Tu(t)Tv(t') &\leq \lambda^2 \int_0^1 \left[\int_0^1 \left[\int_0^1 \left[\int_0^1 k(t,s)k(s,x)k(t',s')k(s',x')\frac{1}{\lambda} f(x',v(x'))dx \right] dx' \right] ds \right] ds' \\ &\leq \lambda^2 \frac{1}{\lambda} \int_0^1 \left[\int_0^1 \left[\int_0^1 \left[\int_0^1 k(t',s')k(s',x')f(x',v(x'))dx \right] dx' \right] ds \right] ds' \\ &= \lambda \int_0^1 \left[\int_0^1 k(t',s')k(s',x')f(x',v(x'))dx' \right] ds' \\ &= T(v(t')) \\ &\leq \max\{T(v(t)), T(v(t'))\}. \end{aligned}$$

These imply that T is $\perp$-preserving.

Step2: We must show that for all $t \in [0,1]$ and $u \in X$,

$$u(t) \leq Bu(t) \Longrightarrow BTu(t) \leq Tu(t).$$

Let $t \in [0,1]$, $u \in X$ and $u(t) \leq Bu(t)$. Applying part (i) of (C_2), we have $g(t,u(t)) \leq f(t,u(t))$. Applying (20), we conclude that $Bu(t) \leq Tu(t)$. Since $u(t) \leq Bu(t) \leq Tu(t)$, part (i) of (C_2) and (C_1) imply that

$$g(t,Tu(t)) \leq g(t,Bu(t)) \leq g(t,u(t)) \leq f(t,u(t)).$$

Therefore $g(t,Tu(t)) \leq f(t,u(t))$. Applying (20), we have $BTu(t) \leq Tu(t)$.

Step3: We must show that for all $t \in [0,1]$ and $u \in X$,

$$Bu(t) \leq u(t) \Longrightarrow Tu(t) \leq BTu(t).$$

Let $t \in [0,1]$, $u \in X$ and $Bu(t) \leq u(t)$. Applying part (ii) of (C_2), we have $f(t,u(t)) \leq g(t,u(t))$. Applying (20), we conclude that $Tu(t) \leq Bu(t)$. Since $Tu(t) \leq Bu(t) \leq u(t)$, part (ii) of (C_2) and (C_1) imply that

$$f(t,u(t)) \leq g(t,u(t)) \leq g(t,Bu(t)) \leq g(t,Tu(t)).$$

Therefore $f(t,u(t)) \leq g(t,Tu(t))$. Applying (20), we have $Tu(t) \leq BTu(t)$.

Step4: We show that there exists $\varphi \in \Phi$ such that for each $\perp$-comparable elements $u, v \in X$

$$d(Tu,Tv) \leq d(u,v) - \varphi(d(u,v)).$$

Let $u, v \in X$ with $u \perp v$. Then for all $t \in [0,1]$, we have $u(t)v(t) \leq v(t)$. Applying (C_4), we obtain that

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \left|\lambda\left[\int_0^1\left[\int_0^1 k(t,s)k(s,x)f(x,u(x))dx\right]ds - \int_0^1\left[\int_0^1 k(t,s)k(s,x)f(x,v(x))dx\right]ds\right]\right| \\
&\leq \lambda\int_0^1\left[\int_0^1 k(t,s)k(s,x)|f(x,u(x)) - f(x,v(x))|dx\right]ds \\
&= \lambda\int_0^1\left[\int_0^1 k(t,s)k(s,x)dx\right]ds\frac{||u-v||}{A} \\
&\leq \lambda||u-v|| \\
&= ||u-v|| - (1-\lambda)||u-v||,
\end{aligned}$$

for all $t \in [0,1]$. By setting $\varphi(t) = (1-\lambda)t$ and applying Corollary 3.2, T and B have a unique common fixed point in X which is a unique positive solution to the boundary value system (19). □

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